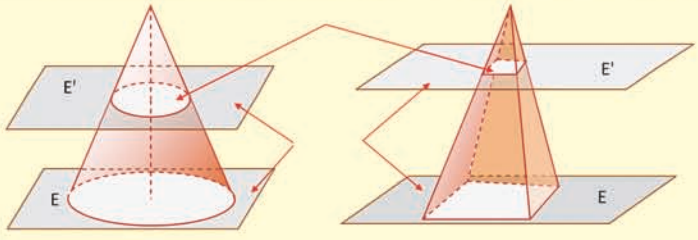
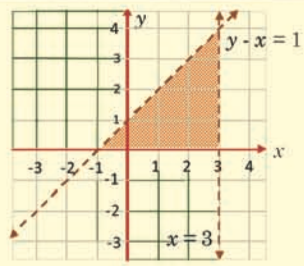
MATHEMATICS

STUDENT TEXTBOOK GRADE 10





FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

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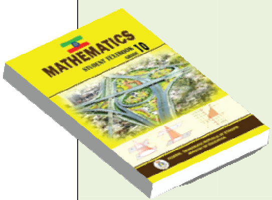

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STUDENT TEXTBOOK

GRADE 10

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Contents

Unit 1

POLYNOMIAL FUNCTIONS ARE THE MOST WIDELY USED FUNCTIONS IN MATHEMATICS. THEY ARISE NATURALLY IN MANY APPLICATIONS. ESSENTIALLY, THE GRAPH OF A POLYNOMIAL FUNCTION HAS NO BREAKS AND GAPS. IT DESCRIBES SMOOTH CURVES AS SHOWN IN THE FIGURE ABOVE.

POLYNOMIAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *define polynomial functions.*
- *perform the four fundamental operations on polynomials.*
- *apply theorems on polynomials to solve related problems.*
- *determine the number of rational and irrational zeros of a polynomial.*
- *sketch and analyse the graphs of polynomial functions.*

Main Contents

INTRODUCTION

THERE IS AN EXTREMELY IMPORTANT FAMILY OF FUNCTIONS IN MATHEMATICS CALLE FUNCTIONS.

STATED QUITE SIMPLY, POLYNOMIAL FUNCTIONS ARE FUNCTIONS WITH x AS AN INPUT VARIABLE, CONSISTING OF THE SUM OF SEVERAL TERMS, EACH TERM IS A PRODUCT OF TWO FAC BEING A REAL NUMBER COEFFICIENT AND THE SECOND BEING x RAISED TO SOME NON-NEGATIVE INTEGER POWER.

IN THIS UNIT YOU WILL BE LOOKING AT THE DIFFERENT COMPONENTS OF POLYNOMI THESE ARE THEOREMS ON POLYNOMIAL FUNCTIONS; ZEROS OF A POLYNOMIAL FU GRAPHS OF POLYNOMIAL FUNCTIONS. BASICALLY THE GRAPH OF A POLYNOMIAL F SMOOTH AND CONTINUOUS CURVE. HOWEVER, YOU WILL BE GOING OVER HOW TO US (EVEN OR ODD) AND THE LEADING COEFFICIENT TO DETERMINE THE END BEHAVIOUR O

1.1 INTRODUCTION TO POLYNOMIAL FUNCTIONS

OPENING PROBLEM

OBVIOUSLY, THE VOLUME OF WATER IN ANY DAM FLUCTUATES FROM SEASON TO SEASON. AN ENGINEER SUGGESTS THAT THE VOLUME OF THE WATER (IN GIGA LITRES) IN A CERTAIN t-MONTHS (STARTING 1ST SEPTEMBER) IS DESCRIBED BY THE MODEL:

$$v(t) = 450 - 170t + 22t^2 - 0.6t^3$$

ELECTRIC POWER CORPORATION RULES THAT IF THE VOLUME FALLS BELOW 200 GIGA WISE PROJECT, "IRRIGATION", IS PROHIBITED. DURING WHICH MONTHS, IF ANY, WAS PROHIBITED IN THE LAST 12 MONTHS?

RECALL THAT, A ***function*** f IS A RELATION IN WHICH NO TWO ORDERED PAIRS HAVE THE SAM ELEMENT, WHICH MEANS THAT FOR ANY GIVEN x IN THE DOMAIN OF f, THERE IS A UNIQUE PAIR (x, y) BELONGING TO THE FUNCTION f

IN UNIT 4 OF GRADE 9 MATHEMATICS, YOU HAVE DISCUSSED FUNCTIONS SUCH AS:

$$f(x) = \frac{2}{3}x + \frac{1}{2},\ g(x) = 5 - 3x,\ h(x) = 8x \text{ AND } l(x) = -\sqrt{3}x + 2.7.$$

SUCH FUNCTIONS ARE ***linear functions***

A FUNCTION f IS A ***linear function***, IF IT CAN BE WRITTEN IN THE FORM

$$f(x) = ax + b,\ a \neq 0,$$

WHERE a AND b ARE REAL NUMBERS.

THE ***domain*** OF f IS THE SET OF ALL REAL NUMBERS AND ITS ***range*** IS ALSO THE SET OF ALL REAL NUMBERS.

IF $a = 0$, THEN f IS CALLED A ***constant function***. IN THIS CASE,

$$f(x) = b.$$

THIS FUNCTION HAS THE SET OF ALL REAL NUMBERS AS ITS ***domain*** AND $\{b\}$ AS ITS ***range***.

ALSO RECALL WHAT YOU STUDIED ABOUT ***quadratic functions***. EACH OF THE FOLLOWING FUNCTIONS IS A QUADRATIC FUNCTION.

$$f(x) = x^2 + 7x - 12,\quad g(x) = 9 + \frac{1}{4}x^2,\quad h(x) = -x^2 + \ ,\quad k(x) = x^2,$$

$$l(x) = 2(x-1)^2 + 3,\quad m(x) = (x+2)(1-x)$$

IF a, b, AND c ARE REAL NUMBERS WITH $a \neq 0$, THEN THE FUNCTION

$f(x) = ax^2 + bx + c$ IS A ***quadratic function***.

SINCE THE EXPRESSION $ax^2 + bx + c$ REPRESENTS A REAL NUMBER FOR ANY REAL NUMBER x, THE ***domain*** OF A QUADRATIC FUNCTION IS THE SET OF ALL REAL NUMBERS. THE RANGE OF A QUADRATIC FUNCTION DEPENDS ON THE VALUES OF a, b AND c.

Exercise 1.1

1 IN EACH OF THE FOLLOWING CASES, CLASSIFY THE FUNCTION AS CONSTANT, LINEAR, QUADRATIC OR NONE OF THESE:

A $f(x) = 1 - x^2$ **B** $h(x) = \sqrt{2x-1}$

C $h(x) = 3 + 2^x$ **D** $g(x) = 5 - \frac{4}{5}x$

E $f(x) = 2\sqrt{3}$ **F** $f(x) = \left(\frac{2}{3}\right)^{-1}$

G $h(x) = 1 - |x|$ **H** $f(x) = (1 - \sqrt{2}x)(1 + \sqrt{2}x)$

I $k(x) = \frac{3}{4}(12 + 8x)$ **J** $f(x) = 12x^{-1}$

K $l(x) = \frac{(x+1)(x-2)}{x-2}$ **L** $f(x) = x^4 - x + 1$

2 FOR WHAT VALUES OF a, b, AND c IS $f(x) = ax^2 + bx + c$ A CONSTANT, A LINEAR OR A QUADRATIC FUNCTION?

1.1.1 Definition of a Polynomial Function

CONSTANT, LINEAR AND QUADRATIC FUNCTIONS ARE ALL SPECIAL CASES OF A W FUNCTIONS CALLED ***polynomial functions.***

Definition 1.1

Let n be a non-negative integer and let $a_n, a_{n-1}, \ldots, a_1, a_o$ be real numbers with $a_n \neq 0$. The function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$$

is called a ***polynomial function in variable*** x ***of degree*** n.

NOTE THAT IN THE DEFINITION OF A POLYNOMIAL FUNCTION

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_O$$

I $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_O$ ARE CALLED THE ***coefficients*** OF THE POLYNOMIAL FUNCTION (OR SIMPLY THE POLYNOMIAL).

II THE NUMBER a_n IS CALLED THE ***leading coefficient*** OF THE POLYNOMIAL FUNCTION AND $a_n x^n$ IS THE ***leading term***.

III THE NUMBER a_0 IS CALLED THE ***constant term*** OF THE POLYNOMIAL.

IV THE NUMBER n (THE EXPONENT OF THE HIGHEST POWER OF x), IS THE ***degree*** OF THE POLYNOMIAL.

NOTE THAT THE DOMAIN OF A POLYNOMIAL FUNCTION IS $\mathbb{R}$.

EXAMPLE 1 WHICH OF THE FOLLOWING ARE POLYNOMIAL FUNCTIONS? FOR THOSE W POLYNOMIALS, FIND THE DEGREE, LEADING COEFFICIENT, AND CONSTANT TE

A $f(x) = \frac{2}{3}x^4 - 12x^2 + x + \frac{7}{8}$

B $f(x) = \frac{x}{x}$

C $g(x) = \sqrt{(x+1)^2}$

D $f(x) = 2x^{-4} + x^2 + 8x + 1$

E $k(x) = \frac{x^2+1}{x^2+1}$

F $g(x) = \frac{8}{5}x^{15}$

G $f(x) = (1 - \sqrt{2}x)(1 + \sqrt{2}x)$

H $k(y) = \frac{6}{y}$

SOLUTION:

A IT IS A POLYNOMIAL FUNCTION OF DEGREE 4 WITH LEADING COEFFICIENT $\frac{2}{3}$ AND CONSTANT TERM $\frac{7}{8}$

B IT IS NOT A POLYNOMIAL FUNCTION BECAUSE ITS DOMAIN IS NOT $\mathbb{R}$

c $g(x)=\sqrt{(x+1)^2}=|x+1|$, so it is not a polynomial function because it cannot be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.

d It is not a polynomial function because one of its terms has a nega

exponent.

e $k(x) = \dfrac{x^2+1}{x^2+1} = 1$, so it is a polynomial function of degree 0 with leading coefficient 1 and constant term 1.

f It is a polynomial function of degree 15 with leading coefficient $\dfrac{8}{5}$ and constant term 0.

g It is a polynomial function of degree 2 with leading coefficient −2 and constant term 1.

h It is not a polynomial function because its domain is not ℝ.

A polynomial expression in x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_o$$

where n is a non negative integer and $a_n \neq 0$. Each individual expression $a_k x^k$ making up the polynomial is called a ***term***.

ACTIVITY 1.1

1 For the polynomial expression $\dfrac{x^2+3-6x^4}{4} + \dfrac{7}{8}x - x^3$,

a What is the degree? **b** What is the leading coefficient?

c What is the coefficient of x^3? **d** What is the constant term?

2 A match box has length x cm, width $x + 1$ cm and height 3 cm.

a Express its surface area as a function of x.

b What is the degree and the constant term of the polynomial obtained above?

We can restate the definitions of ***linear*** and ***quadratic functions*** using the terminology for polynomials. Linear functions are polynomial functions of ***degree 1***. Nonzero ***constant functions*** are polynomial functions of degree 0. Similarly, ***quadratic functions*** are polynomial functions of degree 2. The zero function, $p(x) = 0$, is also considered to be a polynomial function but is not assigned a degree at this l

Note that in expressing a polynomial, we usually omit all terms which app coefficients and write others in decreasing order, or increasing order, of thi

EXAMPLE 2 FOR THE POLYNOMIAL FUNCTION $p(x) = \frac{x^2 - 2x^5 + 8}{4} + \frac{7}{8}x - x^3$,

A WHAT IS ITS DEGREE? **B** FIND a_n, a_{n-1}, a_{n-2} AND a_2.

C WHAT IS THE CONSTANT TERM? **D** WHAT IS THE COEFFICIENT OF x?

SOLUTION: $p(x) = \frac{x^2 - 2x^5 + 8}{4} + \frac{7}{8}x - x^3 = \frac{x^2}{4} - \frac{2}{4}x^5 + \frac{8}{4} + \frac{7}{8}x - x^3$

$= -\frac{1}{2}x^5 - x^3 + \frac{1}{4}x^2 + \frac{7}{8}x + 2$

A THE DEGREE IS 5.

B $a_n = a_5 = \frac{-1}{2}$, $a_{n-1} = a_4 = 0$, $a_{n-2} = a_3 = -1$ AND $a_2 = \frac{1}{4}$.

C THE CONSTANT TERM IS 2.

D THE COEFFICIENT OF x IS $\frac{7}{8}$.

ALTHOUGH THE ***domain*** OF A POLYNOMIAL FUNCTION IS THE SET OF ALL REAL NUMBERS, WE MAY HAVE TO SET A RESTRICTION ON THE DOMAIN BECAUSE OF OTHER CIRCUMSTANCES. FOR EXAMPLE, IN A GEOMETRICAL APPLICATION, IF A RECTANGLE IS x LONG AND $p(x)$ IS THE AREA OF THE RECTANGLE, THE DOMAIN OF THE FUNCTION p IS THE SET OF POSITIVE REAL NUMBERS. SIMILARLY, FOR A POPULATION FUNCTION, THE DOMAIN IS THE SET OF POSITIVE INTEGERS.

Based on the types of coefficients it has, a polynomial function *p* is said to be:

- ✓ A POLYNOMIAL FUNCTION ***over integers***, IF THE COEFFICIENTS OF $p(x)$ ARE ALL INTEGERS.
- ✓ A POLYNOMIAL FUNCTION ***over rational numbers***, IF THE COEFFICIENTS OF $p(x)$ ARE ALL RATIONAL NUMBERS.
- ✓ A POLYNOMIAL FUNCTION ***over real numbers***, IF THE COEFFICIENTS OF $p(x)$ ARE ALL REAL NUMBERS.

Remark: EVERY POLYNOMIAL FUNCTION THAT WE WILL CONSIDER IN THIS UNIT IS A POLYNOMIAL FUNCTION OVER THE REAL NUMBERS.

FOR EXAMPLE, IF $g(x) = \frac{2}{3}x^4 - 13x^2 + \frac{7}{8}$, THEN g IS A POLYNOMIAL FUNCTION OVER RATIONAL AND REAL NUMBERS, BUT NOT OVER INTEGERS.

IF $p(x)$ CAN BE WRITTEN IN THE FORM $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, THEN DIFFERENT EXPRESSIONS CAN DEFINE THE SAME POLYNOMIAL FUNCTION.

FOR EXAMPLE, THE FOLLOWING EXPRESSIONS ALL DEFINE THE SAME POLYNOMIAL
$\frac{1}{2}x^2 - x$.

A $\frac{x^2-2x}{2}$ **B** $-x+\frac{1}{2}x^2$ **C** $\frac{1}{2}(x^2-2x)$ **D** $x\left(\frac{1}{2}x-1\right)$

ANY EXPRESSION WHICH DEFINES A POLYNOMIAL FUNCTION IS CALLED A ***polynomial expression***.

EXAMPLE 3 FOR THE POLYNOMIAL EXPRESSION $6x^3 - x^5 + 2x + 1$,

A WHAT IS THE DEGREE? **B** WHAT IS THE COEFFICIENT OF x^3?

C WHAT IS THE LEADING COEFFICIENT? **D** WHAT IS THE CONSTANT TERM?

SOLUTION:

A THE DEGREE IS 5. **B** THE COEFFICIENT OF x^3 IS 6.

C THE LEADING COEFFICIENT IS – 1. **D** THE CONSTANT TERM IS 1.

CONSIDER THE FUNCTIONS $f(x) = \frac{(x+3)(x-1)}{x-1}$ AND $g(x) = x + 3$.

WHEN f IS SIMPLIFIED IT GIVES $f(x) = x + 3$, WHERE $x \neq 1$. AS THE DOMAIN OF f IS NOT THE SET OF ALL REAL NUMBERS, f IS NOT A POLYNOMIAL FUNCTION. BUT THE DOMAIN OF g IS THE SET OF ALL REAL NUMBERS. THE FUNCTIONS f AND g HAVE DIFFERENT DOMAINS AND YOU CAN CONCLUDE THAT f AND g ARE NOT THE SAME FUNCTIONS.

WHEN YOU ARE TESTING AN EXPRESSION TO CHECK WHETHER OR NOT IT DEFINES A FUNCTION, YOU MUST BE CAREFUL AND WATCH THE ***domain*** OF THE FUNCTION DEFINED

Exercise 1.2

1 WHICH OF THE FOLLOWING ARE POLYNOMIAL FUNCTIONS?

A $f(x) = 3x^4 - 2x^3 + x^2 + 7x - 9$ **B** $f(x) = x^{25} + 1$

C $f(x) = 3x^{-3} + 2x^{-2} + x + 4$ **D** $f(y) = \frac{1}{3}y^2 + \frac{2}{3}y + 1$

E $f(t) = \frac{3}{t} + \frac{2}{t^2}$ **F** $f(y) = 108 - 95y$

G $f(x) = 312x^6$ **H** $f(x) = \sqrt{3}x^2 - x^3 + \sqrt{2}$

I $f(x) = \sqrt{3x} + x + 3$

J $f(x) = \dfrac{4x^2 - 5x^3 + 6}{8}$

K $f(x) = \dfrac{3}{6+x}$

L $f(y) = \dfrac{18}{y}$

M $f(a) = \dfrac{a}{2a}$

N $f(x) = \dfrac{x}{12}$

O $f(x) = 0$

P $f(a) = a^{\frac{1}{2}} + 3a + a^2$

Q $f(x) = \dfrac{9}{17}x^{83} + \sqrt{54}x^{97} +$

R $f(t) = \dfrac{4}{7} - 2$

S $f(x) = (1 - x)(x + 2)$

T $g(x) = \left(x - \dfrac{2}{3}\right)\left(x + \dfrac{3}{4}\right)$

2 GIVE THE DEGREE, THE LEADING COEFFICIENT AND THE CONSTANT TERM OF EACH FUNCTION IN QUESTION 1 ABOVE.

3 WHICH OF THE POLYNOMIAL FUNCTIONS IN QUESTION 1 ABOVE ARE:

A POLYNOMIAL FUNCTIONS OVER INTEGERS?

B POLYNOMIAL FUNCTIONS OVER RATIONAL NUMBERS?

C POLYNOMIAL FUNCTIONS OVER REAL NUMBERS?

4 WHICH OF THE FOLLOWING ARE POLYNOMIAL EXPRESSIONS?

A $2\sqrt{3} - x$ **B** $y(y - 2)$ **C** $\dfrac{(x+3)^2}{x+3}$

D $\sqrt{y^2 + 3} + 2 - 3y^3$ **E** $\dfrac{(y-3)(y-1)}{2}$ **F** $\dfrac{(t-5)(t-1)}{t-1}$

G $\dfrac{(x-3)(x^2+1)}{x^2+1}$ **H** $y + 2y - 3y$ **I** $\dfrac{x^2+4}{x^2+4}$

5 AN OPEN BOX IS TO BE MADE FROM A 20 CM LONG SQUARE PIECE OF MATERIAL, BY CUTTING EQUAL SQUARES OF LENGTH x CM FROM THE CORNERS AND TURNING UP THE SIDES AS SHOWN IN FIGURE 1.1

A VERIFY THAT THE VOLUME OF THE BOX IS GIVEN BY THE FUNCTION $V(x) = 4x^3 - 80x^2 + 400x$.

B DETERMINE THE DOMAIN OF V.

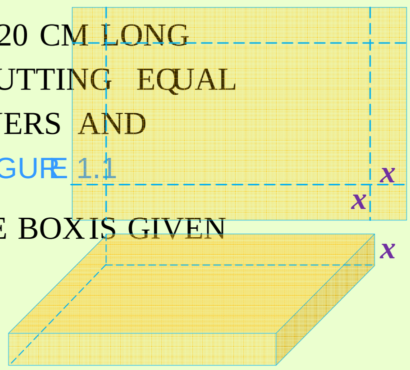
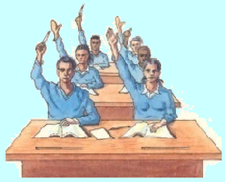


Figure 1.1

1.1.2 Operations on Polynomial Functions

RECALL THAT, IN ALGEBRA, THE FUNDAMENTAL OPERATIONS ARE ADDITION, MULTIPLICATION AND DIVISION. THE FIRST STEP IN PERFORMING OPERATIONS ON FUNCTIONS IS TO USE THE COMMUTATIVE, ASSOCIATIVE AND DISTRIBUTIVE LAWS COMBINE LIKE TERMS TOGETHER.

ACTIVITY 1.2



1. WHAT ARE LIKE TERMS? GIVE AN EXAMPLE.
2. ARE $8a^2$, $2a^3$ AND $5a$ LIKE TERMS? EXPLAIN.
3. FOR ANY THREE REAL NUMBERS a, b AND c, DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE. GIVE REASONS FOR YOUR ANSWERS.
   - A $a - (b + c) = a - b + c$
   - B $a + (b - c) = a + b - c$
   - C $a - (b - c) = a - b + c$
   - D $a - (b - c) = a - b - c$
4. VERIFY EACH OF THE FOLLOWING STATEMENTS:
   - A $(4x + a) + (2a - x) = 3(a + x)$
   - B $5x^2y + 2xy^2 - (x^2y - xy^2) = 4x^2y + 3xy^2$
   - C $8a - (b + 9a) = -(a + b)$
   - D $2x - 4(x - y) + (y - x) = 5y - 3x$
5. IF $f(x) = x^3 - 2x^2 + 1$ AND $g(x) = x^2 - x - 1$, THEN WHICH OF THE FOLLOWING STATEMENTS ARE TRUE?
   - A $f(x) + g(x) = x^3 + x^2 - x$
   - B $f(x) - g(x) = x^3 - 3x^2 + x + 2$
   - C $g(x) - f(x) = 3x^2 + x^3 - x - 2$
   - D $f(x) - g(x) \neq g(x) - f(x)$.
6. IF f AND g ARE POLYNOMIAL FUNCTIONS OF DEGREE 3, THEN WHICH OF THE FOLLO NECESSARILY TRUE?
   - A $f + g$ IS OF DEGREE 3.
   - B $f + g$ IS OF DEGREE 6.
   - C $2f$ IS OF DEGREE 3.
   - D fg IS OF DEGREE 6.

Addition of polynomial functions

YOU CAN ADD POLYNOMIAL FUNCTIONS IN THE SAME WAY AS YOU ADD REAL NUMB ADD THE LIKE TERMS BY ADDING THEIR COEFFICIENTS. NOTE THAT LIKE TERMS ARE TE SAME VARIABLES TO THE SAME POWERS BUT POSSIBLY DIFFERENT COEFFICIENTS.

FOR EXAMPLE, IF $f(x) = 5x^4 - x^3 + 8x - 2$ AND $g(x) = 4x^3 - x^2 - 3x + 5$, THEN THE SUM OF $f(x)$ AND $g(x)$ IS THE POLYNOMIAL FUNCTION:

$$
\begin{aligned}
f(x) + g(x) &= (5x^4 - x^3 + 8x - 2) + (4x^3 - x^2 - 3x + 5)\\
&= 5x^4 + (-x^3 + 4x^3) - x^2 + (8x - 3x) + (-2 + 5) \;...\; \textit{(grouping like terms)}\\
&= 5x^4 + (4 - 1)x^3 - x^2 + (8 - 3)\,x + (5 - 2) \;......\; \textit{(adding their coefficients)}\\
&= 5x^4 + 3x^3 - x^2 + 5x + 3 \;...................\; \textit{(combining like terms).}
\end{aligned}
$$

THEREFORE, THE SUM $f(x) + g(x) = 5x^4 + 3x^3 - x^2 + 5x + 3$ IS A POLYNOMIAL OF DEGREE 4.

THE ***sum*** OF TWO POLYNOMIAL FUNCTIONS f AND g IS WRITTEN AS $f + g$, AND IS DEFINED AS:

$$f + g : (f + g)(x) = f(x) + g(x), \text{ FOR ALL } x \in \mathbb{R}.$$

EXAMPLE 4 IN EACH OF THE FOLLOWING, FIND THE SUM OF $f(x)$ AND $g(x)$:

A $f(x) = x^3 + \frac{2}{3}x^2 - \frac{1}{2}x + 3$ AND $g(x) = -x^3 + \frac{1}{3}x^2 + x - 4$.

B $f(x) = 2x^5 + 3x^4 - 2\sqrt{2}x^3 + x - 5$ AND $g(x) = x^4 + \sqrt{2}x^3 + x^2 + 6x + 8$.

SOLUTION:

A
$$
\begin{aligned}
f(x) + g(x) &= \left(x^3 + \frac{2}{3}x^2 - \frac{1}{2}x + 3\right) + \left(-x^3 + \frac{1}{3}x^2 + x - 4\right)\\
&= (x^3 - x^3) + \left(\frac{2}{3}x^2 + \frac{1}{3}x^2\right) + \left(-\frac{1}{2}x + x\right) + (3 - 4) \;...\; \textit{(grouping like terms)}\\
&= (1 - 1)x^3 + \left(\frac{2}{3} + \frac{1}{3}\right)x^2 + \left(1 - \frac{1}{2}\right)x + (3 - 4) \;..\; \textit{(adding their coefficients)}\\
&= x^2 + \frac{1}{2}x - 1 \;.............................\; \textit{(combining like terms)}
\end{aligned}
$$

SO, $f(x) + g(x) = x^2 + \frac{1}{2}x - 1$, WHICH IS A POLYNOMIAL OF DEGREE 2.

B
$$
\begin{aligned}
f(x) + g(x) &= (2x^5 + 3x^4 - 2\sqrt{2}x^3 + x - 5) + (x^4 + \sqrt{2}x^3 + x^2 + 6x + 8)\\
&= 2x^5 + (3x^4 + x^4) + (-2\sqrt{2}\,x^3 + \sqrt{2}\,x^3) + x^2 + (x + 6x) + (-5 + 8)\\
&= 2x^5 + (3 + 1)x^4 + (-2\sqrt{2} + \sqrt{2})x^3 + x^2 + (1 + 6)\,x + (8 - 5)\\
&= 2x^5 + 4x^4 - \sqrt{2}\,x^3 + x^2 + 7x + 3
\end{aligned}
$$

SO, $f(x) + g(x) = 2x^5 + 4x^4 - \sqrt{2}\,x^3 + x^2 + 7x + 3$, WHICH IS A POLYNOMIAL FUNCTION OF DEGREE 5.

ACTIVITY 1.3

1. WHAT DO YOU OBSERVE IN EXAMPLE 4 ABOUT THE DEGREE OF $f + g$?
2. IS THE DEGREE OF $(f + g)(x)$ EQUAL TO THE DEGREE OF $f(x)$ OR $g(x)$, WHICHEVER HAS THE HIGHEST DEGREE?
3. IF $f(x)$ AND $g(x)$ HAVE SAME DEGREE, THEN THE DEGREE OF $(f + g)(x)$ MIGHT BE LOWER THAN THE DEGREE OF $f(x)$ OR THE DEGREE OF $g(x)$. WHICH PART OF EXAMPLE 4 ILLUSTRATES THIS SITUATION? WHY DOES THIS HAPPEN?
4. WHAT IS THE DOMAIN OF $(f + g)(x)$?

Subtraction of polynomial functions

TO SUBTRACT A POLYNOMIAL FROM A POLYNOMIAL, SUBTRACT THE COEFFICI CORRESPONDING LIKE TERMS. SO, WHICHEVER TERM IS TO BE SUBTRACTED, ITS SIGN IS THEN THE TERMS ARE ADDED.

FOR EXAMPLE, IF $f(x) = 2x^3 - 5x^2 + x - 7$ AND $g(x) = 8x^2 - x^3 + 4x + 5$, THEN THE DIFFERENCE OF $f(x)$ AND $g(x)$ IS THE POLYNOMIAL FUNCTION:

$$f(x) - g(x) = (2x^3 - 5x^2 + x - 7) - (8x^2 - x^3 + 4x + 5)$$

$= 2x^3 - 5x^2 + x - 7 - 8x^2 + x^3 - 4x - 5$ . . . . . . (*removing brackets*)

$= (2 + 1)x^3 + (-5 - 8)x^2 + (1 - 4)x + (-7 - 5)$ . . (*adding coefficients of like terms*)

$= 3x^3 - 13x^2 - 3x - 12$ . . . . (*combining like terms*)

THE ***difference*** OF TWO POLYNOMIAL FUNCTIONS f AND g IS WRITTEN AS $f - g$, AND IS DEFINED AS:

$$(f - g) : (f - g)(x) = f(x) - g(x), \text{ FOR ALL } x \in \mathbb{R}.$$

EXAMPLE 5 IN EACH OF THE FOLLOWING, FIND $f - g$.

A $f(x) = x^4 + 3x^3 - x^2 + 4$ AND $g(x) = x^4 - x^3 + 5x^2 + 6x$

B $f(x) = x^5 + 2x^3 - 8x + 1$ AND $g(x) = x^3 + 2x^2 + 6x - 9$

SOLUTION:

A $f(x) - g(x) = (x^4 + 3x^3 - x^2 + 4) - (x^4 - x^3 + 5x^2 + 6x)$

$= x^4 + 3x^3 - x^2 + 4 - x^4 + x^3 - 5x^2 - 6x$.........(*removing brackets*)

$= (1 - 1)x^4 + (3 + 1)x^3 + (-1 - 5)x^2 - 6x + 4$..(*adding their coefficients*)

$= 4x^3 - 6x^2 - 6x + 4$ ....(*combining like terms*)

THEREFORE, THE DIFFERENCE IS A POLYNOMIAL FUNCTION OF DEGREE 3,

$$f(x) - g(x) = 4x^3 - 6x^2 - 6x + 4$$

B $f(x) - g(x) = (x^5 + 2x^3 - 8x + 1) - (x^3 + 2x^2 + 6x - 9)$

$$= x^5 + 2x^3 - 8x + 1 - x^3 - 2x^2 - 6x + 9$$
$$= x^5 + (2x^3 - x^3) - 2x^2 + (-8x - 6x) + (1 + 9)$$
$$= x^5 + (2 - 1)x^3 - 2x^2 + (-8 - 6)\,x + (1 + 9)$$
$$= x^5 + x^3 - 2x^2 - 14x + 10$$

THEREFORE THE DIFFERENCE $f(x) - g(x) = x^5 + x^3 - 2x^2 - 14x + 10$, WHICH IS A POLYNOMIAL FUNCTION OF DEGREE 5.

NOTE THAT IF THE DEGREE OF f IS NOT EQUAL TO THE DEGREE OF G, THEN THE DEGREE OF $(f + g)(x)$ IS THE DEGREE OF $f(x)$ OR THE DEGREE OF $g(x)$, WHICHEVER HAS THE HIGHEST DEGREE. IF THEY HAVE THE SAME DEGREE, HOWEVER, THE DEGREE OF $(f - g)(x)$ MIGHT BE LOWER THAN THIS COMMON DEGREE WHEN THEY HAVE THE SAME LEADING COEFFICIENT AS ILLUSTRATED IN EXAMPLE 5.

Multiplication of polynomial functions

TO MULTIPLY TWO POLYNOMIAL FUNCTIONS, MULTIPLY EACH TERM OF ONE BY EACH TERM OF THE OTHER, AND COLLECT LIKE TERMS.

FOR EXAMPLE, LET $f(x) = 2x^3 - x^2 + 3x - 2$ AND $g(x) = x^2 - 2x + 3$. THEN THE PRODUCT OF $f(x)$ AND $g(x)$ IS A POLYNOMIAL FUNCTION:

$$f(x)\,.\,g(x) = (2x^3 - x^2 + 3x - 2)\cdot(x^2 - 2x + 3)$$
$$= 2x^3(x^2 - 2x + 3) - x^2(x^2 - 2x + 3) + 3x(x^2 - 2x + 3) - 2(x^2 - 2x + 3)$$
$$= 2x^5 - 4x^4 + 6x^3 - x^4 + 2x^3 - 3x^2 + 3x^3 - 6x^2 + 9x - 2x^2 + 4x - 6$$
$$= 2x^5 + (-4x^4 - x^4) + (6x^3 + 2x^3 + 3x^3) + (-3x^2 - 6x^2 - 2x^2) + (9x + 4x) - 6$$
$$= 2x^5 - 5x^4 + 11x^3 - 11x^2 + 13x - 6$$

THE ***product*** OF TWO POLYNOMIAL FUNCTIONS f AND g IS WRITTEN AS $f.g$ AND IS DEFINED AS:

$$f\cdot g : (f\cdot g)(x) = f(x)\cdot g(x), \text{ FOR ALL } x \in \mathbb{R}.$$

EXAMPLE 6 IN EACH OF THE FOLLOWING, FIND $f.g$ AND GIVE THE DEGREE OF $f.g$:

A $f(x) = \frac{3}{4}x^2 + \frac{9}{2},\ g(x) = 4x$ **B** $f(x) = x^2 + 2x,\ g(x) = x^5 + 4x^2 - 2$

SOLUTION: A $f(x).g(x) = \left(\frac{3}{4}x^2 + \frac{9}{2}\right).(4x) = 3x^3 + 18x$

SO, THE PRODUCT $(f.g)(x) = 3x^3 + 18x$ HAS DEGREE 3.

B $f(x).g(x) = (x^2 + 2x).(x^5 + 4x^2 - 2)$

$= x^2(x^5 + 4x^2 - 2) + 2x(x^5 + 4x^2 - 2)$

$= x^7 + 2x^6 + 4x^4 + 8x^3 - 2x^2 - 4x$

SO, THE PRODUCT $f.g(x) = x^7 + 2x^6 + 4x^4 + 8x^3 - 2x^2 - 4x$ HAS DEGREE 7.

IN EXAMPLE 6, YOU CAN SEE THAT THE DEGREE OF fg IS THE SUM OF THE DEGREES OF THE TWO POLYNOMIAL FUNCTIONS f AND g.

TO FIND THE PRODUCT OF TWO POLYNOMIAL FUNCTIONS, WE CAN ALSO USE A VERTICAL FORMAT FOR MULTIPLICATION.

EXAMPLE 7 LET $f(x) = 3x^2 - 2x^3 + x^5 - 8x + 1$ AND $g(x) = 5 + 2x^2 + 8x$. FIND $f(x).g(x)$ AND THE DEGREE OF THE PRODUCT.

SOLUTION: TO FIND THE PRODUCT fg, FIRST REARRANGE EACH POLYNOMIAL IN DESCENDING POWERS OF x AS FOLLOWS:

$$x^5 - 2x^3 + 3x^2 - 8x + 1$$

$$2x^2 + 8x + 5$$

Like terms are written in the same column.

$5x^5 + 0x^4 - 10x^3 + 15x^2 - 40x + 5$ . . . . .(*multiplying by* 5)

$8x^6 + 0x^5 - 16x^4 + 24x^3 - 64x^2 + 8x$ . . . .(*multiplying by* $8x$)

$2x^7 + 0x^6 - 4x^5 + 6x^4 - 16x^3 + 2x^2$ . . . . (*multiplying by* $2x^2$)

$2x^7 + 8x^6 + x^5 - 10x^4 - 2x^3 - 47x^2 - 32x + 5$ . ...(*adding vertically.*)

THUS, $f(x).g(x) = 2x^7 + 8x^6 + x^5 - 10x^4 - 2x^3 - 47x^2 - 32x + 5$ AND HENCE THE DEGREE OF fg IS 7.

ACTIVITY 1.4

1 FOR ANY NON-ZERO POLYNOMIAL FUNCTION, IF THE DEGREE OF f IS m AND THE DEGREE OF g IS n, THEN WHAT IS THE DEGREE OF fg?

2 IF EITHER f OR g IS THE ZERO POLYNOMIAL, WHAT IS THE DEGREE OF $f.g$?

3 IS THE PRODUCT OF TWO OR MORE POLYNOMIALS ALWAYS A POLYNOMIAL?

EXAMPLE 8 (*Application of polynomial functions*)

A PERSON WANTS TO MAKE AN OPEN BOX BY CUTTING EQUAL SQUARES FROM THE CORNERS OF A PIECE OF METAL 160 CM BY 240 CM AS SHOWN IN FIGURE 1.2. IF THE EDGE OF EACH CUT-OUT SQUARE IS x CM, FIND THE VOLUME OF THE BOX WHEN $x = 3$.

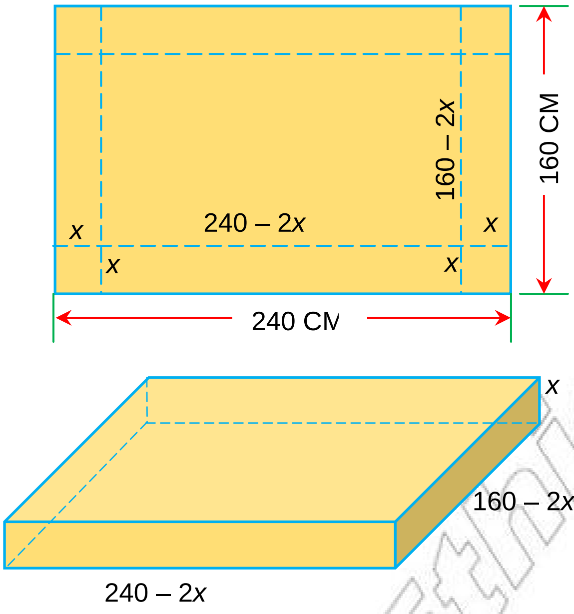


FIGURE 1.2

SOLUTION: THE VOLUME OF A RECTANGULAR BOX IS EQUAL TO THE PRODUCT OF ITS LENGTH, WIDTH AND HEIGHT. FROM THE FIGURE 1.2, THE LENGTH IS $240 - 2x$, THE WIDTH IS $160 - 2x$, AND THE HEIGHT IS x. SO THE VOLUME OF THE BOX IS

$$v(x) = (240 - 2x)(160 - 2x)(x)$$
$$= (38400 - 800x + 4x^2)(x)$$
$$= 38400x - 800x^2 + 4x^3 \text{ (A POLYNOMIAL OF DEGREE 3)}$$

WHEN $x = 1$, THE VOLUME OF THE BOX IS $v(1) = 38400 - 800 + 4 = 37604$ CM^3

WHEN $x = 3$, THE VOLUME OF THE BOX IS

$$v(3) = 38400(3) - 800(3)^2 + 4(3)^3 = 115200 - 7200 + 108 = 108{,}108 \text{ CM}^3$$

Division of polynomial functions

IT IS POSSIBLE TO DIVIDE A POLYNOMIAL BY A POLYNOMIAL USING A PROCESS OF LONG DIVISION SIMILAR TO THAT USED IN ARITHMETIC.

LOOK AT THE CALCULATIONS BELOW, WHERE 939 IS BEING DIVIDED BY 12.

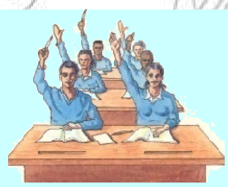
```
      7                78
12 | 939          12 | 939
     84                84
      99                99
                        96
                         3
```

THE SECOND DIVISION CAN BE EXPRESSED BY AN EQUATION WHICH SAYS NOTHING ABO

939 = (78 × 12) + 3. OBSERVE THAT, 939 ÷ 12 = 78 + (3÷12) OR $\frac{939}{12} = 78 + \frac{3}{12}$.

HERE 939 IS THE DIVIDEND, 12 IS THE DIVISOR, 78 IS THE QUOTIENT AND 3 IS THE REMAI THE DIVISION. WHAT WE ACTUALLY DID IN THE ABOVE CALCULATION WAS TO CONTIN AS LONG AS THE QUOTIENT AND THE REMAINDER ARE INTEGERS AND THE REMAINDER DIVISOR.

ACTIVITY 1.5



1 CONSIDER THE FOLLOWING: $f(x) = \frac{x^2 - x + 2}{x - 2} = x + 1 + \frac{4}{x - 2}$. WHICH POLYNOMIALS DO YOU THINK WE SHOULD CALL THE DIVISOR, DIVIDEND, QUOTIENT AND REMAINDER?

2 DIVIDE $x^3 + 1$ BY $x + 1$. (YOU SHOULD SEE THAT THE REMAINDER IS 0)

3 WHEN DO WE SAY THE DIVISION IS EXACT?

4 WHAT MUST BE TRUE ABOUT THE DEGREES OF THE DIVIDEND AND THE DIVISOR BE CAN TRY TO DIVIDE POLYNOMIALS?

5 SUPPOSE THE DEGREE OF THE DIVIDEND IS n AND THE DEGREE OF THE DIVISOR IS m $n > m$, THEN WHAT WILL BE THE DEGREE OF THE QUOTIENT?

WHEN SHOULD WE STOP DIVIDING ONE POLYNOMIAL BY ANOTHER? LOOK AT T CALCULATIONS BELOW:

$$\begin{array}{r|l} & \quad x \\ \hline x+1 & x^2 + 3x + 5 \\ & \underline{x^2 + x} \\ & \qquad 2x + 5 \end{array} \qquad \begin{array}{r|l} & \quad x + 2 \\ \hline x+1 & x^2 + 3x + 5 \\ & \underline{x^2 + x} \\ & \qquad 2x + 5 \\ & \qquad \underline{2x + 2} \\ & \qquad\qquad 3 \end{array} \qquad \begin{array}{r|l} & x + 2 + \frac{3}{x+1} \\ \hline x+1 & x^2 + 3x + 5 \\ & \underline{x^2 + x} \\ & \qquad 2x + 5 \\ & \qquad \underline{2x + 2} \\ & \qquad\qquad 3 \\ & \qquad\qquad \underline{3} \\ & \qquad\qquad 0 \end{array}$$

THE FIRST DIVISION ABOVE TELLS US THAT

$$x^2 + 3x + 5 = x(x + 1) + 2x + 5.$$

IT HOLDS TRUE FOR ALL VALUES OF $x \neq -1$. IN THE MIDDLE ONE OF THE THREE DIVISIONS, YOU CONTINUED AS LONG AS YOU GOT A QUOTIENT AND REMAINDER WHICH ARE BOTH POLY

WHEN YOU ARE ASKED TO DIVIDE ONE POLYNOMIAL BY ANOTHER, STOP THE DIVISI
WHEN YOU GET A QUOTIENT AND REMAINDER THAT ARE POLYNOMIALS AND THE D
REMAINDER IS LESS THAN THE DEGREE OF THE DIVISOR.

STUDY THE EXAMPLE BELOW TO DIVIDE $2x^3 - 3x^2 + 4x + 7$ BY $x - 2$.

Think $\frac{2x^3}{x} = 2x^2$; *Think* $\frac{x^2}{x} = x$; *Think* $\frac{6x}{x} = 6$

$$
\begin{array}{rll}
 & 2x^2 + x + 6 & \leftarrow \textit{Quotient} \\
\textit{Divisor} \rightarrow x - 2 \,\big) & 2x^3 - 3x^2 + 4x + 7 & \leftarrow \textit{Dividend} \\
 & \underline{2x^3 - 4x^2} & \leftarrow \textit{multiply } 2x^2\,(x-2) \\
 & x^2 + 4x + 7 & \leftarrow \textit{subtract} \\
 & \underline{x^2 - 2x} & \leftarrow \textit{multiply } x\,(x-2) \\
 & 6x + 7 & \leftarrow \textit{subtract} \\
 & \underline{6x - 12} & \leftarrow \textit{multiply } 6\,(x-2) \\
\textit{Remainder} \rightarrow & 19 & \leftarrow \textit{subtract}
\end{array}
$$

SO, DIVIDING $2x^3 - 3x^2 + 4x + 7$ BY $x - 2$ GIVES A QUOTIENT OF $2x^2 + x + 6$ AND A REMAINDER OF 19. THAT IS, $\frac{2x^3 - 3x^2 + 4x + 7}{x - 2} = 2x^2 + x + 6 + \frac{19}{x - 2}$

THE ***quotient*** (***division***) OF TWO POLYNOMIAL FUNCTIONS f AND g IS WRITTEN $f \div g$, AND IS DEFINED AS:

$$f \div g : (f \div g)(x) = f(x) \div g(x), \text{ PROVIDED THAT } g(x) \neq 0, \text{ FOR ALL } x \in \mathbb{R}.$$

EXAMPLE 9 DIVIDE $4x^3 - 3x + 5$ BY $2x - 3$

SOLUTION:

$$
\begin{array}{rl}
 & 2x^2 + 3x + 3 \\
2x - 3 \,\big) & 4x^3 + 0x^2 - 3x + 5 \\
 & \underline{4x^3 - 6x^2} \\
 & 6x^2 - 3x + 5 \\
 & \underline{6x^2 - 9x} \\
 & 6x + 5 \\
 & \underline{6x - 9} \\
\text{REMAINDER} \rightarrow & 14
\end{array}
$$

Arrange the dividend and the divisor in descending powers of x.

Insert (with 0 coefficients) for missing terms.

Divide the first term of the dividend by the first term of the divisor.

Multiply the divisor by $2x^2$, line up like terms and, subtract

Repeat the process until the degree of the remainder is less than that of the divisor.

THEREFORE, $4x^3 - 3x + 5 = (2x^2 + 3x + 3)(2x - 3) + 14$

EXAMPLE 10 FIND THE QUOTIENT AND REMAINDER WHEN $x^5 + 4x^3 - 6x^2 - 8$ IS DIVIDED BY $x^2 + 3x + 2$.

SOLUTION:

$$
\begin{array}{r|l}
 & x^3 - 3x^2 + 11x - 33 \\
\hline
x^2 + 3x + 2 & x^5 + 0x^4 + 4x^3 - 6x^2 + 0x - 8 \\
 & \underline{x^5 + 3x^4 + 2x^3} \\
 & \quad -3x^4 + 2x^3 - 6x^2 + 0x - 8 \\
 & \quad \underline{-3x^4 - 9x^3 - 6x^2} \\
 & \qquad\quad 11x^3 + 0x^2 + 0x - 8 \\
 & \qquad\quad \underline{11x^3 + 33x^2 + 22x} \\
 & \qquad\qquad -33x^2 - 22x - 8 \\
 & \qquad\qquad \underline{-33x^2 - 99x - 66} \\
 & \qquad\qquad\qquad 77x + 58
\end{array}
$$

THEREFORE THE QUOTIENT IS $x^3 - 3x^2 + 11x - 33$ AND THE REMAINDER IS $77x + 58$.

WE CAN WRITE THE RESULT AS $\dfrac{x^5 + 4x^3 - 6x^2 - 8}{x^2 + 3x + 2} = x^3 - 3x^2 + 11x - 33 + \dfrac{77x + 58}{x^2 + 3x + 2}$.

Group Work 1.1

1 FIND TWO POLYNOMIAL FUNCTIONS f AND g BOTH OF DEGREE WITH $f + g$ OF DEGREE ONE. WHAT RELATIONS DO BETWEEN THE LEADING COEFFICIENTS OF f AND g?

2 GIVEN $f(x) = x + 2$ AND $g(x) = ax + b$, FIND ALL VALUES OF a AND b SO THAT $\dfrac{f}{g}$ IS A POLYNOMIAL FUNCTION.

3 GIVEN POLYNOMIAL FUNCTIONS $f(x) = x^2 + 3$, $q(x) = x^2 - 5$ AND $r(x) = 2x + 1$, FIND A FUNCTION $g(x)$ SUCH THAT $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$.

Exercise 1.3

1 WRITE EACH OF THE FOLLOWING EXPRESSIONS AS A POLYNOMIAL IN THE FORM $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, IF POSSIBLE:

- **A** $(x^2 - x - 6) - (x + 2)$
- **B** $(x^2 - x - 6)(x + 2)$
- **C** $(x+2) - (x^2 - x - 6)$
- **D** $\dfrac{x^2 - x - 6}{x + 2}$
- **E** $\dfrac{x + 2}{x^2 - x - 6}$
- **F** $(x^2 - x - 6)^2$
- **G** $2^{x-3} + 2^3 - x$
- **H** $(2x + 3)^2$
- **I** $(x^2 - x + 1)(x^2 - 3x + 5)$
- **J** $(x^3 - x^4 + 2x + 1) - (x^4 + x^3 - 2x^2 + 8)$

2 LET f AND g BE POLYNOMIAL FUNCTIONS SUCH THAT $f(x) = x^2 - 5x + 6$ AND $g(x) = x^2 - x + 3$. WHICH OF THE FOLLOWING FUNCTIONS ARE ALSO POLYNOMIAL FUNCTIONS?

- **A** $f + g$ **B** $g - f$ **C** $f \cdot g$ **D** $\frac{f}{g}$
- **E** $f^2 - g$ **F** $2f + 3g$ **G** $\sqrt{f^2}$

3 IF f AND g ARE ANY TWO POLYNOMIAL FUNCTIONS, WHICH OF THE FOLLOWING WILL A POLYNOMIAL FUNCTION?

- **A** $f + g$ **B** $f - g$ **C** $f \cdot g$ **D** $\frac{f}{g}$
- **E** f^2 **F** $\frac{3}{4}g - \frac{1}{3}f$ **G** $\frac{f - g}{f + g}$

4 IN EACH OF THE FOLLOWING, FIND $f + g$, $f - g$ AND GIVE THE DEGREE OF f, THE DEGREE OF g, THE DEGREE OF $f + g$ AND THE DEGREE OF $f - g$:

- **A** $f(x) = 3x - \frac{2}{3}$; $g(x) = 2x + 5$
- **B** $f(x) = -7x^2 + x - 8$; $g(x) = 2x^2 - x + 1$
- **C** $f(x) = 1 - x^3 + 6x^2 - 8x$; $g(x) = x^3 + 10$

5 IN EACH OF THE FOLLOWING,

- **I** FIND THE FUNCTION $f \cdot g$
- **II** GIVE THE DEGREE OF f AND THE DEGREE OF g
- **III** GIVE THE DEGREE OF $f \cdot g$.
  - **A** $f(x) = 2x + 1$; $g(x) = 3x - 5$
  - **B** $f(x) = x^2 - 3x + 5$; $g(x) = 5x + 3$
  - **C** $f(x) = 2x^3 - x - 7$; $g(x) = x^2 + 2x$
  - **D** $f(x) = 0$; $g(x) = x^3 - 8x^2 + 9$

6 IN EACH OF THE FOLLOWING, DIVIDE THE FIRST POLYNOMIAL BY THE SECOND:

- **A** $x^3 - 1$; $x - 1$ **B** $x^3 + 1$; $x^2 - x + 1$
- **C** $x^4 - 1$; $x^2 + 1$ **D** $x^5 + 1$; $x + 1$
- **E** $2x^5 - x^6 + 2x^3 + 6$; $x^3 - x - 2$

7 FOR EACH OF THE FOLLOWING, FIND THE QUOTIENT AND REMAINDER:

- **A** $(5 - 6x + 8x^2) \div (x - 1)$ **B** $(x^3 - 1) \div (x - 1)$
- **C** $(3y - y^2 + 2y^3 - 1) \div (y^2 + 1)$ **D** $(3x^4 + 2x^3 - 4x - 1) \div (x + 3)$
- **E** $(3x^3 - x^2 + x + 2) \div \left(x + \frac{2}{3}\right)$

1.2 THEOREMS ON POLYNOMIALS

1.2.1 Polynomial Division Theorem

RECALL THAT, WHEN WE DIVIDED ONE POLYNOMIAL BY ANOTHER, WE APPLY THE L PROCEDURE, UNTIL THE REMAINDER WAS EITHER THE ZERO POLYNOMIAL OR A POLYN DEGREE THAN THE DIVISOR.

FOR EXAMPLE, IF WE DIVIDE $x^2 + 3x + 7$ BY $x + 1$, WE OBTAIN THE FOLLOWING.

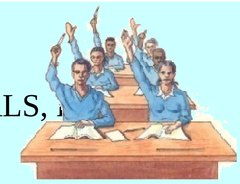
$$\begin{array}{r|l} & x + 2 \quad \leftarrow \text{quotient} \\ \text{Divisor} \rightarrow x + 1 & x^2 + 3x + 7 \quad \leftarrow \text{dividend} \\ & \underline{x^2 + x} \\ & 2x + 7 \\ & \underline{2x + 2} \\ & 5 \quad \leftarrow \text{remainder} \end{array}$$

IN FRACTIONAL FORM, WE CAN WRITE THIS RESULT AS FOLLOWS:

$$\underbrace{\frac{\overbrace{x^2 + 3x + 7}^{\text{dividend}}}{x + 1}}_{\text{divisor}} = \overbrace{x + 2}^{\text{quotient}} + \frac{\overset{\text{remainder}}{5}}{\underbrace{x + 1}_{\text{divisor}}}$$

THIS IMPLIES THAT $x^2 + 3x + 7 = (x + 1)(x + 2) + 5$ WHICH ILLUSTRATES THE THEOREM CALLED THE ***polynomial division theorem***.

ACTIVITY 1.6



1 FOR EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, $q(x)$ AND $r(x)$ THAT SATISFY $f(x) = d(x)\,q(x) + r(x)$.

A $f(x) = x^2 + x - 7;\ d(x) = x - 3$ **B** $f(x) = x^3 - x^2 + 8;\ d(x) = x + 2$

C $f(x) = x^4 - x^3 + x - 1;\ d(x) = x - 1$

2 IN QUESTION 1, WHAT DID YOU OBSERVE ABOUT THE DEGREES OF THE POLYNOMIAL FUNCTIONS $f(x)$ AND $d(x)$?

3 IN QUESTION 1, THE FRACTIONAL EXPRESSION $\frac{f(x)}{d(x)}$ IS IMPROPER. WHY?

4 IS $\frac{r(x)}{d(x)}$ PROPER OR IMPROPER? WHAT CAN YOU SAY ABOUT THE DEGREE OF $r(x)$ AND $d(x)$?

Theorem 1.1 Polynomial division theorem

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)\, q(x) + r(x)$$

$f(x)$ → **Dividend**; $d(x)$ → **Divisor**; $q(x)$ → **Quotient**; $r(x)$ → **Remainder**

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $f(x)$ divides exactly into $d(x)$.

Proof:-

I Existence of the polynomials $q(x)$ and $r(x)$

SINCE $f(x)$ AND $d(x)$ ARE POLYNOMIALS, LONG DIVISION OF $f(x)$ BY $d(x)$ WILL GIVE A QUOTIENT $q(x)$ AND REMAINDER $r(x)$ WITH DEGREE OF $r(x)$ < DEGREE OF $d(x)$ OR $r(x) = 0$.

II The uniqueness of $q(x)$ and $r(x)$

TO SHOW THE UNIQUENESS OF $q(x)$ AND $r(x)$, SUPPOSE THAT

$f(x) = d(x)q_1(x) + r_1(x)$ AND ALSO

$f(x) = d(x)q_2(x) + r_2(x)$ WITH DEG $r_1(x)$ < DEG $d(x)$ AND DEG $r_2(x)$ < DEG $d(x)$.

THEN $r_2(x) = f(x) - d(x)\, q_2(x)$ AND $r_1(x) = f(x) - d(x)\, q_1(x)$

$\Rightarrow r_2(x) - r_1(x) = d(x)\,[q_1(x) - q_2(x)]$

THEREFORE $d(x)$ IS A FACTOR OF $r_2(x) - r_1(x)$

AS DEG$(r_2(x) - r_1(x)) \leq$ MAX {DEG $r_1(x)$, DEG $r_2(x)$} < DEG $d(x)$ IT FOLLOWS THAT,

$r_2(x) - r_1(x) = 0$

AS A RESULT $r_1(x) = r_2(x)$ AND $q_1(x) = q_2(x)$.

THEREFORE $q(x)$ AND $r(x)$ ARE UNIQUE POLYNOMIAL FUNCTIONS.

EXAMPLE 1 IN EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, FIND POLYNOMIALS $q(x)$ AND $r(x)$ SUCH THAT $f(x) = d(x)\,q(x) + r(x)$.

A $f(x) = 2x^3 - 3x + 1;\ d(x) = x + 2$

B $f(x) = x^3 - 2x^2 + x + 5;\ d(x) = x^2 + 1$

C $f(x) = x^4 + x^2 - 2;\ d(x) = x^2 - x + 3$

SOLUTION:

A $$\frac{f(x)}{d(x)} = \frac{2x^3 - 3x + 1}{x + 2} = 2x^2 - 4x + 5 - \frac{9}{x + 2}$$

$\Rightarrow 2x^3 - 3x + 1 = (x+2)(2x^2 - 4x + 5) - 9$

THEREFORE, $q(x) = 2x^2 - 4x + 5$ AND $r(x) = -9$.

B $$\frac{f(x)}{d(x)} = \frac{x^3 - 2x^2 + x + 5}{x^2 + 1} = x - 2 + \frac{7}{x^2 + 1}$$

$\Rightarrow x^3 - 2x^2 + x + 5 = (x^2 + 1)(x - 2) + 7$

THEREFORE, $q(x) = x - 2$ AND $r(x) = 7$.

C $$\frac{f(x)}{d(x)} = \frac{x^4 + x^2 - 2}{x^2 - x + 3} = x^2 + x - 1 + \frac{-4x + 1}{x^2 - x + 3}$$

$\Rightarrow x^4 + x^2 - 2 = (x^2 - x + 3)(x^2 + x - 1) + (-4x + 1)$

GIVING US $q(x) = x^2 + x - 1$ AND $r(x) = -4x + 1$.

Exercise 1.4

1. FOR EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, FIND THE QUOTIENT $q(x)$ AND REMAINDER $r(x)$ THAT SATISFY THE REQUIREMENTS OF THE POLYNOMIAL DIVISION TH

   A $f(x) = x^2 - x + 7;\ d(x) = x + 1$

   B $f(x) = x^3 + 2x^2 - 5x + 3;\ d(x) = x^2 + x - 1$

   C $f(x) = x^2 + 8x - 12;\ d(x) = 2$

2. IN EACH OF THE FOLLOWING, EXPRESS THE FUNCTION $f(x)$ IN THE FORM $f(x) = (x - c)\,q(x) + r(x)$ FOR THE GIVEN NUMBER c.

   A $f(x) = x^3 - 5x^2 - x + 8;\ c = -2$ **B** $f(x) = x^3 + 2x^2 - 2x - 14;\ c = \frac{1}{2}$

3. PERFORM THE FOLLOWING DIVISIONS, ASSUMING THAT n IS A POSITIVE INTEGER:

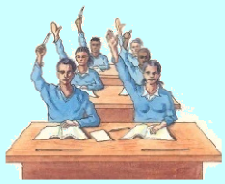
   A $\dfrac{x^{3n} + 5x^{2n} + 12x^n + 18}{x^n + 3}$ **B** $\dfrac{x^{3n} - x^{2n} + 3x^n - 10}{x^n - 2}$

1.2.2 Remainder Theorem

THE EQUALITY $f(x) = d(x)\,q(x) + r(x)$ EXPRESSES THE FACT THAT

Dividend = (divisor) (quotient) + remainder.

ACTIVITY 1.7



1 LET $f(x) = x^4 - x^3 - x^2 - x - 2$.

- **A** FIND $f(-2)$ AND $f(2)$.
- **B** WHAT IS THE REMAINDER IF $f(x)$ IS DIVIDED BY $x + 2$?
- **C** IS THE REMAINDER EQUAL TO $f(-2)$?
- **D** WHAT IS THE REMAINDER IF $f(x)$ IS DIVIDED BY $x - 2$?
- **E** IS THE REMAINDER EQUAL TO $f(2)$?

2 IN EACH OF THE FOLLOWING, FIND THE REMAINDER WHEN THE GIVEN POLYNOMIAL $f(x)$ IS DIVIDED BY THE POLYNOMIAL $x - c$ FOR THE GIVEN NUMBER c. ALSO, FIND $f(c)$.

- **A** $f(x) = 2x^2 + 3x + 1$; $c = -1$
- **B** $f(x) = x^6 + 1$; $c = -1, 1$
- **C** $f(x) = 3x^3 - x^4 + 2$; $c = 2$
- **D** $f(x) = x^3 - x + 1$; $c = -1, 1$

Theorem 1.2 Remainder theorem

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let c be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

Proof:-

WHEN $f(x)$ IS DIVIDED BY $x - c$, THE REMAINDER IS ALWAYS A CONSTANT. WHY?

BY THE POLYNOMIAL DIVISION THEOREM,

$$f(x) = (x - c)\,q(x) + k$$

WHERE k IS CONSTANT. THIS EQUATION HOLDS FOR EVERY REAL NUMBER x. HENCE, IT HOLDS WHEN $x = c$.

IN PARTICULAR, IF $x = c$, YOU OBSERVE A VERY INTERESTING AND USEFUL RELATIONSHIP:

$$f(c) = (c - c)\,q(c) + k$$
$$= 0.\,q(c) + k$$
$$= 0 + k = k$$

IT FOLLOWS THAT THE VALUE OF THE POLYNOMIAL $f(x)$ AT $x = c$ IS THE SAME AS THE REMAINDER k OBTAINED WHEN YOU DIVIDE $f(x)$ BY $x - c$.

EXAMPLE 2 FIND THE REMAINDER BY DIVIDING $f(x)$ BY $d(x)$ IN EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, USING THE POLYNOMIAL DIVISION THEOREM AND THE REMAINDER THEOREM:

A $f(x) = x^3 - x^2 + 8x - 1;\ d(x) = x + 2$

B $f(x) = x^4 + x^2 + 2x + 5;\ d(x) = x - 1$

SOLUTION:

A ***Polynomial division theorem***

$$\frac{x^3 - x^2 + 8x - 1}{x + 2} = x^2 - 3x + 14 - \frac{29}{x + 2}$$

Remainder theorem

$$f(-2) = (-2)^3 - (-2)^2 + 8(-2) - 1,$$

$$= -8 - 4 - 16 - 1 = -29$$

THEREFORE, THE REMAINDER IS –29.

B ***Polynomial division theorem***

$$\frac{x^4 + x^2 + 2x + 5}{x - 1} = x^3 + x^2 + 2x + 4 + \frac{9}{x - 1}$$

Remainder theorem

$$f(1) = (1)^4 + (1)^2 + 2(1) + 5$$

$$= 1 + 1 + 2 + 5 = 9$$

THEREFORE, THE REMAINDER IS 9.

EXAMPLE 3 WHEN $x^3 - 2x^2 + 3bx + 10$ IS DIVIDED BY $x - 3$, THE REMAINDER IS 37. FIND THE VALUE OF b.

SOLUTION: LET $f(x) = x^3 - 2x^2 + 3bx + 10$.

$f(3) = 37$. (BY THE REMAINDER THEOREM)

$\Rightarrow (3)^3 - 2(3)^2 + 3b(3) + 10 = 37$

$27 - 18 + 9b + 10 = 37 \Rightarrow 9b + 19 = 37 \Rightarrow b = 2.$

Exercise 1.5

1 IN EACH OF THE FOLLOWING, EXPRESS THE FUNCTION IN THE FORM

$f(x) = (x - c)\,q(x) + r(x)$

FOR THE GIVEN NUMBER c, AND SHOW THAT $f(c) = r$ IS THE REMAINDER.

A $f(x) = x^3 - x^2 + 7x + 11;\quad c = 2$

B $f(x) = 1 - x^5 + 2x^3 + x;\quad c = -1$

C $f(x) = x^4 + 2x^3 + 5x^2 + 1;\quad c = -\frac{2}{3}$

2 IN EACH OF THE FOLLOWING, USE THE REMAINDER THEOREM TO FIND THE REMAINDER WHEN THE POLYNOMIAL $f(x)$ IS DIVIDED BY $x - c$ FOR THE GIVEN NUMBER c.

A $f(x) = x^{17} - 1;\ c = 1$ **B** $f(x) = 2x^2 + 3x + 1;\ c = -\frac{1}{2}$

C $f(x) = x^{23} + 1;\ c = -1$

3 WHEN $f(x) = 3x^7 - ax^6 + 5x^3 - x + 11$ IS DIVIDED BY $x + 1$, THE REMAINDER IS 15. WHAT IS THE VALUE OF a?

4 WHEN THE POLYNOMIAL $f(x) = ax^3 + bx^2 - 2x + 8$ IS DIVIDED BY $x - 1$ AND $x + 1$ THE REMAINDERS ARE 3 AND 5 RESPECTIVELY. FIND THE VALUES OF a AND b.

1.2.3 Factor Theorem

RECALL THAT ***factorizing*** A POLYNOMIAL MEANS WRITING IT AS A PRODUCT OF TWO OR MORE POLYNOMIALS. YOU WILL DISCUSS BELOW AN INTERESTING THEOREM, KNOWN AS ***factor theorem***, WHICH IS HELPFUL IN CHECKING WHETHER A LINEAR POLYNOMIAL IS A FACTOR OF A GIVEN POLYNOMIAL OR NOT.

ACTIVITY 1.8

1 LET $f(x) = x^3 - 5x^2 + 2x + 8$.

A FIND $f(2)$.

B WHAT IS THE REMAINDER WHEN $f(x)$ IS DIVIDED BY $x - 2$?

C IS $x - 2$ A FACTOR OF $f(x)$?

D FIND $f(-1)$ AND $f(1)$.

E EXPRESS $f(x)$ AS $f(x) = (x - C)q(x)$ WHERE $q(x)$ IS THE QUOTIENT.

2 LET $f(x) = x^3 - 3x^2 - x + 3$.

A WHAT ARE THE VALUES OF $f(-1)$, $f(1)$ AND $f(3)$?

B WHAT DOES THIS TELL US ABOUT THE REMAINDER WHEN $f(x)$ IS DIVIDED BY $x + 1$, $x - 1$ AND $x - 3$?

C HOW CAN THIS HELP US IN FACTORIZING $f(x)$?

Theorem 1.3 Factor theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number, then

I $x - c$ is a factor of $f(x)$, if $f(c) = 0$, and

II $f(c) = 0$, if $x - c$ is a factor of $f(x)$.

TRY TO DEVELOP A PROOF OF THIS THEOREM USING THE REMAINDER THEOREM.

Group Work 1.2

1 Let $f(x) = 4x^4 - 5x^2 + 1$.

a Find $f(-1)$ and show that $x + 1$ is a factor of $f(x)$.

b Show that $2x - 1$ is a factor of $f(x)$.

c Try to completely factorize $f(x)$ into linear factors.

2 Give the proof of the Factor Theorem.

Hint: You have to prove that

i If $f(c) = 0$, then $x - c$ is a factor of $f(x)$

ii If $x - c$ is a factor of $f(x)$, then $f(c) = 0$

Use the Polynomial Division Theorem with factor $(x - c)$ to express $f(x)$ as

$f(x) = d(x)\, q(x) + r(x)$, where $d(x) = x - c$.

Use the Remainder Theorem $r(x) = k = f(c)$, giving you

$f(x) = (x - c)\, q(x) + f(c)$

where $q(x)$ is a polynomial of degree less than the degree of $f(x)$.

If $f(c) = 0$, then what will $f(x)$ be? Complete the proof.

Example 4 Let $f(x) = x^3 + 2x^2 - 5x - 6$. Use the Factor Theorem to determine whether:

a $x + 1$ is a factor of $f(x)$ **b** $x + 2$ is a factor of $f(x)$.

Solution:

a Since $x + 1 = x - (-1)$, it has the form $x - c$ with $c = -1$.

$f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = -1 + 2 + 5 - 6 = 0.$

So, by the Factor Theorem, $x + 1$ is a factor of $f(x)$.

b $f(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6 = -8 + 8 + 10 - 6 = 4 \neq 0.$

By the Factor Theorem, $x + 2$ is not a factor of $f(x)$.

Example 5 Show that $x + 3$, $x - 2$ and $x + 1$ are factors and $x + 2$ is not a factor of $f(x) = x^4 + x^3 - 7x^2 - x + 6$.

Solution: $f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6 = 81 - 27 - 63 + 3 + 6 = 0.$

Hence $x + 3$ is a factor of $f(x)$.

$f(2) = 2^4 + (2)^3 - 7(2)^2 - 2 + 6 = 16 + 8 - 28 - 2 + 6 = 0.$

HENCE $x – 2$ IS A FACTOR OF $f(x)$.

$$f(–1) = (–1)^4 + (–1)^3 – 7(–1)^2 – (–1) + 6 = 1 – 1 – 7 + 1 + 6 = 0$$

HENCE $x + 1$ IS A FACTOR OF $f(x)$.

$$f(–2) = (–2)^4 + (–2)^3 – 7(–2)^2 – (–2) + 6 = 16 – 8 – 28 + 2 + 6 = –12 \neq 0$$

HENCE $x + 2$ IS NOT A FACTOR OF $f(x)$.

Exercise 1.6

1 IN EACH OF THE FOLLOWING, USE THE FACTOR THEOREM TO DETERMINE WHETHER OR NOT $g(x)$ IS A FACTOR OF $f(x)$.

A $g(x) = x + 1;\ f(x) = x^{15} + 1$

B $g(x) = x - 1;\ f(x) = x^7 + x - 1$

C $g(x) = x - \frac{3}{2};\ f(x) = 6x^2 + x - 1$

D $g(x) = x + 2;\ f(x) = x^3 - 3x^2 - 4x - 12$

2 IN EACH OF THE FOLLOWING, FIND THE NUMBER k SATISFYING THE GIVEN CONDITION:

A $x - 2$ IS A FACTOR OF $3x^4 + 8x^2 - kx + 6$

B $x + 3$ IS A FACTOR OF $x^5 - kx^4 - 6x^3 - x^2 + 4x + 29$

C $3x - 2$ IS A FACTOR OF $6x^3 - 4x^2 + kx - k$

3 FIND NUMBERS a AND k SO THAT $x – 2$ IS A FACTOR OF $f(x) = x^4 – 2ax^3 + ax^2 – x + k$ AND $f(–1) = 3$.

4 FIND A POLYNOMIAL FUNCTION OF DEGREE 3 SUCH THAT $f(2) = 24$ AND THAT $x – 1$, x AND $x + 2$ ARE FACTORS OF THE POLYNOMIAL.

5 LET a BE A REAL NUMBER AND n A POSITIVE INTEGER. SHOW THAT $x – a$ IS A FACTOR OF $x^n – a^n$.

6 SHOW THAT $x – 1$ AND $x + 1$ ARE FACTORS AND x IS NOT A FACTOR OF $2x^3 – x^2 – 2x + 1$.

7 IN EACH OF THE FOLLOWING, FIND THE CONSTANT c SUCH THAT THE DENOMINATOR WILL DIVIDE THE NUMERATOR EXACTLY:

A $\dfrac{x^3 + 3x^2 - 3x + c}{x - 3}$ **B** $\dfrac{x^3 - 2x^2 + x + c}{x + 2}$.

8 THE AREA OF A RECTANGLE IN SQUARE FEET IS $x^2 + 13x + 36$. HOW MUCH LONGER IS THE LENGTH THAN THE WIDTH OF THE RECTANGLE?

1.3 ZEROS OF A POLYNOMIAL FUNCTION

IN THIS SECTION, YOU WILL DISCUSS AN INTERESTING CONCEPT KNOWN AS ***zeros of a polynomial***. CONSIDER THE POLYNOMIAL FUNCTION $f(x) = x - 1$.

WHAT IS $f(1)$? NOTE THAT $f(1) = 1 - 1 = 0$.

AS $f(1) = 0$, WE SAY THAT 1 IS THE ZERO OF THE POLYNOMIAL FUNCTION $f(x)$.

TO FIND THE ZERO OF A LINEAR (FIRST DEGREE POLYNOMIAL) FUNCTION OF THE FORM $f(x) = ax + b$, $a \neq 0$, WE FIND THE NUMBER x FOR WHICH $ax + b = 0$.

NOTE THAT EVERY LINEAR FUNCTION HAS EXACTLY ONE ZERO.

$ax + b = 0 \Rightarrow ax = -b$ .... *Subtracting b from both sides*

$\Rightarrow x = -\dfrac{b}{a}$ .... *Dividing both sides by a, since a ≠ 0.*

THEREFORE, $x = -\dfrac{b}{a}$ IS THE ONLY ZERO OF THE LINEAR FUNCTION f, WHENEVER $a \neq 0$.

EXAMPLE 1 FIND THE ZEROS OF THE POLYNOMIAL $f(x) = \dfrac{2x-1}{3} - \dfrac{x+2}{3} - 2$.

SOLUTION: $f(x) = 0 \Rightarrow \dfrac{2x-1}{3} - \dfrac{x+2}{3} = 2$

$2x - 1 - (x + 2) = 6 \Rightarrow 2x - 1 - x - 2 = 6 \Rightarrow x = 9.$

SO, 9 IS THE ZERO OF $f(x)$.

SIMILARLY, TO FIND THE ZEROS OF A QUADRATIC FUNCTION (SECOND DEGREE POLY FORM $f(x) = ax^2 + bx + c$, $a \neq 0$, WE FIND THE NUMBER x FOR WHICH

$ax^2 + bx + c = 0,\ a \neq 0.$

ACTIVITY 1.9

1. FIND THE ZEROS OF EACH OF THE FOLLOWING FUNCTIONS:
   - **A** $h(x) = 1 - \dfrac{3}{5}(x + 2)$ **B** $k(x) = 2 - (x^2 - 4) + x^2 - 4x$
   - **C** $f(x) = 4x^2 - 25$ **D** $f(x) = x^2 + x - 12$
   - **E** $f(x) = x^3 - 2x^2 + x$ **F** $g(x) = x^3 + x^2 - x - 1$
2. HOW MANY ZEROS CAN A QUADRATIC FUNCTION HAVE?
3. STATE TECHNIQUES FOR FINDING ZEROS OF A QUADRATIC FUNCTION.
4. HOW MANY ZEROS CAN A POLYNOMIAL FUNCTION OF DEGREE 3 HAVE? WHAT ABOUT DEGREE 4?

EXAMPLE 2 FIND THE ZEROS OF EACH OF THE FOLLOWING QUADRATIC FUNCTIONS:

A $f(x) = x^2 - 16$ **B** $g(x) = x^2 - x - 6$ **C** $h(x) = 4x^2 - 7x + 3$

SOLUTION:

A $f(x) = 0 \Rightarrow x^2 - 16 = 0 \Rightarrow x^2 - 4^2 = 0 \Rightarrow (x - 4)(x + 4) = 0$

$\Rightarrow x - 4 = 0$ OR $x + 4 = 0 \Rightarrow x = 4$ OR $x = -4$

THEREFORE, – 4 AND 4 ARE THE ZEROS OF f.

B $g(x) = 0 \Rightarrow x^2 - x - 6 = 0$

FIND TWO NUMBERS WHOSE SUM IS – 1 AND WHOSE PRODUCT IS – 6. THESE ARE – 3

$x^2 - 3x + 2x - 6 = 0 \Rightarrow x(x - 3) + 2(x - 3) = 0 \Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow x + 2 = 0$ OR $x - 3 = 0 \Rightarrow x = -2$ OR $x = 3$

THEREFORE, –2 AND 3 ARE THE ZEROS OF g.

C $h(x) = 0 \Rightarrow 4x^2 - 7x + 3 = 0$

FIND TWO NUMBERS WHOSE SUM IS –7 AND WHOSE PRODUCT IS 12. THESE ARE –4 A

HENCE, $4x^2 - 7x + 3 = 0 \Rightarrow 4x^2 - 4x - 3x + 3 = 0 \Rightarrow 4x(x - 1) - 3(x - 1) = 0$

$\Rightarrow (4x - 3)(x - 1) = 0 \Rightarrow 4x - 3 = 0$ OR $x - 1 = 0 \Rightarrow x = \frac{3}{4}$ OR $x = 1$.

THEREFORE, $\frac{3}{4}$ AND 1 ARE THE ZEROS OF h.

> **Definition 1.2**
>
> For a polynomial function f and a real number c, if
>
> $f(c) = 0$, then c is a ***zero*** of f.

NOTE THAT IF $x - c$ IS A FACTOR OF $f(x)$, THEN c IS A ZERO OF $f(x)$.

EXAMPLE 3

A USE THE FACTOR THEOREM TO SHOW THAT $x + 1$ IS A FACTOR OF $f(x) = x^{25} + 1$.

B WHAT ARE THE ZEROS OF $p(x) = (x - 5)(x + 2)(x - 1)$?

C WHAT ARE THE REAL ZEROS OF $x^4 - 1$

D DETERMINE THE ZEROS OF $f(x) = 2x^4 - 3x^2 + 1$.

SOLUTION:

A SINCE $x + 1 = x - (-1)$, WE HAVE $c = -1$ AND

$f(c) = f(-1) = (-1)^{25} + 1 = -1 + 1 = 0$

HENCE, –1 IS A ZERO OF $f(x) = x^{25} + 1$, BY THE FACTOR THEOREM.

SO, $x - (-1) = x + 1$ IS A FACTOR OF $x^{25} + 1$.

B SINCE $(x - 5)$, $(x + 2)$ AND $(x - 1)$ ARE ALL FACTORS OF $p(x)$, 5, –2 AND 1 ARE THE ZEROS OF $p(x)$.

c Factorising the left side, we have

$x^4 - 1 = 0 \Rightarrow (x^2 - 1)(x^2 + 1) = 0 \Rightarrow (x - 1)(x + 1)(x^2 + 1) = 0$

So, the real zeros of $f(x) = x^4 - 1$ are -1 and 1.

d $f(x) = 0 \Rightarrow 2x^4 - 3x^2 + 1 = 0 \Rightarrow 2(x^2)^2 - 3x^2 + 1 = 0$

Let $y = x^2$. Then $2(y)^2 - 3y + 1 = 0 \Rightarrow 2y^2 - 3y + 1 = 0 \Rightarrow (2y - 1)(y - 1) = 0$

$\Rightarrow 2y - 1 = 0$ or $y - 1 = 0$

Hence $y = \dfrac{1}{2}$ or $y = 1$

Since $y = x^2$, we have $x^2 = \dfrac{1}{2}$ or $x^2 = 1$.

Therefore $x = \pm\sqrt{\dfrac{1}{2}}$ or $x = \pm 1$. (*Note that* $\sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2}$.)

Hence, $-\dfrac{\sqrt{2}}{2}$, $\dfrac{\sqrt{2}}{2}$, -1 and 1 are zeros of f.

A polynomial function cannot have more zeros than its degree.

1.3.1 Zeros and Their Multiplicities

If $f(x)$ is a polynomial function of degree n, $n \ge 1$, then a ***root*** of the equation $f(x) = 0$ is called a ***zero*** of f.

By the Factor Theorem, each zero c of a polynomial function $f(x)$ generates a first degree factor $(x - c)$ of $f(x)$. When $f(x)$ is factorized completely, the same factor $(x - c)$ may occur more than once, in which case c is called a ***repeated*** or a ***multiple zero*** of $f(x)$. If $x - c$ occurs only once, then c is called a ***simple zero*** of $f(x)$.

Definition 1.3

If $(x - c)^k$ is a factor of $f(x)$, but $(x - c)^{k+1}$ is not, then c is said to be a ***zero of multiplicity k*** of f.

Example 4 Given that -1 and 2 are zeros of $f(x) = x^4 - x^3 - 3x^2 - 5x - 2$, determine their multiplicity.

Solution: By the Factor Theorem, $(x + 1)$ and $(x - 2)$ are factors of $f(x)$.

Hence, $f(x)$ can be divided by $(x + 1)(x - 2) = x^2 - x - 2$, giving you

$f(x) = (x^2 - x - 2)(x^2 + 2x + 1) = (x + 1)(x - 2)(x + 1)^2 = (x + 1)^3(x - 2)$

Therefore, -1 is a zero of multiplicity 3 and 2 is a zero of multiplicity 1.

Exercise 1.7

1 FIND THE ZEROS OF EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = 1 - \frac{3}{5}x$ **B** $f(x) = \frac{1}{4}(1 - 2x) - (x + 3)$

C $g(x) = \frac{2}{3}(2 - 3x)(x - 2)(x + 1)$ **D** $h(x) = x^4 + 7x^2 + 12$

E $g(x) = x^3 + x^2 - 2$ **F** $f(t) = t^3 - 7t + 6$

G $f(y) = y^5 - 2y^3 + y$ **H** $f(x) = 6x^4 - 7x^2 - 3$

2 FOR EACH OF THE FOLLOWING, LIST THE ZEROS OF THE POLYNOMIAL FUNCTION AND STATE THE MULTIPLICITY OF EACH ZERO.

A $f(x) = x^{12}\left(x - \frac{2}{3}\right)$ **B** $g(x) = 3\left(x - \sqrt{2}\right)^2(x + 1)$

C $h(x) = 3x^6(\quad - x)^5(x - (\quad + 1))^3$ **D** $f(x) = 2\left(x - \sqrt{3}\right)^5(x + 5)^9(1 - 3x)$

E $f(x) = x^3 - 3x^2 + 3x - 1$

3 FIND A POLYNOMIAL FUNCTION f OF DEGREE 3 SUCH THAT $f(10) = 17$ AND THE ZEROS OF f ARE 0, 5 AND 8.

4 IN EACH OF THE FOLLOWING, THE INDICATED NUMBER IS A ZERO OF THE POLYNOMIAL FUNCTION $f(x)$. DETERMINE THE MULTIPLICITY OF THIS ZERO.

A 1; $f(x) = x^3 + x^2 - 5x + 3$ **B** -1; $f(x) = x^4 + 3x^3 + 3x^2 + x$

C $\frac{1}{2}$; $f(x) = 4x^3 - 4x^2 + x$.

5 SHOW THAT IF $3x + 4$ IS A FACTOR OF SOME POLYNOMIAL FUNCTION f, THEN $-\frac{4}{3}$ IS A ZERO OF f.

6 IN EACH OF THE FOLLOWING, FIND A POLYNOMIAL FUNCTION THAT HAS THE GIVEN ZEROS SATISFYING THE GIVEN CONDITION.

A 0, 3, 4 AND $f(1) = 5$ **B** $-1, 1 + \sqrt{2}, 1 - \sqrt{2}$ AND $f(0) =$

7 A POLYNOMIAL FUNCTION f OF DEGREE 3 HAS ZEROS $-2, \frac{1}{2}$ AND AND ITS LEADING COEFFICIENT IS NEGATIVE. WRITE AN EXPRESSION FOR f. HOW MANY DIFFERENT POLYNOMIAL FUNCTIONS ARE POSSIBLE FOR f?

8 IF $p(x)$ IS A POLYNOMIAL OF DEGREE 3 WITH $p(0) = p(1) = p(-1) = 0$ AND $p(2) = 6$, THEN

A SHOW THAT $p(-x) = -p(x)$.

B FIND THE INTERVAL IN WHICH $p(x)$ IS LESS THAN ZERO.

9 FIND THE VALUES OF p AND q IF $x - 1$ IS A COMMON FACTOR OF

$f(x) = x^4 - px^3 + 7qx + 1$, AND $g(x) = x^6 - 4x^3 + px^2 + qx - 3$.

10 THE HEIGHT ABOVE GROUND LEVEL IN METRES OF A MISSILE LAUNCHED VERTICALLY, IS GIVEN BY

$$h(t) = -16t^3 + 100t.$$

AT WHAT TIME IS THE MISSILE 72 M ABOVE GROUND LEVEL? [t IS TIME IN SECONDS].

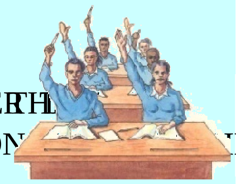
1.3.2 Location Theorem

A POLYNOMIAL FUNCTION WITH RATIONAL COEFFICIENTS MAY HAVE NO RATIONA
EXAMPLE, THE ZEROS OF THE POLYNOMIAL FUNCTION:

$f(x) = x^2 - 4x - 2$ ARE ALL IRRATIONAL.

CAN YOU WORK OUT WHAT THE ZEROS ARE? THE POLYNOMIAL FUNCTION $p(x) = x^3 - x^2 - 2x + 2$ HAS RATIONAL AND IRRATIONAL ZEROS, $-\sqrt{2}$, 1 AND $\sqrt{2}$. CAN YOU CHECK THIS?

ACTIVITY 1.10



1 IN EACH OF THE FOLLOWING, DETERMINE WHETHER THE ZEROS OF THE CORRESPONDING FUNCTION ARE RATIONAL, IRRATIONAL OR NEITHER:

A $f(x) = x^2 + 2x + 2$ **B** $f(x) = x^3 + x^2 - 2x - 2$

C $f(x) = (x + 1)(2x^2 + x - 3)$ **D** $f(x) = x^4 - 5x^2 + 6$

2 FOR EACH OF THE FOLLOWING POLYNOMIALS, MAKE A TABLE OF VALUES FOR $-4 \le x \le 4$:

A $f(x) = 3x^3 + x^2 + x - 2$ **B** $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$

MOST OF THE STANDARD METHODS FOR FINDING THE IRRATIONAL ZEROS OF A POLYN
INVOLVE A TECHNIQUE OF SUCCESSIVE APPROXIMATION. ONE OF THE METHODS IS BA
IDEA OF ***change of sign*** OF A FUNCTION. CONSEQUENTLY, THE FOLLOWING THEOREM IS GIVEN.

Theorem 1.4 Location theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between a and b.

THIS THEOREM HELPS US TO LOCATE THE REAL ZEROS OF A POLYNOMIAL FUNCTION. I
POSSIBLE TO ESTIMATE THE ZEROS OF A POLYNOMIAL FUNCTION FROM A TABLE OF VA

EXAMPLE 5 LET $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$. CONSTRUCT A TABLE OF VALUES AND USE THE LOCATION THEOREM TO LOCATE THE ZEROS OF f BETWEEN SUCCESSIVE INTEGERS.

SOLUTION: CONSTRUCT A TABLE AND LOOK FOR CHANGES IN SIGN AS FOLLOWS:

x	–3	–2	–1	0	1	2	3	4	5	6
$f(x)$	210	38	–10	–6	2	–10	–42	–70	–44	102

SINCE $f(-2) = 38 > 0$ AND $f(-1) = -10 < 0$, WE SEE THAT THE VALUE OF $f(x)$ CHANGES FROM POSITIVE TO NEGATIVE BETWEEN – 2 AND –1. HENCE, BY THE LOCATION THEOREM, THERE IS A ZERO OF $f(x)$ BETWEEN $x = -2$ AND $x = -1$.

SINCE $f(0) = -6 < 0$ AND $f(1) = 2 > 0$, THERE IS ALSO ONE ZERO BETWEEN $x = 0$ AND $x = 1$.

SIMILARLY, THERE ARE ZEROS BETWEEN $x = 1$ AND $x = 2$ AND BETWEEN $x = 5$ AND $x = 6$.

EXAMPLE 6 USING THE LOCATION THEOREM, SHOW THAT THE POLYNOMIAL $f(x) = x^5 - 2x^2 - 1$ HAS A ZERO BETWEEN $x = 1$ AND $x = 2$.

SOLUTION: $f(1) = (1)^5 - 2(1)^2 - 1 = 1 - 2 - 1 = -2 < 0$.

$f(2) = (2)^5 - 2(2)^2 - 1 = 32 - 8 - 1 = 23 > 0$.

HERE, $f(1)$ IS NEGATIVE AND $f(2)$ IS POSITIVE. THEREFORE, THERE IS A ZERO BETWEEN $x = 1$ AND $x = 2$.

Exercise 1.8

1 IN EACH OF THE FOLLOWING, USE THE TABLE FOR THE POLYNOMIAL FUNCTION $f(x)$ TO LOCATE ZEROS OF $y = f(x)$:

A

x	–5	– 3	– 1	0	2	5
$f(x)$	7	4	2	–1	3	– 6

B

x	–6	–5	–4	–3	–2	–1	0	1	2
$f(x)$	–21	–10	8	–1	–5	6	4	–3	18

2 USE THE LOCATION THEOREM TO VERIFY THAT $f(x)$ HAS A ZERO BETWEEN a AND b:

A $f(x) = 3x^3 + 7x^2 + 3x + 7$; $a = -3$, $b = -2$

B $f(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15$; $a = 1$, $b = \frac{3}{2}$

C $f(x) = -x^4 + x^3 + 1$; $a = -1$, $b = 1$

D $f(x) = x^5 - 2x^3 - 1$; $a = 1$, $b = 2$

3 IN EACH OF THE FOLLOWING, USE THE LOCATION THEOREM TO LOCATE EACH REAL ZERO OF $f(x)$ BETWEEN SUCCESSIVE INTEGERS:

A $f(x) = x^3 - 9x^2 + 23x - 14$; FOR $0 \le x \le 6$

B $f(x) = x^3 - 12x^2 + x + 2$; FOR $0 \le x \le 8$

C $f(x) = x^4 - x^2 + x - 1$; FOR $-3 \le x \le 3$

D $f(x) = x^4 + x^3 - x^2 - 11x + 3$; FOR $-3 \le x \le 3$

4 IN EACH OF THE FOLLOWING, FIND ALL THE REAL ZEROS OF THE POLYNOMIAL FUNCTION, FOR $-4 \le x \le 4$:

A $f(x) = x^4 - 5x^3 + \frac{15}{2}x^2 - 2x - 2$ **B** $f(x) = x^5 - 2x^4 - 3x^3 + 6x^2 + 2x - 4$

C $f(x) = x^4 + x^3 - 4x^2 - 2x + 4$ **D** $f(x) = 2x^4 + x^3 - 10x^2 - 5x$

5 IN QUESTION 10 OF EXERCISE 1.7, AT WHAT TIME IS THE MISSILE 50 M ABOVE THE GROUN LEVEL?

6 IS IT POSSIBLE FOR A POLYNOMIAL FUNCTION OF DEGREE 3 WITH INTEGER COEFFICIENT TO HAVE NO REAL ZEROS? EXPLAIN YOUR ANSWER.

1.3.3 Rational Root Test

THE ***rational root test*** RELATES THE POSSIBLE RATIONAL ZEROS OF A POLYNOMIAL WITH INTEGER COEFFICIENTS TO THE LEADING COEFFICIENT AND TO THE CONSTANT TERM OF THE POL

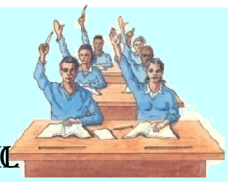
Theorem 1.5 **Rational root test**

IF THE RATIONAL NUMBER $\frac{p}{q}$, IN ITS LOWEST TERMS, IS A ZERO OF THE POLYNOMIAL

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

WITH INTEGER COEFFICIENTS, THEN p MUST BE A FACTOR OF a_0 AND q MUST BE A FACTOR OF a_n.

ACTIVITY 1.11



1 WHAT SHOULD YOU DO FIRST TO USE THE RATIONAL ROOT TEST?

2 WHAT MUST THE LEADING COEFFICIENT BE FOR ALL THE RATIONAL ZEROS TO BE FACTORS OF THE CONSTANT TERM?

3 SUPPOSE THAT ALL OF THE COEFFICIENTS ARE RATIONAL NUMBERS. WHAT COULD BE DONE TO CHANGE THE POLYNOMIAL INTO ONE WITH INTEGER COEFFICIENTS? DOES THE RES POLYNOMIAL HAVE THE SAME ZEROS AS THE ORIGINAL?

4 THERE IS AT LEAST ONE RATIONAL ZERO OF A POLYNOMIAL WHOSE CONSTANT TERM IS ZERO. WHAT IS THIS NUMBER?

EXAMPLE 7 IN EACH OF THE FOLLOWING, FIND ALL THE RATIONAL ZEROS OF THE POLYNOMIAL:

A $f(x) = x^3 - x + 1$ **B** $g(x) = 2x^3 + 9x^2 + 7x - 6$

C $g(x) = \frac{1}{2}x^4 - 2x^3 - \frac{1}{2}x^2 + 2x$

SOLUTION:

A THE LEADING COEFFICIENT IS 1 AND THE CONSTANT TERM IS 1. HENCE, AS THESE ARE FACTORS OF THE CONSTANT TERM, THE POSSIBLE RATIONAL ZEROS ARE ±1.

USING THE REMAINDER THEOREM, TEST THESE POSSIBLE ZEROS.

$$f(1) = (1)^3 - 1 + 1 = 1 - 1 + 1 = 1$$

$$f(-1) = (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1$$

SO, WE CAN CONCLUDE THAT THE GIVEN POLYNOMIAL HAS NO RATIONAL ZEROS.

B $a_n = a_3 = 2$ AND $a_0 = -6$

POSSIBLE VALUES OF p ARE FACTORS OF –6. THESE ARE ±1, ±2, ±3 AND ±6.

POSSIBLE VALUES OF q ARE FACTORS OF 2. THESE ARE ±1, ±2.

THE POSSIBLE RATIONAL ZEROS $\frac{p}{q}$ ARE $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

OF THESE 12 POSSIBLE RATIONAL ZEROS, AT MOST 3 CAN BE THE ZEROS OF g (WHY?).

CHECK THAT $f(-3) = 0$, $f(-2) = 0$ AND $f\left(\frac{1}{2}\right) = 0$.

USING THE FACTOR THEOREM, WE CAN FACTORIZE $g(x)$ AS:

$2x^3 + 9x^2 + 7x - 6 = (x + 3)(x + 2)(2x - 1)$. SO, $g(x) = 0$ AT

$x = -3$, $x = -2$ AND AT $x = \frac{1}{2}$.

THEREFORE –3, –2 AND $\frac{1}{2}$ ARE THE ONLY (RATIONAL) ZEROS OF g.

C LET $h(x) = 2g(x)$. THUS $h(x)$ WILL HAVE THE SAME ZEROS, BUT HAS INTEGER COEFFICIENTS.

$$h(x) = x^4 - 4x^3 - x^2 + 4x$$

x IS A FACTOR, SO $h(x) = x(x^3 - 4x^2 - x + 4) = xk(x)$

$k(x)$ HAS A CONSTANT TERM OF 4 AND LEADING COEFFICIENT OF 1. THE POSSIBLE RATIONAL ZEROS ARE ±1, ±2, ±4.

USING THE REMAINDER THEOREM, $k(1) = 0$, $k(-1) = 0$ AND $k(4) = 0$

SO, BY THE FACTOR THEOREM $k(x) = (x - 1)(x + 1)(x - 4)$.

HENCE, $h(x) = x\,k(x) = x(x - 1)(x + 1)(x - 4)$ AND

$$g(x) = \frac{1}{2}h(x) = \frac{1}{2}x(x - 1)(x + 1)(x - 4).$$

THEREFORE, THE ZEROS OF $g(x)$ ARE 0, ± 1 AND 4.

Exercise 1.9

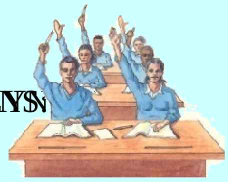
1. IN EACH OF THE FOLLOWING, FIND THE ZEROS AND THE MULTIPLICITY OF EACH ZERO. WHAT IS THE DEGREE OF THE POLYNOMIAL?
   - **A** $f(x) = (x + 6)(x - 3)^2$
   - **B** $f(x) = 3(x + 2)^3(x - 1)^2(x + 3)$
   - **C** $f(x) = \frac{1}{2}(x - 2)^4(x + 3)^3(1 - x)$
   - **D** $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$
   - **E** $f(x) = x^4 - 4x^3 + 7x^2 - 12x + 12$
2. FOR EACH OF THE FOLLOWING POLYNOMIALS, FIND ALL POSSIBLE RATIONAL ZEROS:
   - **A** $p(x) = x^3 - 2x^2 - 5x + 6$
   - **B** $p(x) = x^3 - 3x^2 + 6x + 8$
   - **C** $p(x) = 3x^3 - 11x^2 + 8x + 4$
   - **D** $p(x) = 2x^3 + x^2 - 4x - 3$
   - **E** $p(x) = 12x^3 - 16x^2 - 5x + 3$
3. IN EACH OF THE FOLLOWING, FIND ALL THE RATIONAL ZEROS OF THE POLYNOMIAL, AND EXPRESS THE POLYNOMIAL IN FACTORIZED FORM:
   - **A** $f(x) = x^3 - 5x^2 - x + 5$
   - **B** $g(x) = 3x^3 + 3x^2 - x - 1$
   - **C** $p(t) = t^4 - t^3 - t^2 - t - 2$
4. IN EACH OF THE FOLLOWING, FIND ALL RATIONAL ZEROS OF THE FUNCTION:
   - **A** $p(y) = y^3 + \frac{11}{6}y^2 - \frac{1}{2}y - \frac{1}{3}$
   - **B** $p(x) = x^4 - \frac{25}{4}x^2 + 9$
   - **C** $h(x) = x^4 - \frac{21}{10}x^2 + \frac{3}{5}x$
   - **D** $p(x) = x^4 + \frac{7}{6}x^3 - \frac{7}{3}x^2 - \frac{5}{2}x$
5. FOR EACH OF THE FOLLOWING, FIND ALL RATIONAL ROOTS OF THE POLYNOMIAL EQUATION:
   - **A** $2x^3 - 5x^2 + 1 = 0$
   - **B** $4x^4 + 4x^3 - 9x^2 - x + 2 = 0$
   - **C** $2x^5 - 3x^4 - 2x + 3 = 0$

1.4 GRAPHS OF POLYNOMIAL FUNCTIONS

IN YOUR PREVIOUS GRADES, YOU HAVE DISCUSSED HOW TO DRAW GRAPHS OF FUNCTI ZERO, ONE AND TWO. IN THE PRESENT SECTION, YOU WILL LEARN ABOUT GRAPHS O FUNCTIONS OF DEGREE GREATER THAN TWO.

TO UNDERSTAND PROPERTIES OF POLYNOMIAL FUNCTIONS, TRY THE FOLLOWING ACTIVITY.

ACTIVITY 1.12



1 SKETCH THE GRAPH OF EACH OF THE FOLLOWING POLYNOMIAL FUNCTIONS:

A $f(x) = 3$ **B** $f(x) = -2.5$

C $g(x) = x - 2$ **D** $g(x) = -3x + 1$

2 LET $f(x) = x^2 - 4x + 5$

A COPY AND COMPLETE THE TABLE OF VALUES GIVEN BELOW.

x	–2	–1	0	1	2	3	4
$f(x) = x^2 - 4x + 5$							

B PLOT THE POINTS WITH COORDINATES (x, y), WHERE $y = f(x)$ ON THE xy-COORDINATE PLANE.

C JOIN THE POINTS IN **B** ABOVE BY A SMOOTH CURVE TO GET THE GRAPH OF f. WHAT DO YOU CALL THE GRAPH OF f? GIVE THE DOMAIN AND RANGE OF f.

3 CONSTRUCT A TABLE OF VALUES FOR EACH OF THE FOLLOWING FUNCTIONS AND SKETCH THE GRAPH:

A $f(x) = x^2 - 3$ **B** $g(x) = -x^2 - 2x + 1$

C $h(x) = x^3$ **D** $p(x) = 1 - x^4$

WE SHALL DISCUSS SKETCHING THE GRAPHS OF HIGHER DEGREE POLYNOMIAL FUNC THE FOLLOWING EXAMPLES.

EXAMPLE 1 LET US CONSIDER THE FUNCTION $p(x) = x^3 - 3x - 4$.

THIS FUNCTION CAN BE WRITTEN AS $y = x^3 - 3x - 4$

COPY AND COMPLETE THE TABLE OF VALUES BELOW.

x	–3	–2	–1	0	1	2	3
y		–6	–2		–6		14

OTHER POINTS BETWEEN INTEGERS MAY HELP YOU TO DETERMINE THE SHAPE O BETTER.

FOR INSTANCE, FOR $x = \frac{1}{2}$

$$y = p\left(\frac{1}{2}\right) = -\frac{43}{8}$$

THEREFORE, THE POINT $\left(\frac{1}{2}, -\frac{43}{8}\right)$ IS ON THE GRAPH OF p. SIMILARLY, FOR

$$x = \frac{5}{2},\ y = p\left(\frac{5}{2}\right) = \frac{33}{8}.$$

SO, $\left(\frac{5}{2}, \frac{33}{8}\right)$ IS ALSO ON THE GRAPH OF p.

PLOT THE POINTS WITH COORDINATES (x, y) FROM THE TABLE AS SHOWN IN FIGURE 1.3A. NOW JOIN THESE POINTS BY A SMOOTH CURVE TO GET THE GRAPH OF $p(x)$, AS SHOWN IN FIGURE 1.3B.

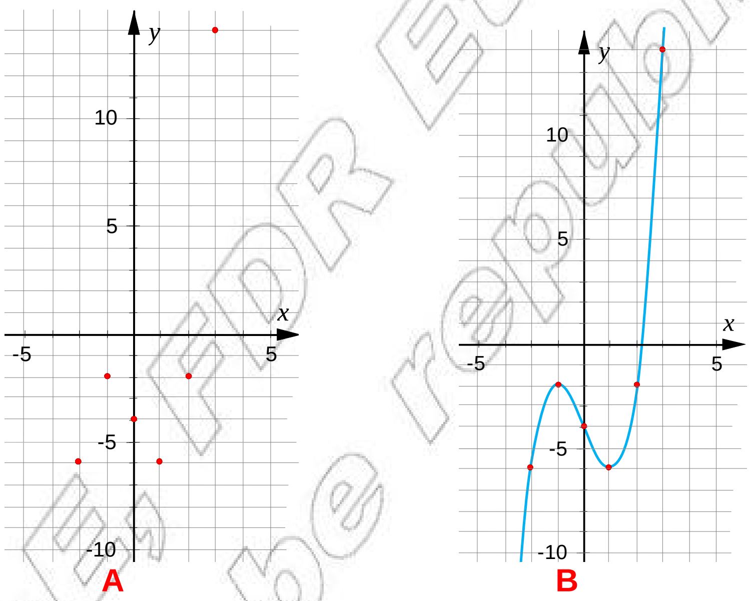


Figure 1.3 Graph of $p(x) = x^3 - 3x - 4$

EXAMPLE 2 SKETCH THE GRAPH OF $f(x) = -x^4 + 2x^2 + 1$

SOLUTION: TO SKETCH THE GRAPH OF f, WE FIND POINTS ON THE GRAPH USING A TABLE OF VALUES.

x	-2	-1	0	1	2
$y = -x^4 + 2x^2 + 1$	-7	2	1	2	-7

PLOT THE POINTS WITH COORDINATES (x, y) FROM THIS TABLE AND JOIN THEM BY A SMOOTH CURVE FOR INCREASING VALUES OF x, AS SHOWN IN FIGURE 1.4

FROM THE GRAPH, FIND THE DOMAIN AND THE RANGE OF f. OBSERVE THAT THE GRAPH OF f OPENS DOWNWARD.

AS OBSERVED FROM THE ABOVE TWO EXAMPLES, THE GRAPH OF A POLYNOMIAL FUNCTION HAS NO JUMPS, GAPS AND HOLES. IT HAS NO SHARP CORNERS. THE GRAPH OF A POLYNOMIAL FUNCTION IS A SMOOTH AND CONTINUOUS CURVE WHICH MEANS THERE IS NO BREAK ANYWHERE ON THE GRAPH.

THE GRAPH ALSO SHOWS THAT FOR EVERY VALUE OF x IN THE DOMAIN OF A POLYNOMIAL FUNCTION $p(x)$, THERE IS EXACTLY ONE VALUE y WHERE $y = p(x)$.

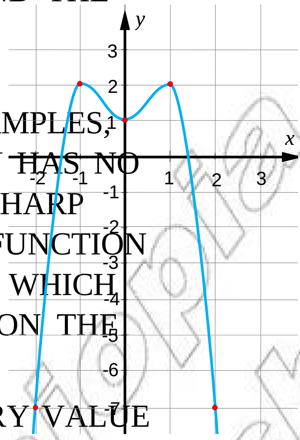


Figure 1.4 Graph of $f(x) = -x^4 + 2x^2 + 1$

THE FOLLOWING ARE NOT GRAPHS OF POLYNOMIAL FUNCTIONS.

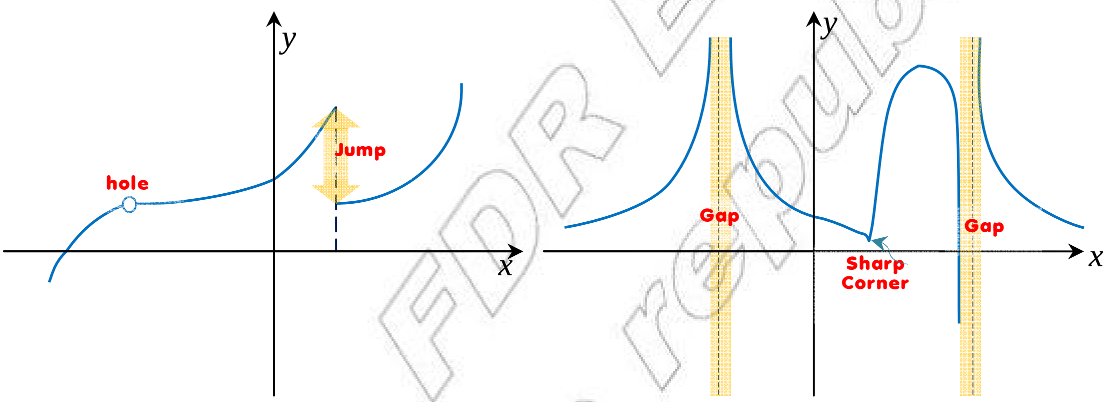


Figure 1.5

FUNCTIONS WITH GRAPHS THAT ARE NOT CONTINUOUS ARE NOT POLYNOMIAL FUNCTIONS. LOOK AT THE GRAPH OF THE FUNCTION $f(x) = |x|$ GIVEN IN FIGURE 1.6. IT HAS A SHARP CORNER AT THE POINT (0, 0) AND HENCE $f(x) = |x|$ IS NOT A POLYNOMIAL FUNCTION.

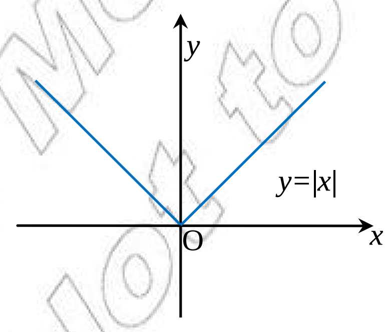


Figure 1.6

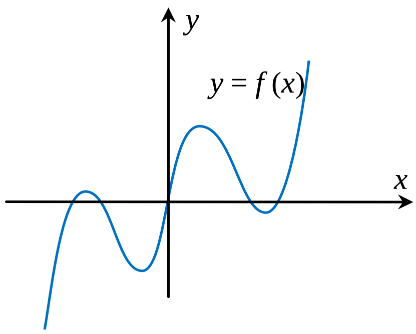


Figure 1.7

Is the function $f(x) = |x - 2|$ a polynomial function? Give reasons for your answer.

THE GRAPH OF THE FUNCTION f IN FIGURE 1.7 IS A SMOOTH CURVE. HENCE IT REPRESENTS A POLYNOMIAL FUNCTION. OBSERVE THAT THE RANGE OF f IS $\mathbb{R}$.

THE POINTS AT WHICH THE GRAPH OF A FUNCTION CROSSES (MEETS) THE COORDINA
IMPORTANT TO NOTE.

IF THE GRAPH OF A FUNCTION f CROSSES THE x-AXIS AT $(x_1, 0)$, THEN x_1 IS THE x-INTERCEPT OF THE GRAPH. IF THE GRAPH OF f CROSSES THE y-AXIS AT THE POINT $(0, y_1)$, THEN y_1 IS THE y-INTERCEPT OF THE GRAPH OF f.

How do we determine the x-intercept and the y-intercept?

SINCE $(x_1, 0)$ LIES ON THE GRAPH OF f, WE MUST HAVE $f(x_1) = 0$. SO x_1 IS A ZERO OF f.

SIMILARLY, $(0, y_1)$ LIES ON THE GRAPH OF f LEADS TO $f(0) = y_1$.

CONSIDER THE FUNCTION

$$f(x) = ax + b,\ a \neq 0$$

What is the x-intercept and the y-intercept?

$f(x_1) = ax_1 + b = 0$. SOLVING FOR x_1 GIVES $ax_1 = -b \Rightarrow x_1 = -\frac{b}{a}$

SO, $-\frac{b}{a}$ IS THE x-INTERCEPT OF THE GRAPH OF f.

AGAIN, $f(0) = a.0 + b = b$. THE NUMBER b IS THE y-INTERCEPT.

TRY TO FIND THE x-INTERCEPT AND THE y-INTERCEPT OF $f(x) = -3x + 5$.

THE ABOVE METHOD CAN ALSO BE APPLIED TO A QUADRATIC FUNCTION. CONSIDER T
EXAMPLE.

EXAMPLE 3 FIND THE x-INTERCEPTS AND THE y-INTERCEPT OF THE GRAPH OF

$$f(x) = x^2 - 4x + 3$$

SOLUTION: $f(x_1) = x_1^2 - 4x_1 + 3 = 0 \Rightarrow (x_1 - 1)(x_1 - 3) = 0 \quad \therefore x_1 = 1$ OR $x_1 =$

THEREFORE, THE GRAPH OF f HAS TWO x-INTERCEPTS, 1 AND 3.

NEXT, $f(0) = 0^2 - 4.0 + 3 = 3$. HERE, $y_1 = 3$ IS THE y-INTERCEPT.

THE GRAPH OF f CROSSES THE x-AXIS AT (1, 0) AND (3, 0). IT CROSSES THE y-AXIS AT (0, 3).

THE GRAPH OPENS UPWARD AND THE POINT $(2, -1)$ IS THE VERTEX OR TURNING POINT OF THE GRAPH OF f. THE MINIMUM VALUE OF f IS -1. THE RANGE OF f IS $\{y : y \geq -1\}$.

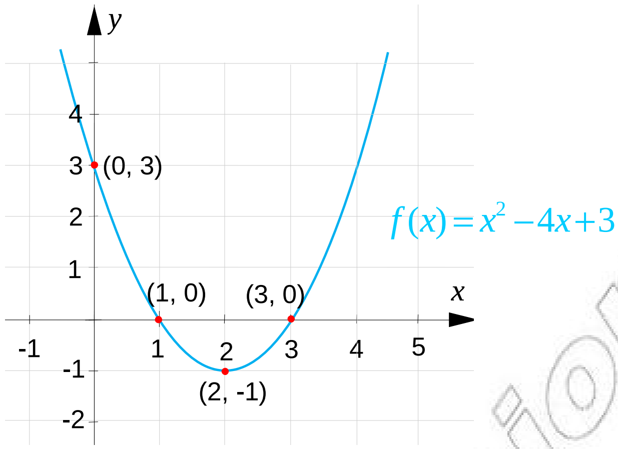


Figure 1.8

NOTE THAT THE GRAPH OF ANY QUADRATIC FUNCTION $f(x)=ax^2+bx+c$ HAS AT MOST TWO x-INTERCEPTS AND EXACTLY ONE y-INTERCEPT. TRY TO FIND THE REASON.

AS SEEN FROM FIGURE 1.8, $a = 1$ IS POSITIVE AND THE PARABOLA OPENS UPWARD.

What can be stated about the graph of $g(x) = -2x^2 + 4x$?

Does the graph open upward?

The coefficient of x^2 is negative. What is the range of g?

TO STUDY SOME PROPERTIES OF POLYNOMIALS, WE WILL LOOK AT GRAPHS OF SOME POLYNOMIAL FUNCTIONS OF HIGHER DEGREE OF THE FORM $f(x)=ax^n+b$, $n\geq 3$.

EXAMPLE 4 BY SKETCHING THE GRAPHS OF $g(x)=x^3+1$ AND $h(x)=-2x^3+1$, OBSERVE THEIR BEHAVIOURS AND GENERALIZE WHEN $|x|$ IS LARGE.

SOLUTION: PLOT THE POINTS OF THE GRAPHS OF g AND h.

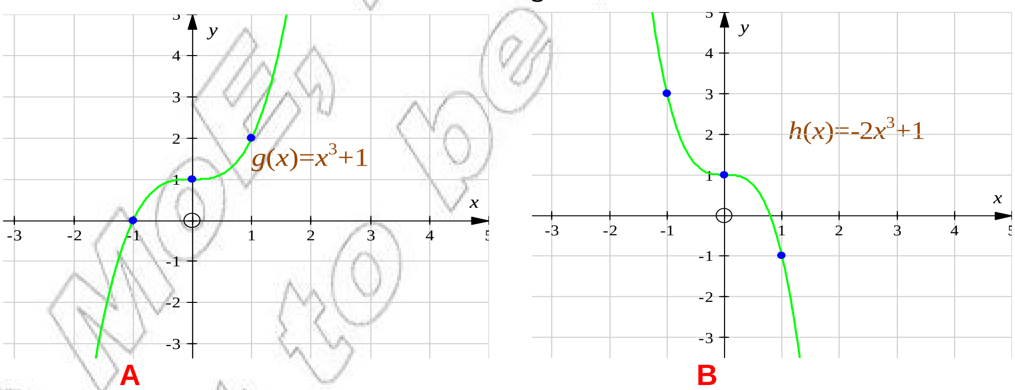


Figure 1.9

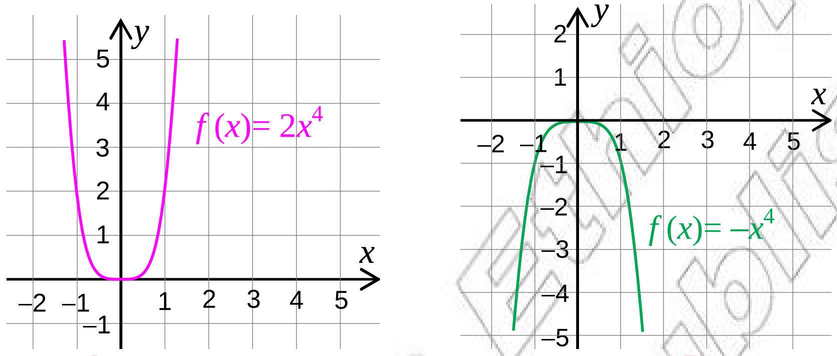
AS SHOWN IN FIGURE 1.9A, WHEN x BECOMES LARGE IN ABSOLUTE VALUE AND NEGATIVE, $g(x)$ IS NEGATIVE BUT LARGE IN ABSOLUTE VALUE (*The graph moves down*). WHEN x TAKES LARGE POSITIVE VALUES, $g(x)$ BECOMES LARGE POSITIVE.

In Figure 1.9B, the coefficient of the leading term is –2 which is negative. As a result, when x becomes large in absolute value and negative, $h(x)$ becomes large positive. When x takes large positive values, $h(x)$ becomes negative but large in absolute value.

The graph of $f(x) = a_n x^n + b$ shows the same behaviour when $|x|$ is large as the graph of g for $a_n > 0$ and as the graph of h for $a_n < 0$ and n odd.

Example 5 By sketching the graphs of $g(x) = 2x^4$ and $h(x) = -x^4$, observe their behaviour and generalize for even n when $|x|$ is large.

Solution: The sketches of the graphs of g and h are as follows.



A B

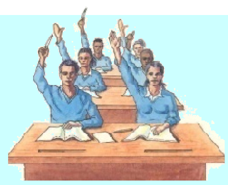
Figure 1.10

From Figure 1.10A, when $|x|$ takes large values, $g(x)$ becomes large positive.

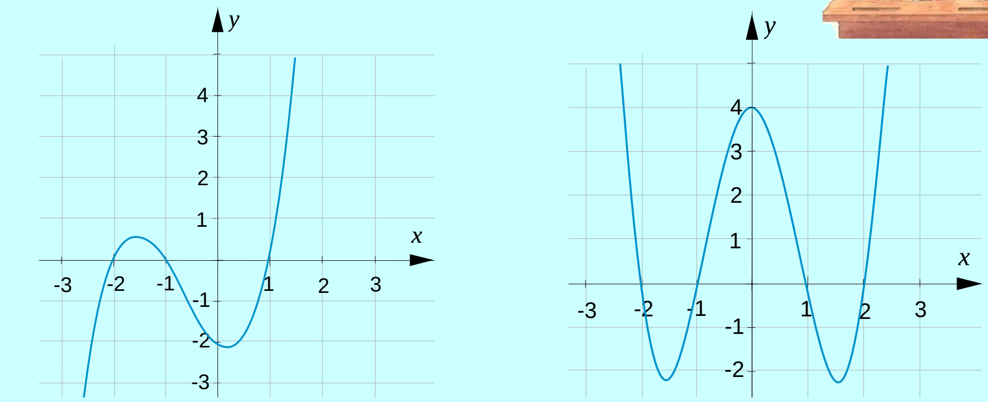
On the other hand, from Figure 1.10B, when $|x|$ takes large values, $h(x)$ becomes negative but large in absolute value and the graph opens downward.

When n is even, the graph of f opens upward for $a_n > 0$ and opens downward for $a_n < 0$. Draw and observe the graphs of $g(x) = 2(x-1)^4$ and $h(x) = -(x-1)^4$.

ACTIVITY 1.13



1 Consider the following graphs:



A Graph of $g(x) = x^3 + 2x^2 - x - 2$. **B** Graph of $f(x) = x^4 - 5x^2 + 4$.

Figure 1.11

A WHAT ARE THE DOMAINS OF f AND g?

B WHAT CAN BE SAID ABOUT THE VALUES OF $f(x)$ AND $g(x)$ WHEN $|x|$ IS LARGE AND POSITIVE, OR LARGE AND NEGATIVE?

C IF $x = 2^{10}$, WILL THE TERM x^3 IN $g(x)$ AND x^4 IN $f(x)$ BE POSITIVE OR WILL THEY BE NEGATIVE? WHAT HAPPENS WHEN $x = -2^{10}$?

2 **A** DO YOU THINK THAT THE RANGE OF EVERY POLYNOMIAL FUNCTION IS THE SET OF ALL REAL NUMBERS?

B WILL THE GRAPH OF EVERY POLYNOMIAL FUNCTION CROSS THE x-AXIS AT EXACTLY ONE POINT? WHY?

Group Work 1.3

1 ON THE GRAPH OF $g(x) = x^4 - 5x^2 + 4$

A WHAT ARE THE VALUES OF x AT THE POINTS WHERE THE GRAPH CROSSES THE x-AXIS? AT HOW MANY POINTS DOES THE GRAPH OF $g(x)$ CROSS THE x-AXIS?

B WHAT IS THE VALUE OF $g(x)$ AT EACH OF THESE POINTS OBTAINED IN A?

C WHAT IS THE TRUTH SET OF THE EQUATION $g(x) = 0$?

2 CONSIDER THE FUNCTION $h(x) = (x + 2)(x + 1)(x - 1)(x - 2)$

A ON THE GRAPH OF THE FUNCTION h, WHAT ARE THE COORDINATES OF THE POINTS WHERE THE GRAPH CROSSES THE x-AXIS AND THE y-AXIS?

B DO YOU THINK THAT $g(x)$ IN QUESTION 1 ABOVE AND $h(x)$ ARE THE SAME FUNCTION?

3 AS SHOWN IN FIGURE 1.11, THE GRAPH OF THE POLYNOMIAL FUNCTION DEFINED BY

$f(x) = x^4 - 5x^2 + 4$ CROSSES THE x-AXIS FOUR TIMES AND THE GRAPH OF

$g(x) = x^3 + 2x^2 - x - 2$ CROSSES THE x-AXIS THREE TIMES.

IN A SIMILAR WAY, HOW MANY TIMES DOES THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS INTERSECT THE x-AXIS?

A $p(x) = 2x + 1$ **B** $p(x) = x^2 + 4$

C $p(x) = x^2 - 8$ **D** $f(x) = (x - 2)(x - 1)(x^2 + 4)$.

4 DO YOU THINK THAT THE GRAPH OF EVERY POLYNOMIAL FUNCTION OF DEGREE FOUR CROSSES THE x-AXIS FOUR TIMES?

NOTE THAT THE GRAPH OF A POLYNOMIAL FUNCTION OF DEGREE n MEETS THE x-AXIS AT MOST n TIMES. SO (AS STATED PREVIOUSLY), EVERY POLYNOMIAL FUNCTION OF DEGREE n HAS AT MOST n ZEROS.

IN GENERAL, THE BEHAVIOUR OF THE GRAPH OF A POLYNOMIAL FUNCTION AS x DECREASES WITHOUT BOUND TO THE LEFT OR AS x INCREASES WITHOUT BOUND TO THE RIGHT CAN BE DETERMINED BY ITS DEGREE (EVEN OR ODD) AND BY ITS LEADING COEFFICIENT.

THE GRAPH OF THE POLYNOMIAL FUNCTION $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$ EVENTUALLY RISES OR FALLS. OBSERVE THE EXAMPLES GIVEN BELOW.

EXAMPLE 6 DESCRIBE THE BEHAVIOUR OF THE GRAPH OF $f(x) = -x^3 + x$, AS x DECREASES TO THE LEFT AND INCREASES TO THE RIGHT.

SOLUTION: BECAUSE THE DEGREE OF f IS ODD AND THE LEADING COEFFICIENT IS NEGATIVE, THE GRAPH RISES TO THE LEFT AND FALLS TO THE RIGHT AS SHOWN IN FIGURE 1.12.

A AND **B** ARE THE TURNING POINTS OF THE GRAPH OF f.

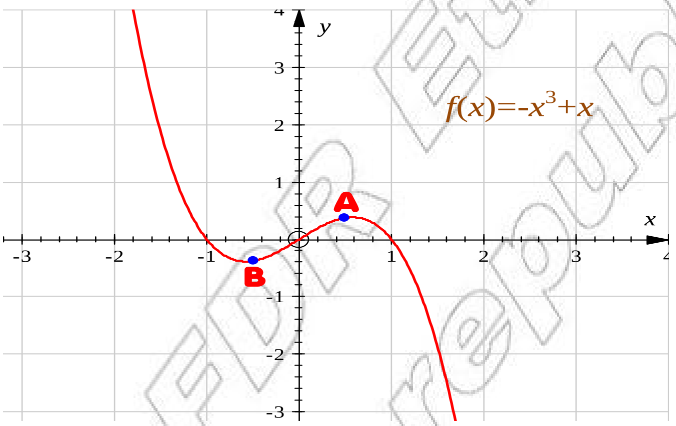


Figure 1.12

FIGURE 1.13 SHOWS AN EXAMPLE OF A POLYNOMIAL FUNCTION WHOSE GRAPH HAS ***peaks*** AND ***valleys***. THE TERM PEAK REFERS TO A ***local maximum*** AND THE TERM VALLEY REFERS TO A ***local minimum.*** SUCH POINTS ARE OFTEN CALLED ***turning points*** OF THE GRAPH.

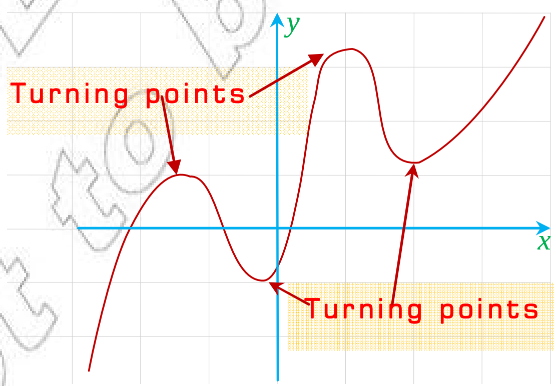


Figure1.13

A POINT OF f THAT IS EITHER A MAXIMUM POINT OR MINIMUM POINT ON ITS DOMAIN IS
local extremum point OF f.

NOTE THAT THE GRAPH OF A POLYNOMIAL FUNCTION OF DEGREE n HAS AT MOST $n – 1$ TURNING POINTS.

EXAMPLE 7 CONSIDER THE POLYNOMIAL

$$f(x) = x(x-2)^2(x+2)^4.$$

THE FUNCTION f HAS A SIMPLE ZERO AT 0, A ZERO OF MULTIPLICITY 2 AT 2 AND A ZERO OF MULTIPLICITY 4 AT –2, AS SHOWN IN FIGURE 1.14, IT HAS A LOCAL MAXIMUM AT $x = –2$ AND DOES NOT CHANGE SIGN AT $x = –2$. ALSO, f HAS A RELATIVE (LOCAL) MINIMUM AT $x = 2$ AND DOES NOT CHANGE SIGN HERE. BOTH $x = –2$ AND $x = 2$ ARE ZEROS OF EVEN MULTIPLICITY.

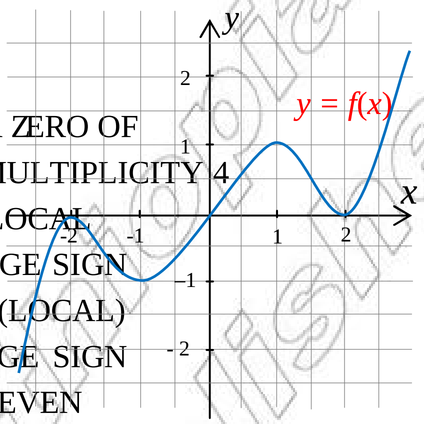


Figure 1.14

ON THE OTHER HAND, $x = 0$ IS A ZERO OF ODD MULTIPLICITY, $f(x)$ CHANGES SIGN AT $x = 0$, AND DOES NOT HAVE A TURNING POINT AT $x = 0$.

EXAMPLE 8 TAKE THE POLYNOMIAL $f(x) = 3x^4 + 4x^3$. IT CAN BE EXPRESSED AS

$$f(x) = x^3(3x+4).$$

THE DEGREE OF f IS EVEN AND THE LEADING COEFFICIENT IS POSITIVE. HENCE, THE
RISES UP AS $|x|$ BECOMES LARGE.

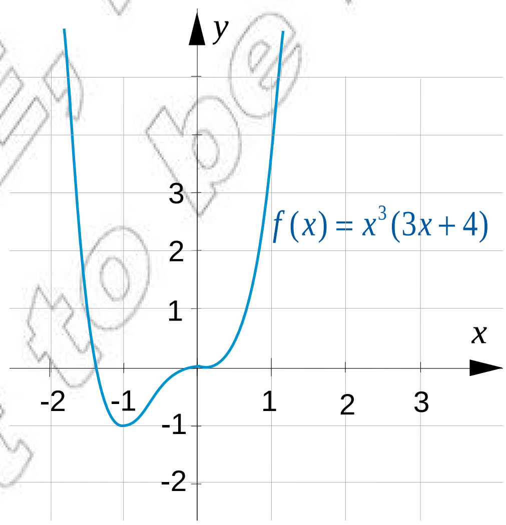


Figure 1.15

THE FUNCTION HAS A SIMPLE ZERO AT $x = -\frac{4}{3}$ AND CHANGES SIGN AT POINT $\left(-\frac{4}{3}, 0\right)$.

THE GRAPH OF f HAS A LOCAL MINIMUM AT POINT (–1, –1).

ALSO f HAS A ZERO AT $x = 0$ AND CHANGES SIGN HERE. SO, 0 IS OF ODD MULTIPLICITY.

THERE IS NO LOCAL MINIMUM OR MAXIMUM AT (0, 0).

The above observations can be generalized as follows:

1 IF c IS A ZERO OF ODD MULTIPLICITY OF A FUNCTION f, THEN THE GRAPH OF THE FUNCTION CROSSES THE x-AXIS AT $x = c$ AND DOES NOT HAVE A RELATIVE EXTREMUM AT $x = c$.

2 IF c IS A ZERO OF EVEN MULTIPLICITY, THEN THE GRAPH OF THE FUNCTION TOUCHES BUT DOES NOT CROSS THE x-AXIS AT $x = c$ AND HAS A LOCAL EXTREMUM AT $x = c$.

Group Work 1.4

1 GIVE SOME EXAMPLES OF POLYNOMIAL FUNCTIONS ... THE BEHAVIOUR OF THEIR GRAPHS AS x INCREASES WITHOUT BOUND TO THE LEFT (x IS NEGATIVE BUT LARGE IN ABSOLUTE VALUE) OR AS x INCREASES WITHOUT BOUND TO THE RIGHT (x BECOMES LARGE POSITIVE).

DID YOU NOTE THAT FOR $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$ IF $a_n > 0$ AND n IS ODD, $p(x)$ BECOMES LARGE POSITIVE AS x TAKES LARGE POSITIVE VALUES AND $p(x)$ BECOMES NEGATIVE BUT LARGE IN ABSOLUTE VALUE AS THE ABSOLUTE VALUE OF x BECOMES LARGE FOR x NEGATIVE?

DISCUSS THE CASES WHERE:

I $a_n > 0$ AND n IS EVEN	II $a_n < 0$ AND n IS EVEN
III $a_n < 0$ AND n IS ODD	IV $a_n > 0$ AND n IS ODD

2 ANSWER THE FOLLOWING QUESTIONS:

A WHAT IS THE LEAST NUMBER OF TURNING POINTS AN ODD DEGREE POLYNOMIAL FUNCTION CAN HAVE? WHAT ABOUT AN EVEN DEGREE POLYNOMIAL FUNCTION?

B WHAT IS THE MAXIMUM NUMBER OF x-INTERCEPTS THE GRAPH OF A POLYNOMIAL FUNCTION OF DEGREE N CAN HAVE?

C WHAT IS THE MAXIMUM NUMBER OF REAL ZEROS A POLYNOMIAL FUNCTION OF DEGREE N CAN HAVE?

D WHAT IS THE LEAST NUMBER OF x-INTERCEPTS THE GRAPH OF A POLYNOMIAL FUNCTION OF ODD DEGREE/EVEN DEGREE CAN HAVE?

Exercise 1.10

1 MAKE A TABLE OF VALUES AND DRAW THE GRAPH OF EACH OF THE FOLLOWING POLYNOMIAL FUNCTIONS:

A $f(x) = 4x^2 - 11x + 3$ **B** $f(x) = -1 - x^2$

C $f(x) = 8 - x^3$ **D** $f(x) = x^3 + x^2 - 6x - 10$

E $f(x) = 2x^2 - 2x^4$ **F** $f(x) = \frac{1}{4}(x-2)^2(x+2)^2$.

2 WITHOUT DRAWING THE GRAPHS OF THE FOLLOWING POLYNOMIAL FUNCTIONS, STATE FOR EACH, AS MUCH AS YOU CAN, ABOUT:

I THE BEHAVIOUR OF THE GRAPH AS x TAKES VALUES FAR TO THE RIGHT AND FAR TO THE LEFT.

II THE NUMBER OF INTERSECTIONS OF THE GRAPH WITH THE x-AXIS.

III THE DEGREE OF THE FUNCTION AND WHETHER THE DEGREE IS EVEN OR ODD.

IV THE LEADING COEFFICIENT AND WHETHER $a_n > 0$ OR $a_n < 0$.

A $f(x) = (x-1)(x-1)$ **B** $f(x) = x^2 + 3x + 2$

C $f(x) = 16 - 2x^3$ **D** $f(x) = x^3 - 2x^2 - x + 1$

E $f(x) = 5x - x^3 - 2$ **F** $f(x) = (x-2)(x-2)(x-3)$

G $f(x) = 2x^5 + 2x^2 - 5x + 1$

3 FOR THE GRAPHS OF EACH OF THE FUNCTIONS GIVEN IN QUESTION 1 (A-F) ABOVE:

I DISCUSS THE BEHAVIOUR OF THE GRAPH AS x TAKES VALUES FAR TO THE RIGHT AND FAR TO THE LEFT.

II GIVE THE NUMBER OF TIMES THE GRAPH INTERSECTS THE x-AXIS.

III FIND THE VALUE OF THE FUNCTION WHERE THE GRAPH CROSSES THE y-AXIS.

IV GIVE THE NUMBER OF TURNING POINTS.

4 IN EACH OF THE FOLLOWING, DECIDE WHETHER THE GRAPH COULD POSSIBLY BE THE GRAPH OF A POLYNOMIAL FUNCTION:

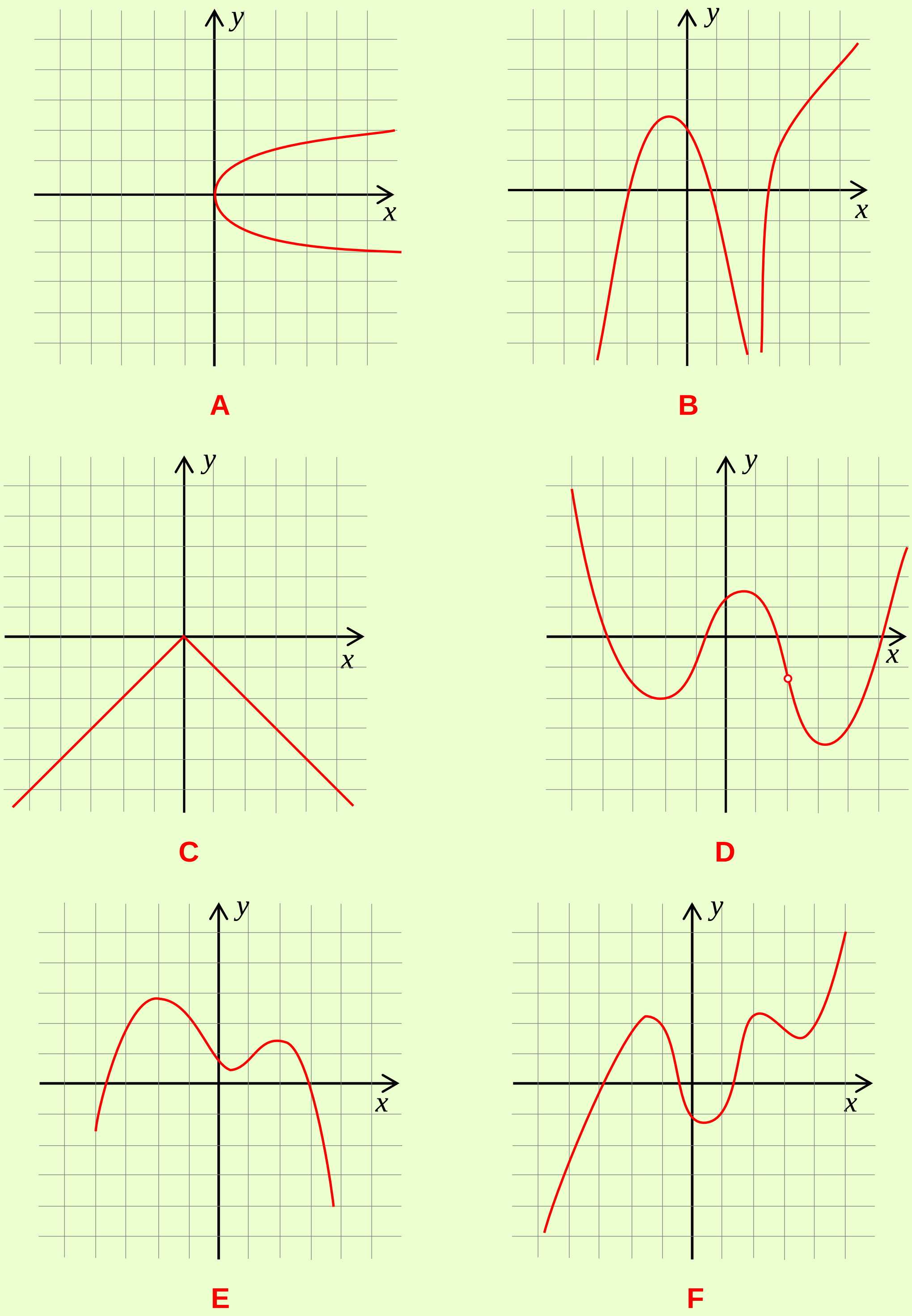


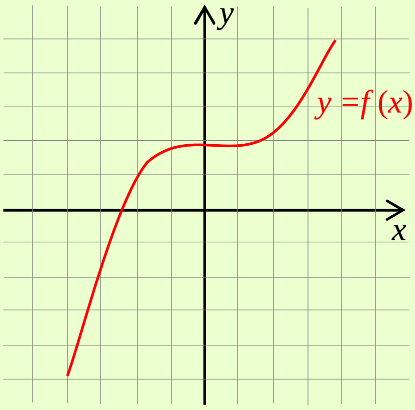
Figure 1.16

5 GRAPHS OF SOME POLYNOMIAL FUNCTIONS ARE GIVEN BELOW. IN EACH CASE:

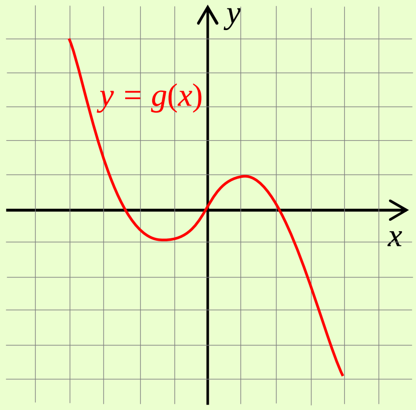
I IDENTIFY THE SIGN OF THE LEADING COEFFICIENT.

ii IDENTIFY THE POSSIBLE DEGREE OF EACH FUNCTION, AND STATE WHETHER THE DEGREE IS EVEN OR ODD.

iii DETERMINE THE NUMBER OF TURNING POINTS.



A



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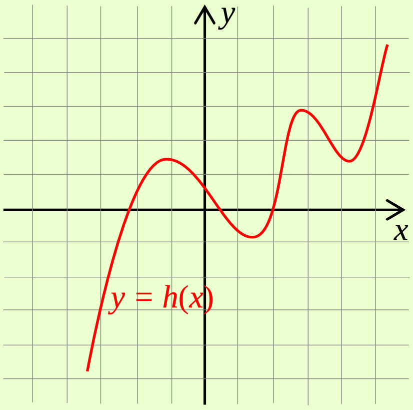

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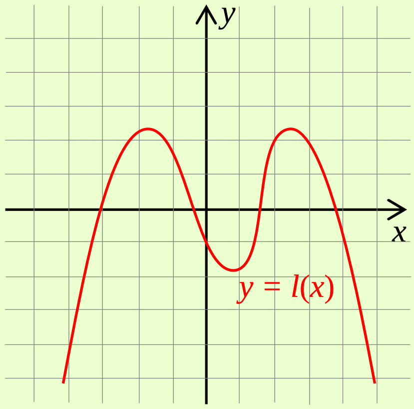

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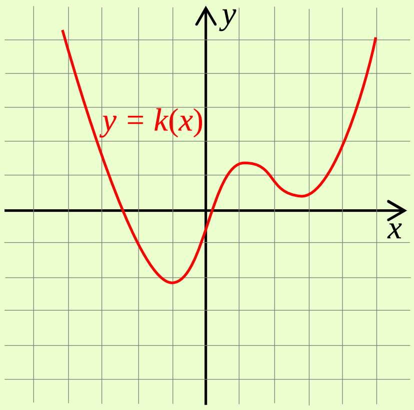
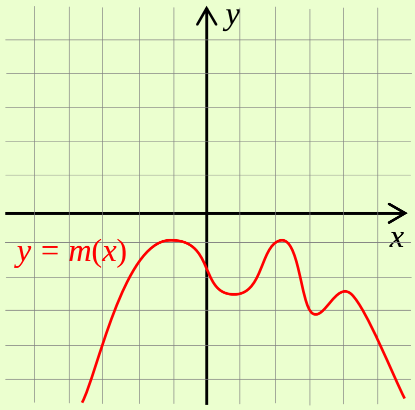


E



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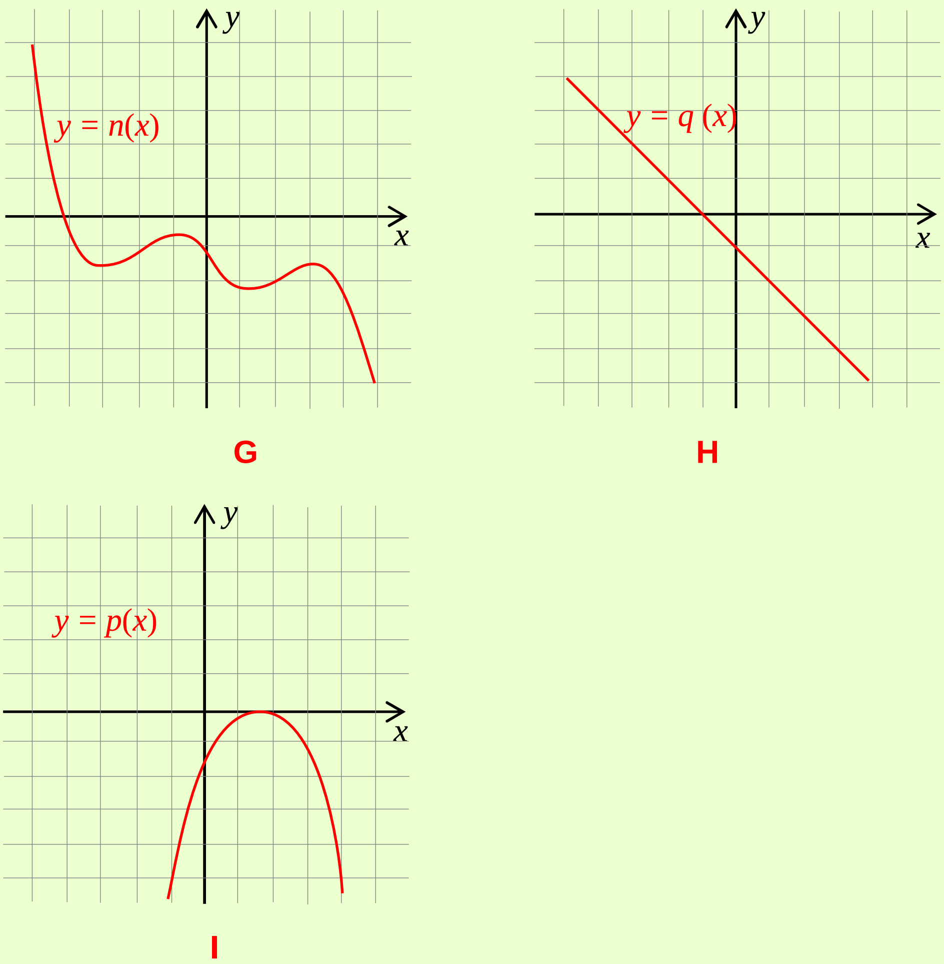


Figure 1.17

6 DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE. JUSTIFY YOUR ANSWER:

A A POLYNOMIAL FUNCTION OF DEGREE 6 CAN HAVE 5 TURNING POINTS.

B IT IS POSSIBLE FOR A POLYNOMIAL FUNCTION OF DEGREE 6 TO INTERSECT THE x-AXIS AT ONE POINT.

Key Terms

constant function	***linear function***	***rational root***
constant term	***local extremum***	***remainder theorem***
degree	***location theorem***	***turning points***
domain	***multiplicity***	***x-intercept***
factor theorem	***polynomial division theorem***	***y-intercept***
leading coefficient	***polynomial function***	***zero(s) of a polynomial***
leading term	***quadratic function***	

Summary

1. A ***linear function*** IS GIVEN BY $f(x) = ax + b$; $a \neq 0$.
2. A ***quadratic function*** IS GIVEN BY $f(x) = ax^2 + bx + c$; $a \neq 0$
3. LET n BE A NON-NEGATIVE INTEGER AND LET $a_n, a_{n-1}, \ldots, a_1, a_0$ BE REAL NUMBERS WITH $a_n \neq 0$. THE FUNCTION $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ IS CALLED ***a polynomial function in x of degree n***.
4. A POLYNOMIAL FUNCTION IS OVER INTEGERS IF ITS COEFFICIENTS ARE ALL INTEGERS.
5. A POLYNOMIAL FUNCTION IS OVER RATIONAL NUMBERS IF ITS COEFFICIENTS ARE ALL RATIONAL NUMBERS.
6. A POLYNOMIAL FUNCTION IS OVER REAL NUMBERS IF ITS COEFFICIENTS ARE ALL REAL NUMBERS.
7. OPERATIONS ON POLYNOMIAL FUNCTIONS:
   - **I** ***Sum:*** $(f + g)(x) = f(x) + g(x)$
   - **II** ***Difference:*** $(f - g)(x) = f(x) - g(x)$
   - **III** ***Product:*** $(f \cdot g)(x) = f(x) \cdot g(x)$
   - **IV** ***Quotient:*** $(f \div g)(x) = f(x) \div g(x)$, IF $g(x) \neq 0$
8. IF $f(x)$ AND $d(x)$ ARE POLYNOMIALS SUCH THAT $d(x) \neq 0$, AND THE DEGREE OF $d(x)$ IS LESS THAN OR EQUAL TO THE DEGREE OF $f(x)$, THEN THERE EXIST UNIQUE POLYNOMIALS $q(x)$ AND $r(x)$ SUCH THAT $f(x) = d(x)\, q(x) + r(x)$, WHERE $r(x) = 0$ OR THE DEGREE OF $r(x)$ IS LESS THAN THE DEGREE OF $d(x)$.
9. IF A POLYNOMIAL $f(x)$ IS DIVIDED BY A FIRST DEGREE POLYNOMIAL OF THE FORM $x - c$, THEN THE ***remainder*** IS THE NUMBER $f(c)$.

10 GIVEN THE POLYNOMIAL FUNCTION

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

IF $p(c) = 0$, THEN c IS A ***zero of the polynomial*** AND A ***root*** OF THE EQUATION $p(x) = 0$. FURTHERMORE, $x - c$ IS A ***factor*** OF THE POLYNOMIAL.

11 FOR EVERY POLYNOMIAL FUNCTION f AND REAL NUMBER c, IF $f(c) = 0$, THEN $x = c$ IS A ZERO OF THE POLYNOMIAL FUNCTION f.

12 IF $(x - c)^k$ IS A FACTOR OF $f(x)$, BUT $(x - c)^{k+1}$ IS NOT, WE SAY THAT c IS A ZERO OF ***multiplicity k of f***.

13 IF THE RATIONAL NUMBER $\frac{p}{q}$, IN ITS LOWEST TERM, IS A ZERO OF THE POLYNOMIAL $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ WITH INTEGER COEFFICIENTS, THEN p MUST BE AN INTEGER FACTOR OF a_0 AND q MUST BE AN INTEGER FACTOR OF a_n.

14 LET a AND b BE REAL NUMBERS SUCH THAT $a < b$. IF $f(x)$ IS A POLYNOMIAL FUNCTION SUCH THAT $f(a)$ AND $f(b)$ HAVE OPPOSITE SIGNS, THEN THERE IS AT LEAST ONE ZERO OF $f(x)$ BETWEEN a AND b.

15 THE GRAPH OF A POLYNOMIAL FUNCTION OF DEGREE n HAS AT MOST $n - 1$ ***turning points*** AND INTERSECTS THE x-AXIS AT MOST n TIMES.

16 THE GRAPH OF EVERY POLYNOMIAL FUNCTION HAS NO ***sharp corners***; IT IS A SMOOTH AND CONTINUOUS CURVE.

Review Exercises on Unit 1

1 IN EACH OF THE FOLLOWING, FIND THE QUOTIENT AND THE REMAINDER WHEN THE FIRST POLYNOMIAL IS DIVIDED BY THE SECOND:

A $x^3 + 7x^2 - 6x - 5;\ x + 1$ **B** $3x^3 - 2x^2 - 4x + 4;\ x + 1$

C $3x^4 + 16x^3 + 6x^2 - 2x - 13;\ x + 5$ **D** $2x^3 + 3x^2 - 6x + 1;\ x - 1$

E $2x^5 + 5x^4 - 4x^3 + 8x^2 + 1;\ 2x^2 - x + 1$ **F** $6x^3 - 4x^2 + 3x - 2;\ 2x^2 + 1$

2 PROVE THAT WHEN A POLYNOMIAL $p(x)$ IS DIVIDED BY A FIRST DEGREE POLYNOMIAL $ax + b$, THE REMAINDER IS $p\left(-\frac{b}{a}\right)$.

3 PROVE THAT $x + 1$ IS A FACTOR OF $x^n + 1$ WHERE N IS AN ODD POSITIVE INTEGER.

4 SHOW THAT $\sqrt{2}$ IS AN IRRATIONAL NUMBER.

Hint: $\sqrt{2}$ IS A ROOT OF $x^2 - 2$. DOES THIS POLYNOMIAL HAVE ANY RATIONAL ROOTS?

5 FIND ALL THE RATIONAL ZEROS OF:

A $f(x) = x^5 + 8x^4 + 20x^3 + 9x^2 - 27x - 27$

B $f(x) = (x - 1)(x(x + 1) + 2x)$

6 FIND THE VALUE OF k SUCH THAT:

- **A** $2x^3 - 3x^2 - kx - 17$ DIVIDED BY $x - 3$ HAS A REMAINDER OF –2.
- **B** $x - 1$ IS A FACTOR OF $x^3 - 6x^2 + 2kx - 3$.
- **C** $5x - 2$ IS A FACTOR OF $x^3 - 3x^2 + kx + 15$.

7 SKETCH THE GRAPH OF EACH OF THE FOLLOWING:

- **A** $f(x) = x^3 - 7x + 6;\ -4 \le x \le 3$
- **B** $f(x) = x^4 - x^3 - 4x^2 + x + 1;\ -2 \le x \le 3$
- **C** $f(x) = x^3 - 3x^2 + 4$
- **D** $f(x) = \frac{1}{4}(1 - x)(1 + x^2)(x - 2)$

8 SKETCH THE GRAPH OF THE FUNCTION $f(x) = x^4$. EXPLAIN FOR EACH OF THE FOLLOWING CASES HOW THE GRAPH OF g IS DIFFER FROM THE GRAPH OF f. DETERMINE WHETHER G IS ODD, EVEN OR NEITHER.

- **A** $g(x) = f(x) + 3$
- **B** $g(x) = f(-x)$
- **C** $g(x) = -f(x)$
- **D** $g(x) = f(x + 3)$

9 THE POLYNOMIAL $f(x) = A(x-1)^2 + B(x+2)^2$ IS DIVIDED BY $x + 1$ AND $x - 2$. THE REMAINDERS ARE 3 AND –15 RESPECTIVELY. FIND THE VALUES OF A AND B.

10 IF $x^2 + (c-2)x - c^2 - 3c + 5$ IS DIVIDED BY $x + c$, THE REMAINDER IS –1. FIND THE VALUE OF c.

11 IF $x - 2$ IS A COMMON FACTOR OF THE EXPRESSIONS $x^2 + (m+n)x - n$ AND $2x^2 + m(x-1) + n(x+2)$, FIND THE VALUES OF m AND n.

12 FACTORIZE FULLY:

- **A** $x^3 - 4x^2 - 7x + 10$
- **B** $2x^5 + 6x^4 + 7x^3 + 21x^2 + 5x + 15$.

13 A PSYCHOLOGIST FINDS THAT THE RESPONSE TIME TO A CERTAIN STIMULUS VARIES WITH AGE GROUP ACCORDING TO

$$R = y^4 + 2y^3 - 4y^2 - 5y + 14,$$

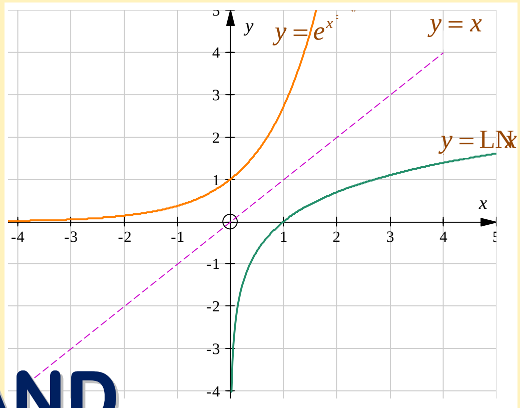
WHERE R IS RESPONSE IN MICROSECONDS AND y IS AGE GROUP IN YEARS. FOR WHAT AGE GROUP IS THE RESPONSE EQUAL TO 8 MICROSECONDS?

14 THE PROFIT OF A FOOTBALL CLUB AFTER A TAKEOVER IS MODELLED BY

$$p(t) = t^3 - 14t^2 + 20t + 120,$$

WHERE t IS THE NUMBER OF YEARS AFTER THE TAKEOVER. IN WHICH YEARS WAS MAKING A LOSS?

Unit 2



EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *understand the laws of exponents for real exponents.*
- *know specific facts about logarithms.*
- *know basic concepts about exponential and logarithmic functions.*
- *solve mathematical problems involving exponents and logarithms.*

Main Contents

INTRODUCTION

EXPONENTIAL AND LOGARITHMIC FUNCTIONS COME INTO PLAY WHEN A VARIABLE A
EXPONENT, FOR EXAMPLE, IN AN EXPRESSION SUCH AS 2^x. SUCH EXPRESSIONS ARISE IN MANY
APPLICATIONS AND ARE POWERFUL MATHEMATICAL TOOLS FOR SOLVING REAL LIFE P
ANALYZING GROWTH OF POPULATIONS OF PEOPLE, ANIMALS, AND BACTERIA; DECAY O
SUBSTANCES; GROWTH OF MONEY AT COMPOUND INTEREST; ABSORPTION OF LIGHT
THROUGH AIR, WATER OR GLASS, ETC.

IN THIS UNIT, YOU WILL STUDY THE VARIOUS PROPERTIES OF EXPONENTIAL AND
FUNCTIONS AND LEARN HOW THEY CAN BE USED IN SOLVING REAL LIFE PROBLEMS.

2.1 EXPONENTS AND LOGARITHMS

2.1.1 Exponents

WHILE SOLVING MATHEMATICAL PROBLEMS, THERE ARE OCCASIONS, YOU NEED TO
NUMBER BY ITSELF. FOR EXAMPLE,

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64.$$

MATHEMATICIANS USE THE IDEA OF ***exponents*** TO REPRESENT A PRODUCT INVOLVING THE SAME
FACTOR. FOR EXAMPLE,

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6.$$

EXPONENTS ARE FREQUENTLY USED IN MANY AREAS OF PHYSICS, ENGINEERING, FINAN
ETC., TO REPRESENT SITUATIONS WHERE QUANTITIES INCREASE OR DECREASE OVER TIM

OPENING PROBLEM

ETHIOPIA HAS A POPULATION OF AROUND 80 MILLION PEOPLE AND IT IS ESTIMATED THA
POPULATION GROWS EVERY YEAR AT AN AVERAGE GROWTH RATE OF 2.3%. IF THE POPULA
CONTINUES AT THE SAME RATE,

A WHAT WILL BE THE POPULATION AFTER

I 10 YEARS? **II** 20 YEARS?

B HOW MANY YEARS WILL IT TAKE FOR THE POPULATION TO DOUBLE?

C WHAT WILL THE GRAPH OF THE NUMBER OF PEOPLE PLOTTED AGAINST TIME L

IT IS HOPED THAT AFTER STUDYING THE CONCEPTS DISCUSSED IN THIS CHAPTER, YOU W
SOLVE PROBLEMS LIKE THE ONE GIVEN ABOVE.

Exponent notation

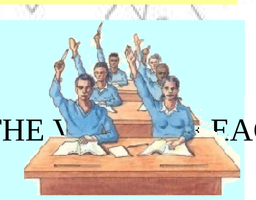
THE PRODUCT $2 \times 2 \times 2 \times 2 \times 2 \times 2$ IS WRITTEN AS 2^6 (READ *"two to the power of six"*).
SIMILARLY, $3^4 = 3 \times 3 \times 3 \times 3$ AND $4^5 = 4 \times 4 \times 4 \times 4 \times 4$.

IF n IS A POSITIVE INTEGER, THEN a^n IS THE PRODUCT OF n FACTORS OF a.

I.E. $a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ FACTORS}}$

IN a^n, a IS CALLED THE **base**, n IS CALLED THE **exponent** AND a^n IS THE n^{TH} **power** OF a.

ACTIVITY 2.1



1 IDENTIFY THE BASE AND THE EXPONENT AND FIND THE VALUE OF EACH OF THE FOLLOWING POWERS:

A 4^3 **B** $(-2)^8$ **C** $\left(\frac{2}{7}\right)^4$ **D** $-(-1)^{23}$ **E** $(5T)^4$

2 FIND THE VALUES OF THE FOLLOWING POWERS:

A $(-1)^1$ **B** $(-1)^2$ **C** $(-1)^3$ **D** $(-1)^4$ **E** $(-1)^5$
F $(-1)^6$ **G** $(-2)^1$ **H** $(-2)^2$ **I** $(-2)^3$ **J** $(-2)^4$
K $(-2)^5$ **L** $(-2)^6$

3 WHICH ONES GIVE YOU A NEGATIVE VALUE: A NEGATIVE BASE RAISED TO AN ODD EXPONENT OR A NEGATIVE BASE RAISED TO AN EVEN EXPONENT?

EXAMPLE 1 EVALUATE:

A $(-3)^4$ **B** -3^4 **C** $(-3)^5$ **D** $-(-3)^5$

SOLUTION:

A $(-3)^4 = -3 \times -3 \times -3 \times -3 = 81$

B $-3^4 = -1 \times 3^4 = -1 \times 3 \times 3 \times 3 \times 3 = -81$

C $(-3)^5 = -3 \times -3 \times -3 \times -3 \times -3 = -243$

D $-(-3)^5 = -1 \times (-3)^5 = -1 \times -243 = 243$

REMEMBER THAT, IN $(-3)^4$ THE BASE IS -3 BUT IN -3^4 ONLY 3 IS THE BASE.

WHAT IS THE BASE IN $(-4t)^3$? THE BASE IS $-4t$ AND $(-4t)^3 = (-4t) \times (-4t) \times (-4t) = -64t^3$

TO WHAT BASE DOES THE EXPONENT 3 REFER IN $-4t^3$? $-4t^3 = -4 \times t \times t \times t$. THEREFORE THE EXPONENT 3 IN $-4t^3$ REFERS TO THE BASE t ONLY.

Laws of exponents

THE FOLLOWING GROUP WORK WILL HELP YOU RECALL THE LAWS OF EXPONENTS DISCUSSI GRADE 9:

Group Work 2.1

1 SIMPLIFY EACH OF THE FOLLOWING:

A $2^3 \times 2^5$ **B** $4^3 \times 4^8$ **C** $\frac{2^7}{2^3}$

D $\frac{2^{-5}}{2^{-9}}$ **E** $(2 \times 3)^3$ **F** $5^{-2} \times 3^{-2}$

G $(3^2)^5$ **H** $\left(\frac{2}{3}\right)^3$ **I** $a^c \times a^d$

2 WHICH LAW OF EXPONENTS DID YOU APPLY TO SIMPLIFY EACH OF THE ABOVE EXPRESSIONS? (DISCUSS WITH YOUR FRIENDS).

IF THE BASES a AND b ARE NON-ZERO REAL NUMBERS AND THE EXPONENTS m AND n ARE INTEGERS, THEN,

1	$a^m \times a^n = a^{m+n}$	TO MULTIPLY POWERS OF THE SAME BASE, KEEP THE BASE AND ADD THE EXPONENTS.
2	$\frac{a^m}{a^n} = a^{m-n}$	TO DIVIDE POWERS OF THE SAME BASE, KEEP THE BASE AND SUBTRACT THE EXPONENTS.
3	$(a^m)^n = a^{m \times n} = a^{mn}$	TO TAKE A POWER OF A POWER, KEEP THE BASE AND MULTIPLY THE EXPONENTS.
4	$(a \times b)^n = a^n \times b^n$	THE POWER OF A PRODUCT IS THE PRODUCT OF THE POWERS.
5	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	THE POWER OF A QUOTIENT IS THE QUOTIENT OF THE POWERS

EXAMPLE 2 SIMPLIFY EACH OF THE FOLLOWING:

A $(4t)^2 \times (4t)^7$ **B** $r^8 \times r^{-3}$ **C** $\frac{10^3}{10^5}$

D $(x^2)^m$ **E** 16×4^{3t} **F** $\left(\frac{2y}{25}\right)^2$

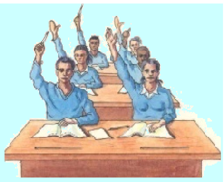
SOLUTION:

A $(4t)^2 \times (4t)^7 = (4t)^{2+7} = (4t)^9$ **B** $r^8 \times r^{-3} = r^{8+(-3)} = r^5$

C $\frac{10^3}{10^5} = 10^{3-5} = 10^{-2}$ **D** $(x^2)^m = x^{2 \times m} = x^{2m}$

E $16 \times 4^{3t} = 2^4 \times (2^2)^{3t} = 2^4 \times 2^{6t} = 2^{4+6t}$ **F** $\left(\frac{2y}{25}\right)^2 = \frac{2^2 \times y^2}{25^2} = \frac{4y^2}{625}$

ACTIVITY 2.2



1 EVALUATE EACH OF THE FOLLOWING USING THE LAW $\frac{a^m}{a^n} = a^{m-n}$

A $\frac{2^3}{2^3}$; IS 2^0 EQUAL TO 1? WHY?

B $\frac{10^5}{10^5}$; IS 10^0 EQUAL TO 1? WHY?

C $\frac{(-8)^3}{(-8)^3}$; IS $(-8)^0$ EQUAL TO 1? WHY?

2 FROM YOUR ANSWERS, CAN YOU SUGGEST WHAT ANY NON-ZERO NUMBER RAISED TO ZERO IS?

ANY NON-ZERO NUMBER RAISED TO ZERO IS ONE.
THAT IS, $a^0 = 1$, IF $a \neq 0$

EXAMPLE 3

A $8^0 = 1$

B $(-100)^0 = 1$

C $\left(\frac{3}{5}\right)^0 = 1$

D $\left(\sqrt{23}\right)^0 = 1$

E $(0.153)^0 = 1$

Group Work 2.2

OBSERVE THE FOLLOWING:

- $\frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2 \times 2} = \frac{1}{2^3}$
- IF WE USE THE RULE $\frac{a^m}{a^n} = a^{m-n}$, $\frac{2^2}{2^5} = 2^{2-5} = 2^{-3}$

A USING THE ABOVE TWO STEPS TRY TO SIMPLIFY $\frac{3^5}{3^7}$

B DISCUSS THE RELATIONSHIP BETWEEN:

I $\frac{1}{2^3}$ AND 2^{-3} **II** $\frac{1}{3^2}$ AND 3^{-2}

C WHAT CAN YOU CONCLUDE ABOUT a^{-n} AND $\frac{1}{a^n}$?

FOR $a \neq 0$ AND $n > 0$

$$a^{-n} = \frac{1}{a^n}$$

ANY NON-ZERO NUMBER RAISED TO A NEGATIVE EXPONENT IS THE RECIPROCAL OF THE SAME POWER WITH POSITIVE EXPONENT.

EXAMPLE 4 SIMPLIFY AND WRITE YOUR ANSWER AS A NON-NEGATIVE EXPO

A 2^{-3}

B $\frac{2^4}{2^9}$

C $\left(\frac{3}{2}\right)^{-3}$

SOLUTION:

A $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ **B** $\frac{2^4}{2^9} = 2^{(4-9)} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

C $\left(\frac{3}{2}\right)^{-3} = \frac{1}{\left(\frac{3}{2}\right)^3} = \frac{1}{\left(\frac{3^3}{2^3}\right)} = 1 \times \frac{2^3}{3^3} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

IN EXAMPLE 4C ABOVE YOU HAVE SEEN THAT $\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^3$. USE THIS TECHNIQUE TO SIMPLIFY THE FOLLOWING:

EXAMPLE 5

A $\left(\frac{4}{5}\right)^{-1}$ **B** $\left(\frac{2}{5}\right)^{-4}$ **C** $\left(\frac{3}{10}\right)^{-2}$

SOLUTION:

A $\left(\frac{4}{5}\right)^{-1} = \frac{5}{4}$ **B** $\left(\frac{2}{5}\right)^{-4} = \left(\frac{5}{2}\right)^4 = \frac{625}{16}$ **C** $\left(\frac{3}{10}\right)^{-2} = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

Note: FOR $a \neq 0,\ a^{-1} = \frac{1}{a}$

THE ABOVE EXAMPLES LEAD YOU TO THE FOLLOWING FACT:

IF a AND b ARE NON-ZERO REAL NUMBERS THEN IT IS ALWAYS TRUE THAT FOR $n \in \mathbb{Z}$,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Exercise 2.1

1 USE THE LAWS OF EXPONENTS TO SIMPLIFY EACH OF THE FOLLOWING EXPONENTIAL EXPRESSIONS:

A $t^2 \times t$ **B** $t^3 \times t \times t^5$ **C** $r \times r^4 \times r^5 \times r$ **D** $a^3 \times a \times a^{-5}$

E $\frac{7^6}{7^4}$ **F** $\frac{(-3y)^2}{(-3y)^5}$ **G** $\frac{(2x)^7}{(2x)^8}$ **H** $b^{2x} \div b$

I $(5^5)^{2n}$ **J** $(b^y)^x$ **K** $(7^3)^{-2}$ **L** $(a^{3x})^2$

2 WRITE EACH OF THE FOLLOWING WITH A PRIME NUMBER AS THE BASE:

A 81 **B** $\frac{16^{2x+3}}{16^{2x-3}}$ **C** $\frac{49^x}{7^y}$ **D** $64^a \times 4^a$

3 REMOVE THE BRACKETS FROM EACH OF THE FOLLOWING EXPRESSIONS:

A $(xyz)^2$ **B** $(2ab^2)^5$ **C** $\left(\frac{9}{3}\right)^2$ **D** $\left(-\frac{2}{2n}\right)^6$

4 SIMPLIFY AND GIVE YOUR ANSWERS IN SIMPLEST RATIONAL FORM:

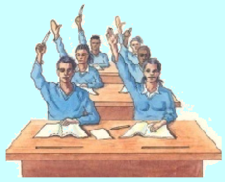
A $\left(\frac{3}{2}\right)^0$ **B** $\left(\frac{8}{3}\right)^{-2}$ **C** $\left(\frac{1}{4^{-3}}\right)^{-1}$ **D** $(-2)^{-5}$ **E** $(3x^2)^{-3}$

Rational exponents

SO FAR WE HAVE CONSIDERED EXPRESSIONS WITH INTEGRAL EXPONENTS. YOU KNOW WHAT 3^5, 2^{-3} AND 7^0 MEAN. BUT WHAT DO EXPRESSIONS SUCH AS $5^{\frac{1}{3}}$ AND $6^{\frac{2}{3}}$ MEAN?

WE NOW EXTEND THE LAWS OF EXPONENTS TO RATIONAL NUMBERS.

ACTIVITY 2.3



USING THE LAW $a^m \times a^n = a^{m+n}$, DO THE FOLLOWING:

1 **A** SIMPLIFY

I $6^{\frac{1}{2}} \times 6^{\frac{1}{2}}$ **II** $\sqrt{6} \times \sqrt{6}$

B COMPARE THE RESULT IN I WITH THE RESULT IN II. WHAT DO YOU NOTICE?

2 **A** SIMPLIFY

I $6^{\frac{1}{3}} \times 6^{\frac{1}{3}} \times 6^{\frac{1}{3}}$ **II** $\sqrt[3]{6} \times \sqrt[3]{6} \times \sqrt[3]{6}$

B COMPARE THE RESULT IN I WITH THE RESULT IN II. WHAT DO YOU NOTICE?

3 **A** SIMPLIFY

I $2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}}$ **II** $\sqrt[4]{2} \times \sqrt[4]{2} \times \sqrt[4]{2} \times \sqrt[4]{2}$.

B COMPARE THE RESULT IN I WITH THE RESULT IN II. WHAT DO YOU NOTICE?

4 IN GENERAL, WHAT DO YOU THINK IS TRUE ABOUT $a^{\frac{1}{n}}$ AND $\sqrt[n]{a}$?

IF $a \ge 0$ AND n IS AN INTEGER WITH $n \ge 2$, THEN $a^{\frac{1}{n}} = \sqrt[n]{a}$. THIS ALSO HOLDS WHEN $a < 0$ AND n IS ODD. (READ $\sqrt[n]{a}$ AS “THE n^{TH} ROOT OF a”.)

EXAMPLE 6 EXPRESS EACH OF THE FOLLOWING IN THE FORM $a^{\frac{1}{n}}$:

A $\sqrt[4]{3}$ **B** $\sqrt[5]{64}$ **C** $\frac{1}{\sqrt{9}}$ **D** $\frac{\left(\sqrt[3]{32}\right)^2}{4^{\frac{5}{3}}}$

SOLUTION:

A $\sqrt[4]{3} = 3^{\frac{1}{4}}$ **B** $\sqrt[5]{64} = 64^{\frac{1}{5}}$ **C** $\dfrac{1}{\sqrt{9}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{(3^2)^{\frac{1}{2}}} = \dfrac{1}{3} = 3^{-1}$

D $\dfrac{\left(\sqrt[3]{32}\right)^2}{4^{\frac{5}{3}}} = \dfrac{\left(32^{\frac{1}{3}}\right)^2}{\left(2^2\right)^{\frac{5}{3}}} = \dfrac{32^{\frac{2}{3}}}{2^{\frac{10}{3}}} = \dfrac{\left(2^5\right)^{\frac{2}{3}}}{2^{\frac{10}{3}}} = \dfrac{2^{\frac{10}{3}}}{2^{\frac{10}{3}}} = 2^{\left(\frac{10}{3}-\frac{10}{3}\right)} = 2^0 = 1$

WHAT IS THE RESULT OF $6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}}$?

$6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}} = 6^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 6^{\frac{6}{3}} = 6^2$

ALSO $6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}} = \left(6^{\frac{2}{3}}\right)^3 = 6^2$ …. *using the law* $(a^m)^n = a^{m \times n}$

THEREFORE, $6^{\frac{2}{3}} = \left(6^2\right)^{\frac{1}{3}} = \sqrt[3]{6^2}$

IN GENERAL, IF $a > 0$ AND m, n ARE INTEGERS WITH $n > 1$, THEN $a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

EXAMPLE 7 EXPRESS IN THE FORM $b^{\frac{m}{n}}$, WITH b BEING A PRIME NUMBER.

A $\sqrt[5]{64}$ **B** $\sqrt[3]{16}$ **C** $\sqrt[8]{27}$

SOLUTION:

A $\sqrt[5]{64} = 64^{\frac{1}{5}} = \left(2^6\right)^{\frac{1}{5}} = 2^{\frac{6}{5}}$ **B** $\sqrt[3]{16} = 16^{\frac{1}{3}} = \left(2^4\right)^{\frac{1}{3}} = 2^{\frac{4}{3}}$

C $\sqrt[8]{27} = 27^{\frac{1}{8}} = \left(3^3\right)^{\frac{1}{8}} = 3^{\frac{3}{8}}$

REMEMBER THAT $\sqrt[n]{a}$ IS NOT A REAL NUMBER IF a IS NEGATIVE AND n IS AN EVEN NATURAL NUMBER.
HOWEVER, $\sqrt[n]{a}$ IS A REAL NUMBER IF a IS NEGATIVE AND n IS AN ODD NATURAL NUMBER.

FOR EXAMPLE, $\sqrt{-4}$, $\sqrt[4]{-5}$, $\sqrt[6]{-9}$, $\sqrt[8]{-8}$, ETC, ARE NOT REAL NUMBERS, WHEREAS, $\sqrt[3]{-27}$, $\sqrt[5]{-32}$, $\sqrt[3]{-8}$, $\sqrt[9]{-81}$, ETC, ARE REAL NUMBERS.

EXAMPLE 8 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt[3]{-27}$ **B** $\sqrt[7]{-128}$ **C** $\dfrac{\sqrt[5]{-32}}{\sqrt[3]{-64}}$

SOLUTION:

A $\sqrt[3]{-27} = \sqrt[3]{(-3)\times(-3)\times(-3)} = -3$

B $\sqrt[7]{-128} = \sqrt[7]{(-2)^7} = \left(-2^7\right)^{\frac{1}{7}} = -2$

C $\dfrac{\sqrt[5]{-32}}{\sqrt[3]{-64}} = \dfrac{\sqrt[5]{(-2^5)}}{\sqrt[3]{(-4)^3}} = \dfrac{-2}{-4} = \dfrac{1}{2}$

WE CONCLUDE OUR DISCUSSION OF RATIONAL EXPONENTS BY THE FOLLOWING REMAR

ALL RULES FOR INTEGRAL EXPONENTS DISCUSSED EARLIER ALSO HOLD TRUE FOR RATI

Irrational exponents

NOW CONSIDER EXPRESSIONS WITH IRRATIONAL EXPONENTS, SUCH AS $2^{\sqrt{5}}$, 3, $5^{\sqrt{3}}$

EXAMPLE 9 WHICH NUMBER IS THE LARGEST: $2^{\sqrt{5}}$, 4, 3,

SOLUTION: THE ANSWER WILL NOT BE SIMPLE BECAUSE WE DO NOT KNOW THE EXACT VALUE OF $2^{\sqrt{5}}$.

TO APPROXIMATE THE NUMBER $2^{\sqrt{5}}$, LET US CONSIDER THE FOLLOWING TABLE FOR 2^x

x	−4	−3	−2	−1	0	1	2	3	4	5
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

FROM THE TABLE WE SEE THAT FOR ANY VALUES OF $x_1 < x_2$, THEN $2^{x_1} < 2^{x_2}$.

THEREFORE, SINCE $2.2 < \sqrt{5} < 2.3$, WE HAVE $2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$.

LET US NOW TAKE CLOSER APPROXIMATIONS TO $2^{\sqrt{5}}$ BY USING A CALCULATOR.

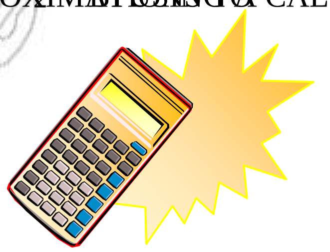
$2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$

$2^{2.23} < 2^{\sqrt{5}} < 2^{2.24}$

$2^{2.236} < 2^{\sqrt{5}} < 2^{2.237}$

$2^{2.2360} < 2^{\sqrt{5}} < 2^{2.2361}$

$2^{2.23606} < 2^{\sqrt{5}} < 2^{2.23607}$



AS WE CAN SEE FROM THE ABOVE LIST, THE NUMBERS $2^{2.2}, 2^{2.23}, 2^{2.236}, \ldots$ APPROACH TO $2^{\sqrt{5}}$. SIMILARLY, THE NUMBERS $2^{2.3}, 2^{2.24}, 2^{2.237}, \ldots$ ALSO APPROACH TO THE SAME NUMBER $2^{\sqrt{5}}$.

SO $2^{\sqrt{5}}$ IS BOUNDED BY TERMS OF CONVERGING RATIONAL APPROXIMATIONS. USING A WE FIND THAT $2^{\sqrt{5}} \approx 4.7111$, TO FOUR DECIMAL PLACES; HENCE $2^{\sqrt{5}}$ IS A NUMBER BETWEEN 4.7 AND 4.8. SO THE LARGEST OF THE NUMBERS 3, $2^{\sqrt{5}}$ AND 4 MUST BE $2^{\sqrt{5}}$.

EXAMPLE 10 GIVE AN APPROXIMATION TO 3^{π}.

SOLUTION: RECALL THAT $\pi \approx 3.1415926$. A CALCULATOR GIVES THE ROUNDED VALUES:

$3^{3.1} \approx 30.1353$

$3^{3.14} \approx 31.4891$

$3^{3.141} \approx 31.5237$

$2^{3.1415} \approx 31.5411$

$3^{3.14159} \approx 31.5442$

$3^{3.141592} \approx 31.5443$

$3^{3.1415926} \approx 31.5443$

HENCE $3^{\pi} \approx 31.5443$, ROUNDED TO FOUR DECIMAL PLACES. A TEN-PLACE CALCULATOR AC
APPROXIMATES 3^{π} BY $3^{3.141592654} \approx 31.5442807002$.

THE ABOVE TWO EXAMPLES SUGGEST THE FOLLOWING:

IF x IS AN IRRATIONAL NUMBER AND $a > 0$, THEN a^x IS THE REAL NUMBER BETWEEN a^{x_1} AND a^{x_2} FOR ALL POSSIBLE CHOICES OF RATIONAL NUMBERS x_1 AND x_2 SUCH THAT $x_1 < x < x_2$.

THE ABOVE STATEMENT ABOUT IRRATIONAL EXPONENTS SHOWS THAT THE EXPRESSION a^x IS DEFINED NOT ONLY FOR INTEGRAL AND RATIONAL EXPONENTS BUT ALSO FOR IRRATION

EXAMPLE 11 SIMPLIFY EACH OF THE FOLLOWING:

A $4^{\sqrt{3}} \times 4^{\sqrt{12}}$ **B** $\dfrac{2^{\sqrt{5}} \times 2^{\sqrt{20}}}{8^{\sqrt{5}}}$ **C** $\dfrac{3^{\sqrt{2}} \times 3^{-\sqrt{2}} \times 27^{\sqrt{2}}}{3^{\sqrt{8}}}$.

SOLUTION:

A $4^{\sqrt{3}} \times 4^{\sqrt{12}} = 4^{\sqrt{3}} \times 4^{2\sqrt{3}} = 4^{\sqrt{3}+2\sqrt{3}} = 4^{3\sqrt{3}} = (4^3)^{\sqrt{3}} = 64^{\sqrt{3}}$

B $\dfrac{2^{\sqrt{5}} \times 2^{\sqrt{20}}}{8^{\sqrt{5}}} = \dfrac{2^{\sqrt{5}+2\sqrt{5}}}{8^{\sqrt{5}}} = \dfrac{2^{3\sqrt{5}}}{8^{\sqrt{5}}} = \dfrac{\left(2^3\right)^{\sqrt{5}}}{8^{\sqrt{5}}} = \dfrac{8^{\sqrt{5}}}{8^{\sqrt{5}}} = 1$

C $\dfrac{3^{\sqrt{2}} \times 3^{-\sqrt{2}} \times 27^{\sqrt{2}}}{3^{\sqrt{8}}} = \dfrac{3^0 \times 3^{3\sqrt{2}}}{3^{\sqrt{8}}} = \dfrac{3^{3\sqrt{2}}}{3^{\sqrt{8}}} = \dfrac{3^{3\sqrt{2}}}{3^{2\sqrt{2}}} = 3^{(3\sqrt{2}-2\sqrt{2})} = 3^{\sqrt{2}}$

THE LAWS OF EXPONENTS DISCUSSED EARLIER FOR INTEGRAL AND RATIONAL EXPONEN
HOLD TRUE FOR IRRATIONAL EXPONENTS.

IN GENERAL, IF a AND b ARE POSITIVE NUMBERS AND r AND s ARE REAL NUMBERS, THEN

1 $a^r \times a^s = a^{r+s}$ **2** $\dfrac{a^r}{a^s} = a^{r-s}$ **3** $(a^r)^s = a^{rs}$

4 $(a \times b)^s = a^s \times b^s$ **5** $\left(\dfrac{a}{b}\right)^s = \dfrac{a^s}{b^s}$

Group Work 2.3

DISCUSS IN GROUPS AND ANSWER EACH OF THE FOLLO

1 **A** $24 > 23$; IS $24^2 > 23^2$?

B $81 > 16$; IS $81^{\frac{1}{4}} > 16^{\frac{1}{4}}$?

C $20 > 10$; IS $20^{-2} > 10^{-2}$?

D $\frac{1}{100} < \frac{1}{10}$; IS $\left(\frac{1}{100}\right)^2 < \left(\frac{1}{10}\right)^2$?

E $\frac{1}{100} < \frac{1}{10}$; IS $\left(\frac{1}{100}\right)^{-2} < \left(\frac{1}{10}\right)^{-2}$?

2 **A** LET $a > b > 1$.

IS $a^x > b^x$, FOR $x > 0$?

IS $a^x > b^x$, FOR $x < 0$?

B LET $0 < a < b < 1$.

IS $a^x < b^x$, FOR $x > 0$?

IS $a^x < b^x$, FOR $x < 0$?

Exercise 2.2

SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS USING ONE OR MORE OF THE LAWS O

A $a^2 \times a \times a^3$ **B** $(2^{-3} + 3^{-2})^{-1}$ **C** $\left(\sqrt[3]{343}\right)^{-2}$

D $(2a^{-3} \times b^2)^{-2}$ **E** $\frac{(3a)^4}{(3a)^3}$ **F** $\left(\frac{a^2}{b}\right)^3$

G $\left(\frac{a^3}{b^5}\right)^{-2}$ **H** $\frac{(n^2)^4 \times (n^3)^{-2}}{n^{-1}}$ **I** $\left(\frac{m^{-3}m^3}{n^{-2}}\right)^{-2}$

J $\left(\frac{m^{-\frac{2}{3}}}{n^{-\frac{1}{2}}}\right)^{-6}$ **K** $\left(\frac{a^{-\frac{1}{3}}b^{\frac{1}{2}}}{a^{-\frac{1}{4}}b^{\frac{1}{3}}}\right)^6$ **L** $\frac{\left(3^{\sqrt{2}}\right)^2 \times 9^{-\sqrt{3}}}{3^{-\sqrt{12}}}$

M $\left(2^{\sqrt{3}}\right)^2 \div \left(4^{\sqrt{3}}\right)^{-2}$ **N** $\left(\frac{2^{\sqrt{5}} \times 2^{-\sqrt{5}}}{\sqrt{2}}\right)^2$

O $\frac{2^{\sqrt{2}} \times 2^{-\sqrt{2}} \times 32^{\sqrt{2}}}{4^{\sqrt{8}}}$ **P** $\sqrt[6]{64a^6b^{-2}}$

2.1.2 Logarithms

Logarithms CAN BE THOUGHT OF AS ***"the inverse"*** OF EXPONENTS.

FOR EXAMPLE, WE KNOW THAT THE FOLLOWING EXPONENTIAL EQUATION IS TRUE: $3^2 = 9$

IN THIS CASE, THE ***base*** IS 3 AND THE ***exponent*** IS 2. WE WRITE THIS EQUATION IN ***logarithm form*** (WITH IDENTICAL MEANING) AS $\log_3 9 = 2$

WE READ THIS AS "THE LOGARITHM OF 9 TO THE BASE 3 IS 2".

HISTORICAL NOTE:

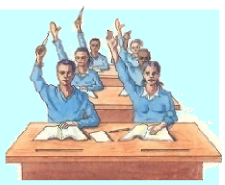
LOGARITHMS ***were developed in the 17th century by the Scottish mathematician, John Napier (1550-1617). They were clever methods of reducing long multiplications into much simpler additions and reducing divisions into subtractions. While he was young, Napier had to help his father, who was a tax collector. John got sick of multiplying and dividing large numbers all day and devised logarithms to make his life easier!***

SINCE $2^4 = 16$, WE CAN SAY THAT $\log_2 16 = 4$.

AS $10^3 = 1000$, $3 = \log_{10} 1000$.

THE FOLLOWING ACTIVITY WILL HELP YOU LEARN HOW TO CONVERT EXPONENTIAL STATEMENTS INTO LOGARITHMIC STATEMENTS AND VICE VERSA.

ACTIVITY 2.4



COMPLETE THE FOLLOWING TABLE:

Exponential statement	Logarithmic statement
$2^3 = 8$	$\log_2 8 = 3$
$2^5 = 32$	
$2^6 = 64$	
	$\log_{10} 100 = 2$
$2^x = y$	

IN GENERAL,

FOR A FIXED ***positive*** NUMBER $b \neq 1$, AND FOR EACH $a > 0$,

$b^c = a$, IF AND ONLY IF $c = \log_b a$.

OBSERVE FROM THE ABOVE NOTE THAT EVERY LOGARITHMIC STATEMENT CAN BE TRA EXPONENTIAL STATEMENT AND VICE VERSA.

Note: THE VALUE OF $\log_b a$ IS THE ANSWER TO THE QUESTION: " TO WHAT POWER MUST b BE RAISED TO PRODUCE a?"

EXAMPLE 1 WRITE AN EQUIVALENT LOGARITHMIC STATEMENT FOR:

A $3^4 = 81$ **B** $4^3 = 64$ **C** $8^{\frac{1}{3}} = 2$

SOLUTION:

A FROM $3^4 = 81$, WE DEDUCE THAT $\log_3 81 = 4$

B FROM $4^3 = 64$, WE HAVE $\log_4 64 = 3$

C SINCE $8^{\frac{1}{3}} = 2$, $\log_8 2 = \frac{1}{3}$

EXAMPLE 2 WRITE AN EQUIVALENT EXPONENTIAL STATEMENT FOR:

A $\log_{12} 144 = 2$ **B** $\log_4\left(\frac{1}{64}\right) = -3$ **C** $\log_{10}\sqrt{10} = \frac{1}{2}$

SOLUTION:

A FROM $\log_{12} 144 = 2$, WE DEDUCE THAT $12^2 = 144$.

B $\log_4 \frac{1}{64} = -3$ IS THE SAME AS SAYING $4^{-3} = \frac{1}{64}$.

C $\log_{10}\sqrt{10} = \frac{1}{2}$ CAN BE WRITTEN IN EXPONENTIAL FORM AS $10^{\frac{1}{2}} = \sqrt{10}$.

EXAMPLE 3 FIND:

A $\log_2 64$ **B** $\log_3 \frac{1}{9}$ **C** $\log_{1000} 1$

SOLUTION:

A TO FIND $\log_2 64$, YOU ASK "*what power must* 2 *be raised to get* 64?"

AS $2^6 = 64$, $\log_2 64 = 6$ OR FROM THE EXPONENTIAL EQUATIONS DISCUSSED IN GRADE 9, YOU CAN FORM THE EQUATION $2^x = 64$.

SOLVING THIS GIVES $2^x = 2^6 \Rightarrow x = 6$.

... remember that $b^x = b^y$, if and only if $x = y$, for $b > 0$, $b \neq 1$.

B TO FIND $\log_3 \frac{1}{9}$, WE ASK "*to what power must* 3 *be raised to get* $\frac{1}{9}$?"

AS $3^{-2} = \frac{1}{9}$, $\log_3 \frac{1}{9} = -2$ OR $3^x = \frac{1}{9} \Rightarrow 3^x = 3^{-2} \Rightarrow x = -2$.

c TO FIND $\log_{1000} 10$, WE ASK "*to what* POWER *must 1000 be raised to get 10?*"

AS $1000^{\frac{1}{3}} = 10$, $\log_{1000} 10 = \frac{1}{3}$ OR $1000^x = 10 \Rightarrow 10^{3x} = 10^1 \Rightarrow 3x = 1$

$\Rightarrow x = \frac{1}{3}$.

Exercise 2.3

1 WRITE AN EQUIVALENT LOGARITHMIC STATEMENT FOR:

a $100^2 = 10000$ **b** $2^{-5} = \frac{1}{32}$ **c** $125^{\frac{1}{3}} = 5$ **d** $8^{\frac{-2}{3}} = \frac{1}{4}$

2 WRITE AN EQUIVALENT EXPONENTIAL STATEMENT FOR:

a $\log_{10} 10000 = 4$ **b** $\log_7 \sqrt{49} = 1$

c $\log_{10} 0.1 = -1$ **d** $\log_2 \frac{1}{4} = -$

3 FIND:

a $\log_2$ **b** $\log_9$ 8

c $\log_{100} 1000$ **d** $\log_{49}$

Laws of logarithms

THE FOLLOWING GROUP WORK WILL HELP YOU OBSERVE DIFFERENT LAWS WHILE USING LOG ARITHMS.

Group Work 2.4

1 FIND:

a $\log_2 8 + \log_2 2$; COMPARE THE RESULT WITH $\log_2 (8 \times 2)$

b $\log_{10} 100 + \log_{10} 1000$; COMPARE THE RESULT WITH $\log_{10}(100 \times 1000)$

c $\log_3 \quad + \log_3 \left(\frac{1}{27}\right)$; COMPARE THE RESULT WITH $\log_3 (9 \times \frac{1}{27})$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b x + \log_b y$?

2 FIND:

a $\log_2 8 - \log_2 2$; COMPARE THE RESULT WITH $\log_2 \left(\frac{8}{2}\right)$

b $\log_{10} 100 - \log_{10} 1000$; COMPARE THE RESULT WITH $\log_{10} \left(\frac{100}{1000}\right)$.

c $\log_3 \quad - \log_3 \frac{1}{27}$; COMPARE THE RESULT WITH $\log_3 \left(9 \div \frac{1}{27}\right)$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b x - \log_b y$?

3 FIND:

A $3\log_2$; COMPARE THE RESULT WITH $\log_2 (2^3)$

B $2\log_{10} 100$; COMPARE THE RESULT WITH $\log_{10} (100^2)$

C $\frac{1}{2}\log_2 16$; COMPARE THE RESULT WITH $\log_2 16^{\left(\frac{1}{2}\right)}$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $k\log_b x$?

4 FIND:

A $\log_3 3$ **B** $\log_8 8$ **C** $\log_{100} 100$ **D** $\log_{\frac{1}{3}} \frac{1}{3}$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b b$, $b > 0$ AND $b \neq 1$?

5 FIND:

A $\log_3 1$ **B** $\log_4 1$ **C** $\log_{\frac{1}{3}} 1$ **D** $\log_{1000} 1$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b 1$, $b > 0$ AND $b \neq 1$?

THE FOLLOWING ARE LAWS OF LOGARITHMS:

IF b, x AND y ARE POSITIVE NUMBERS AND $b \neq 1$, THEN

I $\log_b xy = \log_b x + \log_b y$ **II** $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

III FOR ANY REAL NUMBER k, $\log_b (x^k) = k\log_b x$

Note: IF $b > 0$ AND $b \neq 1$, THEN

I $\log_b b = 1$ **II** $\log_b 1 = 0$

EXAMPLE 4 USE THE LAWS OF LOGARITHMS TO FIND:

A $\log_2 16 + \log_2$ **B** $\log_4 \sqrt{16} - \log_4 4$

C $2(\log_{10} 100)$ **D** $\log_{10} \sqrt[4]{0.01}$

SOLUTION:

A $\log_2 16 + \log_2 4 = \log_2 (16 \times 4) = \log_2 64 = 6$

… *using the law* $\log_b xy = \log_b x + \log_b y$

B $\log_4 \sqrt{16} - \log_4 4 = \log_4 \frac{\sqrt{16}}{4} = \log_4 \frac{4}{4} = \log_4 1 = 0$

… *using the law* $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

c $2(\log_{10} 100 - 1) = 2(\log_{10}100 - \log_{10}10) = 2\log_{10}\left(\frac{100}{10}\right) = 2\log_{10}10 = 2$

… using the law $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

d $\log_{10}\sqrt[4]{0.01} = \log_{10}(0.01)^{\frac{1}{4}} = \log_{10}\left(\frac{1}{100}\right)^{\frac{1}{4}} = \log_{10}(10^{-2})^{\frac{1}{4}} = \log_{10}10^{\frac{-2}{4}}$

$= \frac{-2}{4}\log_{10}10 = \frac{-2}{4}\times 1 = \frac{-2}{4} = \frac{-1}{2}$

… using the law $\log_b\left(x^k\right) = k\log_b x$

Two additional laws of logarithms

If a, b and c are positive real numbers, $a \neq 1$, $b \neq 1$, then

i $\log_a c = \frac{\log_b c}{\log_b a}$ ("change of base law") **ii** $b^{\log_b c} = c$

Example 5 Using the above two laws find

a $\log_{16} 64$

b $\log_3 2$ (Given that $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$)

c $10^{\log_{10}7}$

Solution

a $\log_{16} 64 = \frac{\log_2 64}{\log_2 16} = \frac{6}{4} = \frac{3}{2}$ OR

$\log_{16} 64 = \frac{\log_{10}64}{\log_{10}16} = \frac{\log_{10}4^3}{\log_{10}4^2} = \frac{3\log_{10}4}{2\log_{10}4} = \frac{3}{2}\times 1 = \frac{3}{2}$

… you can use any base $b > 0$, $b \neq 1$

b $\log_3 2 = \frac{\log_{10}2}{\log_{10}3} = \frac{0.3010}{0.4771} = 0.6309$ **c** $10^{\log_{10}7} = 7$

Exercise 2.4

1 Find:

a $\log_{11} 121$ **b** $\log_6 6$ **c** $\log_{10}100000$ **d** $\log_5 125$

e $\log_3\sqrt{3}$ **f** $\log_9 3$ **g** $\log_{100}\sqrt[5]{100}$ **h** $\log_{\frac{1}{5}}125$

2 SIMPLIFY:

A $\log_2(64 \times 1024)$ **B** $\log_2 \frac{32}{256}$ **C** $\log_2 512^3$

D $\log_{10}(2 \times 10^{-3})$ **E** $\log_2\left(\frac{128\times 64}{512}\right)$ **F** $\log_3 + \log_3 \frac{1}{27}$

G $\log_2 64^2 \div \log_2 1^7$

3 USING THE LAWS $\log_a c = \frac{\log_b c}{\log_b a}$ OR $b^{\log_b c} = c$ FIND:

A $\log_{\left(\frac{1}{3}\right)} 8$ **B** $\log_{\left(\frac{1}{2}\right)}$ **C** $\log_{\frac{1}{3}} \frac{1}{27}$

D $5^{\log_5 3}$ **E** $6^{\log_6 10}$

4 If $\log_{10} 2 = 0.3010$ AND $\log_{10} 3 = 0.4771$, THEN FIND:

A $\log_2 \sqrt{3}$ **B** $\log_{\frac{1}{2}} 5$ **C** $\log_{\frac{1}{3}} 0.00$

Logarithms in base 10 (common logarithms)

OUR DECIMAL SYSTEM IS BASED ON NUMBERS OF THE FORM 10^n. FOR EXAMPLE,

$10000 = 10^4$	$0.0001 = 10^{-4}$
$1000 = 10^3$	$0.001 = 10^{-3}$
$100 = 10^2$	$0.01 = 10^{-2}$
$10 = 10^1$	$0.1 = 10^{-1}$
$1 = 10^0$	

ALSO NUMBERS LIKE $\sqrt{10}$, $\sqrt{100}$, $10\sqrt{10}$ AND $\frac{1}{\sqrt[5]{10}}$ CAN BE WRITTEN AS

$10^{\frac{1}{2}}$, $100^{\frac{1}{2}}$, $10^1 \times 10^{\frac{1}{2}} = 10^{\frac{3}{2}}$ AND $10^{\frac{-1}{5}}$ RESPECTIVELY.

IN FACT, ALL POSITIVE NUMBERS CAN BE WRITTEN IN THE FORM 10^n. BY INTRODUCING THE CONCEPT OF LOGARITHMS. THE LOGARITHM OF A POSITIVE NUMBER TO BASE 10 IS CALLED A ***common logarithm***.

THE COMMON LOGARITHM IS USUALLY THE MOST CONVENIENT ONE TO USE FOR CO INVOLVING SCIENTIFIC NOTATIONS BECAUSE WE USE THE BASE 10 NUMBER SYSTEM.

ONE IMPORTANT USAGE OF COMMON LOGARITHMS IS IN THEIR USE IN SIMPLIFYING COMPUTATIONS. DUE TO THE EXTENSIVE USAGE OF VARIOUS ADVANCED CALCULATORS, OF THE USAGE OF LOGARITHMS AT PRESENT IS NOT AS IT WAS IN THE PAST. HOWEVER, TH OPERATIONS LIKE 1.27 THAT YOU ARE ABLE TO PERFORM USING COMMON LOGARITHMS

THIS IS DUE TO THE FACT THAT ANY LOGARITHM TO BASE OTHER THAN 10 CAN BE E COMMON LOGARITHM SO THAT ONE CAN USE THE TABLE OF COMMON LOGARITHM F STANDARD BOOKS AND MATHEMATICAL TABLES.

A COMMON LOGARITHM IS USUALLY WRITTEN WITHOUT INDICATING ITS BASE. FOR EXA $\log_{10} x$ IS SIMPLY DENOTED BY $\log x$.

SO IF A LOGARITHM IS GIVEN WITH NO BASE, WE TAKE IT TO BE BASE 10.

ACTIVITY 2.5

FIND THE FOLLOWING COMMON LOGARITHMS:

A $\log \sqrt{1}$ **B** $\log 0.000$ **C** $\log 1$ **D** $\log\left(\frac{10}{10^n}\right)$

EXAMPLE 6 FIND THE FOLLOWING COMMON LOGARITHMS:

A $\log 100,000$ **B** $\log \sqrt[3]{100}$ **C** $\log 0.001$

SOLUTION

A $\log 100,000 = 5$ BECAUSE $10^5 = 100,000$ OR $\log 100,000 = \log 10^5 = 5\log 1$

B $\log \sqrt[3]{100} = \frac{2}{3}$ BECAUSE $\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{\frac{2}{3}}$ OR

$\log \sqrt[3]{100} = \log 100^{\frac{1}{3}} = \left(\log 10^2\right)^{\frac{1}{3}} = \log 10^{\frac{2}{3}} = \frac{2}{3}\log 10 = \frac{2}{3} \times \log 10 = \frac{2}{3} \times 1$

C $\log 0.001 = -3$ BECAUSE $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$ OR

$\log 0.001 = \log \frac{1}{1000} = \log \frac{1}{10^3} = \log 10^{-3} = -3\log$

EXAMPLE 7 FIND THE COMMON LOGARITHM OF 526.

SOLUTION: $\log 526 = \log(5.26 \times 10^2) = \log 5.26 + \log 10^2$ … *by* $\log_b xy = \log_b x + \log_b y$

$= \log 5.26 + 2$ … NOW WE STILL NEED TO FIND $\log 5.26$.

SINCE $\log 1 = 0$ AND $\log 10 = 1$, WE KNOW THAT $0 < \log 5.26 < 1$.

SO, THE COMMON LOGARITHM OF A NUMBER BETWEEN 1 AND 10 IS A NUMBER BE AND 1. THE SPECIFIC COMMON LOGARITHMIC VALUES FOR NUMBERS BETWEEN 1 A GIVEN IN WHAT IS CALLED A TABLE OF COMMON LOGARITHMS.

A COPY OF THE TABLE IS ATTACHED AT THE END OF THIS BOOK

FROM THE COMMON LOGARITHM TABLE, WE READ THAT $\log 5.26 = 0.721$

(*It should be noted that this value is only an approximate value.*)

HENCE,

$\log 526 = \log(5.26 \times 10^2) = \log 5.26 + \log 10^2 = \log 5.26 + 2 = 0.7210 + 2 = 2.7210$

Mantissa Characteristic M C

IF WE WRITE A NUMBER x AS $x = m \times 10^{c}$, $0 \le m < 10$, THEN THE LOGARITHM OF x CAN BE READ FROM A COMMON LOGARITHM TABLE. THE LOG m IS CALLED THE ***mantissa*** OF THE LOGARITHM OF THE NUMBER x AND c IS CALLED THE ***characteristic*** OF THE LOGARITHM. THEREFORE, THE COMMON LOGARITHM OF A NUMBER IS EQUAL TO ITS ***characteristic*** PLUS ITS ***mantissa***.

EXAMPLE 8 IDENTIFY THE CHARACTERISTIC AND MANTISSA OF EACH OF THE FOLLOWING COMMON LOGARITHMS:

A LOG 0.000415 **B** LOG 239 **C** LOG 0.001

SOLUTION:

A $0.000415 = 4.15 \times 10^{-4}$

THEREFORE, THE CHARACTERISTIC IS –4 AND THE MANTISSA IS LOG 4.15.

B $239 = 2.39 \times 10^{2}$

THEREFORE, THE CHARACTERISTIC IS 2 AND THE MANTISSA IS LOG 2.39.

C $0.001 = 1 \times 10^{-3}$

THEREFORE, THE CHARACTERISTIC IS –3 AND THE MANTISSA IS LOG 1 = 0.

Using the logarithm table

THE LOGARITHM OF ANY TWO DECIMAL PLACE NUMBER BETWEEN 1.00 AND 9.99 CAN BE READ DIRECTLY FROM THE COMMON LOGARITHM TABLE (A PART OF THE TABLE IS GIVEN BELOW FOR REFERENCE).

x	0	1	2	. . .	9
1.0	0.0000	0.0043	0.0086	. . .	0.0374
1.1	0.0414	0.0453	0.0492	. . .	0.0755
1.2	0.0792	0.0828	0.0864	. . .	0.1106
1.3	0.1139	0.1173	0.1206	. . .	0.1430
⋮	⋮	⋮	⋮	⋮	⋮
1.9	0.2788	0.2810	0.2833	. . .	0.2989
2.0	0.3010	0.3032	0.3054	. . .	0.3201
2.1	0.3222	0.3243	0.3263	. . .	0.3404
2.2	0.3424	0.3444	0.3464	. . .	0.3598
⋮	⋮	⋮	⋮	⋮	⋮
9.9	0.9956	0.9961	0.9965	. . .	0.9996

EXAMPLE 9 USE THE TABLE OF LOGARITHMS TO FIND:

A LOG 2.29 **B** LOG 1.21 **C** LOG 1.386 **D** LOG 21,200

SOLUTION:

A READ THE NUMBER AT THE INTERSECTION OF ROW 2.2 AND COLUMN 9.

READING THE NUMBER IN ROW 2.2 UNDER COLUMN 9, WE GET 0.3598.

$\therefore$ LOG 2.29 = 0.3598 .

B READING THE NUMBER AT THE INTERSECTION OF ROW 1.2 AND COLUMN 1, WE GET 0.0828

$\therefore$ LOG 1.21 = 0.0828 .

C 1.386 IS BETWEEN 1.38 AND 1.39 .

SO, ROUND (TO 2 DECIMAL PLACES) LOG 1.386 AS LOG 1.39 . READING IN ROW 1.3 UN
COLUMN 9, WE GET 0.1430. LOG 1.386 $\cong$ 0.1430.

D FIRST WRITE 21,200 AS 2.12×10^4

$\therefore$ LOG 21,200 = LOG (2.12×10^4) = LOG 2.12 + LOG 10^4 = LOG 2.12 + 4

= 0.3263 + 4 = 4.3263.

Note: NUMBERS GREATER THAN 10 HAVE LOGARITHMS GREATER THAN 1.

Antilogarithms

SUPPOSE LOG x = 0.6665. WHAT IS THE VALUE OF x?

IN SUCH CASES, WE APPLY WHAT IS CALLED THE *antilogarithm of the logarithm of x*, WRITTEN AS *antilog* (LOG x). THUS ANTILOG (LOG x) = ANTILOG (0.6665).

WE HAVE TO SEARCH THROUGH THE LOGARITHM TABLE, FOR THE VALUE 0.6665 .WE
NUMBER LOCATED WHERE THE ROW WITH HEADING 4.6 MEETS THE COLUMN WITH
THEREFORE LOG 4.64 = 0.6665, AND WE HAVE x = 4.64.

In general, Antilog (log c) = c.

EXAMPLE 10 FIND:

A ANTILOG 0.7348 **B** ANTILOG 0.9335

C ANTILOG 0.8175 **D** ANTILOG 2.4771

SOLUTION:

A THE NUMBER 0.7348 IS FOUND IN THE TABLE WHERE ROW 5.4 AND COLUMN 3 MEET .

$\therefore$ ANTILOG 0.7348 = 5.43 .

B THE NUMBER 0.9335 IS FOUND IN THE TABLE WHERE ROW 8.5 AND COLUMN 8 MEET.

$\therefore$ ANTILOG 0.9335 = 8.58 .

c THE NUMBER 0.8175 DOES NOT APPEAR IN THE TABLE. THE CLOSEST VALUE IS AND 0.8176 = LOG 6.57.

$\therefore$ ANTILOG 0.8175 CAN BE APPROXIMATED BY 6.57 .

d ANTILOG 2.4771 = ANTILOG (0.4771 + 2) = 3×10^2 = 300

(*The antilogarithm of the decimal part* 0.4771 *is found using the table of logarithms and equals* 3. *The antilogarithm of* 2 *is* 10^2 *because* LOG 10^2 = 2.)

EXAMPLE 11 FIND:

a ANTILOG 3.9058 **b** ANTILOG 5.9586. **c** ANTILOG (−1.0150)

SOLUTION:

a ANTILOG 3.9058 = ANTILOG (0. 9058 + 3) = 8.05×10^3 = 8050.

b ANTILOG 5.9586 = ANTILOG (0.9586 + 5) = 9.09×10^5 = 909000.

c ANTILOG(−1.0150) = ANTILOG(2 − 1.0150 − 2) = ANTILOG(0.9850 − 2)

$= 9.66 \times 10^{-2} = 0.0966.$

Note: DO NOT WRITE −1.0150 AS 0.0150. THE ARITHMETIC IS NOT CORRECT!

Computation with logarithms

IN THIS SECTION YOU WILL SEE HOW LOGARITHMS ARE USED IN COMPUTATIONS.

FOR INSTANCE, TO FIND THE PRODUCT OF 32 AND 128 USING LOGARITHM TO THE BASE 2 IT AS FOLLOWS:

LET $x = 32 \times 128$

$\log_2 x = \log_2(32 \times 128)$ .... WHY?

$\log_2 x = \log_2 32 + \log_2 128$ .... WHY?

$\log_2 x = 5 + 7 \Rightarrow \log_2 x = 12$ .... WHY?

$\therefore x = 2^{12}$

IN THE NEXT EXAMPLES YOU WILL SEE HOW COMMON LOGARITHMS ARE USED IN MATH COMPUTATIONS:

REMEMBER THAT ANTILOG (LOG c) = c

In order to compute c you can perform the following two steps:

Step 1 FIND LOG c USING THE LAWS OF LOGARITHMS.

Step 2 FIND THE ANTILOGARITHM OF LOG c

EXAMPLE 12 COMPUTE $\dfrac{354 \times 605}{8450}$ USING LOGARITHMS.

SOLUTION:

Step 1 LET $x = \dfrac{354 \times 605}{8450}$

$\text{LOG } x = \text{LOG } \dfrac{354 \times 605}{8450}$

$\text{LOG } x = \text{LOG } (354 \times 605) - \text{LOG } 8450$

$\text{LOG } x = \text{LOG } 354 + \text{LOG } 605 - \text{LOG } 8450$

$\text{LOG } x = (0.5490 + 2 + 0.7818 + 2) - (0.9269 + 3)$

$\text{LOG } x = 0.4039 + 1$

SO $x = \text{ANTILOG } (0.4039 + 1) \Rightarrow x \approx 2.53 \times 10 \approx 25.3$

$\therefore \dfrac{354 \times 605}{8450} \approx 25.3$

EXAMPLE 13 COMPUTE $\sqrt{35}$ USING LOGARITHMS.

SOLUTION: LET $x = \sqrt{35}$

$\text{LOG } x = \text{LOG } \sqrt{35} \Rightarrow \text{LOG } x = \text{LOG } 35^{\frac{1}{2}} \Rightarrow \text{LOG } x = \dfrac{1}{2}[\text{LOG } 3.5 + 1]$

$\text{LOG } x = \dfrac{1}{2}[0.5441 + 1] \Rightarrow \text{LOG } x \approx 0.77205$; $\text{LOG } x \approx 0.7721$

SO $x = \text{ANTILOG } (0.7721) \Rightarrow x \approx 5.92$

$\therefore \sqrt{35} \approx 5.92$

EXAMPLE 14 COMPUTE $380^{\frac{1}{3}}$ USING LOGARITHMS.

SOLUTION: LET $x = 380^{\frac{1}{3}}$

$\text{LOG } x = \text{LOG } 380^{\frac{1}{3}}$; $\text{LOG } x = \dfrac{1}{3}[\text{LOG } 3.80 + 2]$; $\text{LOG } x = \dfrac{1}{3}[0.5798 + 2]$;

$\text{LOG } x = 0.8599$ SO $x = \text{ANTILOG } (0.8599) \Rightarrow x \approx 7.24$ $\therefore 380^{\frac{1}{3}} \approx 7.24$

Group Work 2.5

DISCUSS

1. WHICH BASE IS PREFERABLE FOR MATHEMATICAL CALCULATIONS? WHY? PRESENT YOUR FINDINGS TO YOUR GROUP.
2. APPROXIMATE $\sqrt{3}$ USING LOGARITHM.
3. USE YOUR RESULT IN 2 TO COMPUTE $10^{\sqrt{3}}$. COMPARE YOUR RESULTS. WHAT DIFFERENCES DO YOU GET?

Exercise 2.5

1 FIND EACH OF THE FOLLOWING COMMON LOGARITHMS:

A $\log(100 \times \sqrt[4]{10})$ **B** $\log\frac{100}{\sqrt{10}}$ **C** $\log\frac{1}{\sqrt[4]{10}}$ **D** $\log\left(\frac{10^m}{10^n}\right)$

2 IDENTIFY THE CHARACTERISTIC AND MANTISSA OF THE LOGARITHM OF EACH OF THE FOLLOWING:

A 0.000402 **B** 203 **C** 5.5 **D** 2190

E $\frac{1}{4}$ **F** 8 **G** 23 **H** 35.902

3 USE THE TABLE OF LOGARITHMS TO FIND:

A LOG 3.12 **B** LOG 1.99 **C** LOG 7.2 **D** LOG 5.436

E LOG 0.12 **F** LOG 9.99 **G** LOG 0.00007 **H** LOG 300

4 FIND:

A ANTILOG 0.8998 **B** ANTILOG 0.8 **C** ANTILOG 1.3010

D ANTILOG 0.9953 **E** ANTILOG 5.721 **F** ANTILOG 1.9999

G ANTILOG (-6) **H** ANTILOG(-0.2)

5 COMPUTE USING LOGARITHMS:

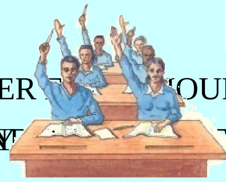
A 6.24×37.5 **B** $\sqrt[9]{125}$ **C** $2^{1.42}$

D $(2.4)^{1.3} \times (0.12)^{4.1}$ **E** $\frac{37.9\sqrt{488}}{(1.28)^3}$ **F** $\sqrt[5]{0.0641}$

2.2 THE EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

IN THIS SECTION YOU WILL DRAW GRAPHS AND INVESTIGATE THE MAJOR PROPERTIES OF FUNCTIONS OF THE FORM $f(x) = 2^x$, $f(x) = 10^x$, $f(x) = 3^{-x}$, $f(x) = (0.5)^x$, ETC.

ACTIVITY 2.6



SUPPOSE AN AMOEBA CELL DIVIDES ITSELF INTO TWO AFTER EVERY HOUR.

A CALCULATE THE NUMBER OF CELLS CREATED BY THE END OF ONE, TWO, THREE, FOUR, FIVE, ..., t HOURS.

B COMPLETE THE FOLLOWING TABLE.

Time in hour (t)	0	1	2	3	4	5	. . .	t
Number of cells created (y)	1							

C WRITE A FORMULA TO CALCULATE THE NUMBER OF CELLS CREATED AFTER t HOURS.

THE FUNCTION $f(x) = b^x$, $b > 0$ AND $b \neq 1$ DEFINES AN EXPONENTIAL FUNCTION.

THE FOLLOWING FUNCTIONS ARE ALL EXPONENTIAL:

A $f(x) = 2^x$ **B** $g(x) = \left(\frac{3}{2}\right)^x$ **C** $h(x) = 3^x$ **D** $k(x) = 10^x$

E $f(x) = \left(\frac{1}{10}\right)^x$ **F** $g(x) = \left(\frac{1}{3}\right)^x$ **G** $h(x) = \left(\frac{1}{2}\right)^x$ **H** $k(x) = \left(\frac{2}{3}\right)^x$

2.2.1 Graphs of Exponential Functions

LET US NOW CONSIDER THE GRAPHS OF SOME OF THE ABOVE EXPONENTIAL FUNCTIONS CAN EXPLORE SOME OF THEIR PROPERTIES.

EXAMPLE 1 DRAW THE GRAPH OF $f(x) = 2^x$.

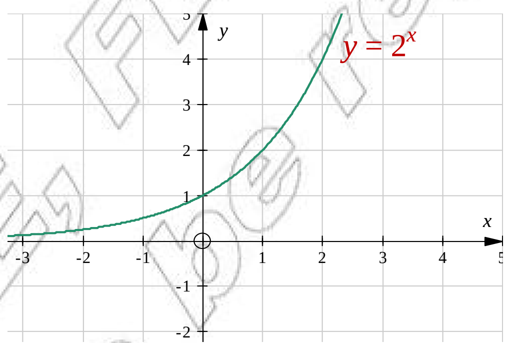
SOLUTION: EVALUATE $y = 2^x$ FOR SOME INTEGRAL VALUES OF x AND PREPARE A TABLE OF VALUES.

FOR EXAMPLE: $f(-3) = 2^{-3} = \frac{1}{8}$; $f(-2) = 2^{-2} = \frac{1}{4}$; $f(-1) = 2^{-1} = \frac{1}{2}$;

$f(0) = 2^0 = 1$; $f(1) = 2^1 = 2$; $f(2) = 2^2 = 4$; $f(3) = 2^3 = 8$.

x	−3	−2	−1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

NOW PLOT THESE POINTS ON THE CO-ORDINATE SYSTEM AND JOIN THEM BY A SMOO OBTAIN THE GRAPH OF $f(x) = 2^x$.



Graph of $f(x) = 2^x$

Figure 2.1

ACTIVITY 2.7

1 WHAT IS THE DOMAIN OF THE FUNCTION $f(x) = 2^x$?

2 FOR WHAT VALUES OF x IS 2^x NEGATIVE?

3 CAN 2^x EVER BE 0?

4 WHAT IS THE RANGE OF THE FUNCTION $f(x) = 2^x$?

5 WHAT IS THE y-INTERCEPT OF $f(x) = 2^x$?

6 FOR WHICH VALUES OF x IS 2^x GREATER THAN 1?

7 WHAT CAN YOU SAY ABOUT THE VALUE OF 2^x IF $x < 0$?

8 DOES 2^x INCREASE AS x INCREASES?

9 WHAT HAPPENS TO THE GRAPH OF f WHEN WE TAKE LARGER AND LARGER POSITIVE VALUES OF x?

10 WHAT HAPPENS TO THE GRAPH OF f IF x IS NEGATIVE AND $|x|$ IS VERY LARGE?

11 DOES THE GRAPH CROSS THE x-AXIS?

12 WHAT IS THE ASYMPTOTE OF THE GRAPH OF $f(x) = 2^x$?

EXAMPLE 2 DRAW THE GRAPH OF $g(x) = \left(\frac{3}{2}\right)^x$

SOLUTION:

x	−3	−2	−1	0	1	2	3
$g(x) = \left(\frac{3}{2}\right)^x$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

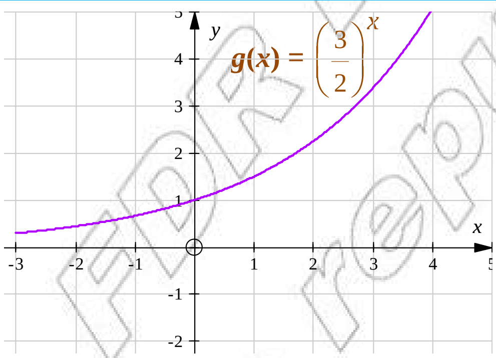


Figure 2.2 Graph of $g(x) = \left(\frac{3}{2}\right)^x$

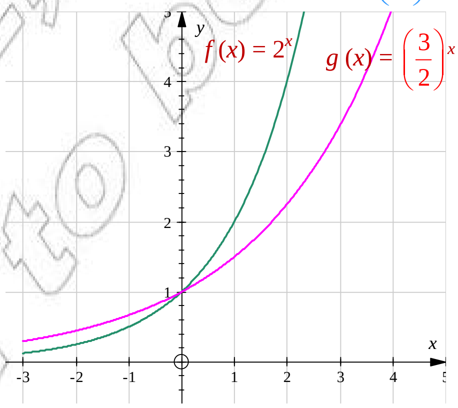


Figure 2.3 Graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{3}{2}\right)^x$ drawn using the same co-ordinate system

IN GENERAL, THE GRAPH OF $f(x) = b^x$, FOR ANY $b > 1$ HAS SIMILAR SHAPE AS THE GRAPHS OF $y = 2^x$ AND $y = \left(\frac{3}{2}\right)^x$.

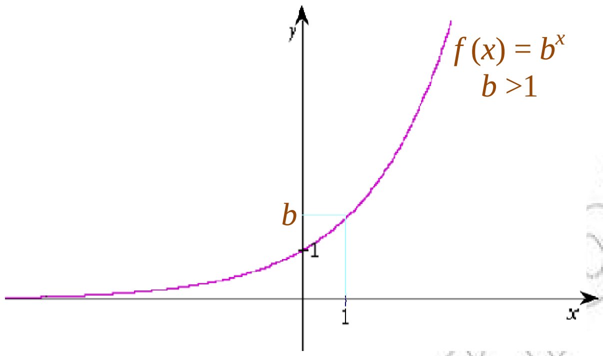


Figure 2.4 Graph of $f(x) = b^x$, for any $b > 1$

Basic properties

THE GRAPH OF $f(x) = b^x$, $b > 1$ *has the following basic properties:*

1. THE DOMAIN IS THE SET OF ALL REAL NUMBERS.
2. THE RANGE IS THE SET OF ALL POSITIVE REAL NUMBERS.
3. THE GRAPH INCLUDES THE POINT (0, 1), I.E. THE y-INTERCEPT IS 1.
4. THE FUNCTION IS INCREASING.
5. THE VALUES OF THE FUNCTION ARE GREATER THAN 1 FOR $x > 0$ AND BETWEEN 0 AND 1 FOR $x < 0$.
6. THE GRAPH APPROACHES THE x-AXIS AS AN ASYMPTOTE ON THE LEFT AND INCREASES WITHOUT BOUND ON THE RIGHT.

WE WILL NEXT DISCUSS HOW THE GRAPH OF THE FUNCTION $f(x) = b^x$ LOOKS LIKE WHEN $0 < b < 1$.

EXAMPLE 3 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

i DIFFERENT COORDINATE AXES. **ii** THE SAME COORDINATE AXES.

A $h(x) = \left(\frac{1}{2}\right)^x$ **B** $k(x) = \left(\frac{2}{3}\right)^x$

SOLUTION: AS BEFORE, CALCULATE THE VALUES OF THE GIVEN FUNCTION FOR SOME INTEGRAL VALUES OF x AS SHOWN IN THE TABLES BELOW. THEN PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY SMOOTH CUR GET THE GRAPHS AS INDICATED BELOW.

A

x	−3	−2	−1	0	1	2	3
$H(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

B

x	-3	-2	-1	0	1	2	3
$k(x)=\left(\frac{2}{3}\right)^x$	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$

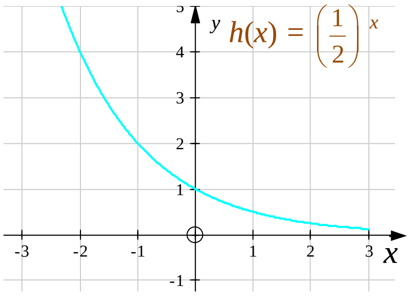


Figure 2.5 Graph of $h(x)=\left(\frac{1}{2}\right)^x$

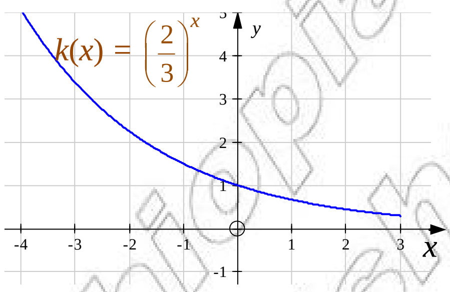


Figure 2.6 Graph of $k(x)=\left(\frac{2}{3}\right)^x$

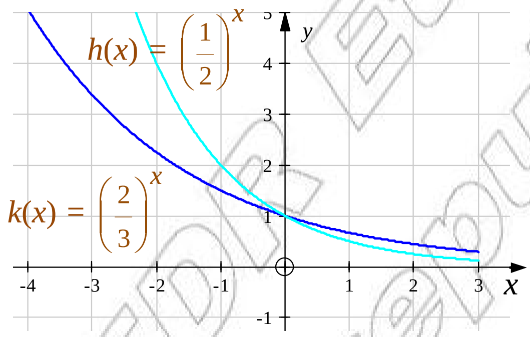


Figure 2.7 Graphs of $h(x)=\left(\frac{1}{2}\right)^x$ and $k(x)=\left(\frac{2}{3}\right)^x$ drawn using the same coordinate axes

THE GRAPH OF $f(x) = b^x$, FOR ANY $0 < b < 1$ HAS SIMILAR SHAPE TO THE GRAPHS OF $y=\left(\frac{1}{2}\right)^x$ AND $y=\left(\frac{2}{3}\right)^x$.

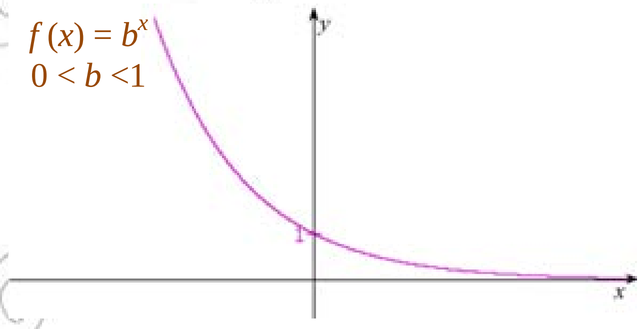


Figure 2.8 Graph of $f(x) = b^x$, for any $0 < b < 1$

Basic properties

THE GRAPH OF $f(x) = b^x$, $0 < b < 1$ *has the following basic properties:*

1 THE DOMAIN IS THE SET OF ALL REAL NUMBERS.

2 THE RANGE IS THE SET OF ALL POSITIVE REAL NUMBERS.

3 THE GRAPH INCLUDES THE POINT (0, 1), I.E., THE y-INTERCEPT IS 1.

4 THE FUNCTION IS DECREASING.

5 THE VALUES OF THE FUNCTION ARE GREATER THAN 1 FOR $x < 0$ AND BETWEEN 0 AND 1 FOR $x > 0$.

6 THE GRAPH APPROACHES THE x-AXIS AS AN ASYMPTOTE ON THE RIGHT AND INCREASES WITHOUT BOUND ON THE LEFT.

Exercise 2.6

1 GIVE THREE EXAMPLES OF EXPONENTIAL FUNCTIONS.

2 GIVEN THE GRAPH OF $y = 2^x$ (*see* FIGURE 2.9), WE CAN FIND APPROXIMATE VALUES OF 2^x FOR VARIOUS VALUES OF x. FOR EXAMPLE,

$2^{1.8} \approx 3.5$ (SEE POINT A).

$2^{2.3} \approx 5$ (SEE POINT B).

USE THE GRAPH TO DETERMINE APPROXIMATE VALUES OF

A $2^{\frac{1}{2}}$ (I.E. $\sqrt{2}$) **B** $2^{0.8}$ **C** $2^{1.5}$ **D** $2^{-1.6}$.

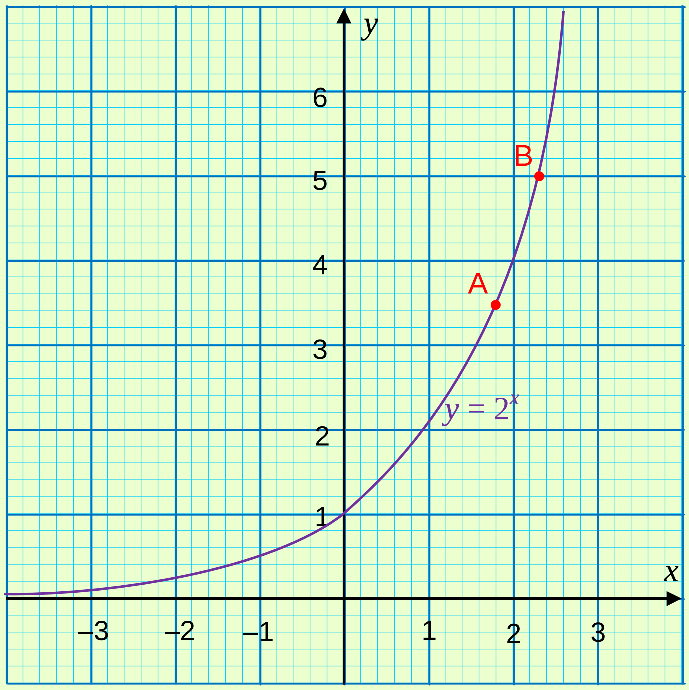


Figure 2.9

3 CONSTRUCT SUITABLE TABLES OF VALUES AND HENCE DRAW THE GRAPHS OF

A $h(x) = 3^x$ AND $g(x) = \left(\frac{1}{3}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

B $k(x) = 10^x$ AND $f(x) = \left(\frac{1}{10}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

C $f(x) = 4^x$ AND $g(x) = \left(\frac{1}{4}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

4 REFERRING TO THE FUNCTIONS IN QUESTION 3:

A FIND THE DOMAIN AND THE RANGE OF EACH FUNCTION,

B WHAT IS THE y-INTERCEPT OF EACH FUNCTION?

C WHICH FUNCTIONS ARE INCREASING AND WHICH ARE DECREASING?

D FIND THE ASYMPTOTE FOR EACH GRAPH

The exponential function with base *e*

UNTIL NOW THE NUMBER π HAS PROBABLY BEEN THE MOST IMPORTANT IRRATIONAL NUMBER YOU HAVE ENCOUNTERED. NEXT, WE WILL INTRODUCE ANOTHER USEFUL IRRATIONAL NUMBER, e, WHICH IS IMPORTANT IN THE FIELD OF MATHEMATICS AND OTHER SCIENCES.

2.2.2 The Number *e*

DO YOU KNOW THAT SOME BANKS CALCULATE INTEREST EVERY MONTH? THIS IS CALLED ***monthly compounding***. OTHER BANKS EVEN ADVERTISE ***continuous*** COMPOUNDING. TO ILLUSTRATE THE IDEA OF *continuous* COMPOUNDING, WE WILL STUDY HOW 1 BIRR GROWS AFTER 1 YEAR AT 100 PERCENT ANNUAL INTEREST, USING VARIOUS PERIODS OF COMPOUNDING.

IN THIS CASE, WE USE THE AMOUNT FORMULA $A = P(1 + i)^n$, WHERE THE PRINCIPAL $P = 1$.

TAKING THE ANNUAL RATE $r = 100\% = 1$, $i = \frac{1}{n}$ IF THERE ARE n PERIODS OF COMPOUNDING PER YEAR, THEN THE AMOUNT AFTER 1 YEAR IS GIVEN BY THE FORMULA:

$$A = \left(1 + \frac{1}{n}\right)^n$$

THE FOLLOWING TABLE GIVES THE AMOUNTS (IN BIRR) AFTER 1 YEAR USING VARIOUS PERIODS OF COMPOUNDING.

Number of compounding periods per year	Amount after one year
yearly	$\left(1+\frac{1}{1}\right)^{1}=2$
semi-annually	$\left(1+\frac{1}{2}\right)^{2}=2.25$
quarterly	$\left(1+\frac{1}{4}\right)^{4}=2.44140625$
monthly	$\left(1+\frac{1}{12}\right)^{12}\approx 2.61303529022...$
weekly	$\left(1+\frac{1}{52}\right)^{52}\approx 2.69259695444...$
daily	$\left(1+\frac{1}{365}\right)^{365}\approx 2.71456748202...$
hourly	$\left(1+\frac{1}{8760}\right)^{8760}\approx 2.71812669063...$
every minute	$\left(1+\frac{1}{525600}\right)^{525600}\approx 2.7182792154...$
every second	$\left(1+\frac{1}{31536000}\right)^{31536000}=2.7182817853...$

THE LAST ROW OF THE ABOVE TABLE SHOWS THE EFFECT OF COMPOUNDING APPROXI SECOND. THE IDEA OF CONTINUOUS COMPOUNDING IS THAT THE TABLE IS CONTINUED F LARGER VALUES OF n. AS n CONTINUES TO INCREASE, THE AMOUNT AFTER 1 YEAR TENDS TOWARDS THE NUMBER ***2.718281828459...***

THIS IRRATIONAL NUMBER IS REPRESENTED BY THE LETTER e.

$$e = 2.718281828459...$$

e IS THE NUMBER THAT $\left(1+\frac{1}{n}\right)^{n}$ APPROACHES AS $n \to \infty$. WHO FIRST DISCOVERED e IS STILL BEING DEBATED. THE NUMBER IS NAMED AFTER THE SWISS MATHEMATICIAN LEO (1707 – 1783), WHO COMPUTED e TO 23 DECIMAL PLACES USING $\left(1+\frac{1}{n}\right)^{n}$.

2.2.3 The Natural Exponential Function

FOR ANY REAL NUMBER x, THE FUNCTION $f(x) = e^x$ DEFINES THE EXPONENTIAL FUNCTION WITH BASE e, USUALLY CALLED THE ***natural exponential function***.

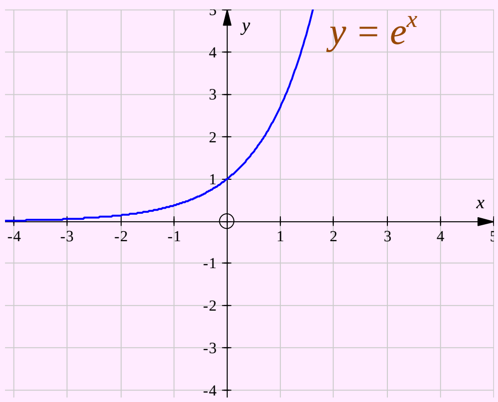


Figure 2.10 The graph of $y = e^x$.

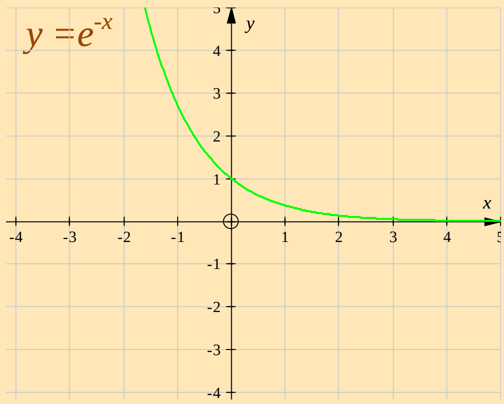


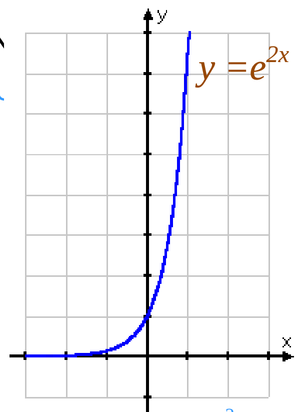
Figure 2.11 The graph of $y = e^{-x}$.

THE DOMAIN OF $y = e^x$ is $\mathbb{R}$.	THE DOMAIN OF $y = e^{-x}$ IS $\mathbb{R}$
THE RANGE IS $(0, \infty)$	THE RANGE IS $(0, \infty)$
$y = e^x$ IS AN INCREASING FUNCTION.	$y = e^{-x}$ IS A DECREASING FUNCTION.
THE GRAPH OF $y = e^x$ INTERSECTS THE y-AXIS AT $(0, 1)$.	THE GRAPH OF $y = e^{-x}$ INTERSECTS THE y-AXIS AT $(0, 1)$.
$e^x > 1$, $if\ x > 0$	$e^{-x} > 1$, $if\ x < 0$
$0 < e^x < 1$, $if\ x < 0$	$0 < e^{-x} < 1$, $if\ x > 0$

EXAMPLE 1 SKETCH THE GRAPH OF $y = e^{2x}$.

SOLUTION: WE CALCULATE AND PLOT SOME POINTS AND CONNECT THEM TO OBTAIN THE REQUIRED GRAPH, AS SHOWN IN FIGURE 2.12.

x	$y = e^{2x}$
–3	≈ 0.0025
–2	≈ 0.0183
–1	≈ 0.1353
0	= 1
1	≈ 7.7391
2	≈ 54.5981



Graph of $y = e^{2x}$
Figure 2.12

Exercise 2.7

1 SKETCH THE GRAPHS OF EACH OF THE FOLLOWING FUNCTIONS:

   a $f(x) = 2^{x-1}$ b $g(x) = 3^{x-2}$ c $k(x) = 3^{2-x}$

2 USE THE e^x KEY ON YOUR CALCULATOR TO EVALUATE EACH OF THE FOLLOWING EXPR 7 DECIMAL PLACES:

   a e^3 b $e^{\sqrt{3}}$ c $e^{-7.3011}$ d $e^{\sqrt{5}}$

3 CONSTRUCT TABLES OF VALUES FOR SOME INTEGER VALUES OF x AND THEN GRAPH EACH OF THE FOLLOWING FUNCTIONS:

   a $y = -e^x$ b $y = -e^{-x}$ c $y = 10e^{0.2x}$

4 STATE THE DOMAIN AND RANGE OF EACH OF THE FUNCTIONS IN QUESTION 3.

2.3 THE LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

FROM SECTION 2.1.2 YOU SHOULD REMEMBER THAT $b^y = x$, IF AND ONLY IF $\log_b x = y$ ($b > 0$, $b \neq 1$ AND $x > 0$)

HENCE, THE FUNCTION $y = \log_b x$, WHERE $x > 0$, $b > 0$ AND $b \neq 1$ IS CALLED ***a logarithmic function with base b.***

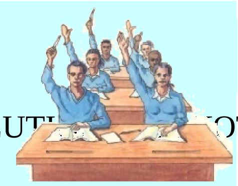
THE FOLLOWING FUNCTIONS ARE ALL LOGARITHMIC:

   A $f(x) = \log_2 x$ B $g(x) = \log_{\frac{3}{2}} x$ C $h(x) = \log_3 x$

   D $k(x) = \log_{10} x$ E $f(x) = \log_{\frac{1}{10}} x$ F $g(x) = \log_{\frac{1}{3}} x$

   G $h(x) = \log_{\frac{1}{2}} x$ H $k(x) = \log_{\frac{2}{3}} x$

ACTIVITY 2.8



THE CONCENTRATION OF HYDROGEN IONS IN A GIVEN SOLUTION IS DENOTED BY $[H^+]$ AND IS MEASURED IN MOLES PER LITER.

FOR EXAMPLE, $[H^+] = 0.0000501$ FOR BEER AND $[H^+] = 0.0004$ FOR WINE.

CHEMISTS DEFINE THE PH OF THE SOLUTION AS THE NUMBER PH $= -\log[H^+]$. THE SOLUTION IS SAID TO BE AN ACID IF PH < 7 AND A BASE IF PH > 7. PURE WATER HAS A PH OF 7, WHICH IT IS NEUTRAL.

   A IS BEER AN ACID OR A BASE? WHAT ABOUT WINE?

   B WHAT IS THE HYDROGEN ION CONCENTRATION $[H^+]$ OF EGGS IF THE PH OF EGGS IS 7.8?

2.3.1 Graphs of Logarithmic Functions

IN THIS SECTION, WE CONSIDER THE GRAPHS OF SOME LOGARITHMIC FUNCTIONS, SO
EXPLORE THEIR PROPERTIES.

EXAMPLE 1 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

I DIFFERENT COORDINATE SYSTEMS **II** THE SAME COORDINATE SYSTEM.

A $f(x) = \log_2 x$ **B** $g(x) = \log_{\frac{3}{2}} x$.

SOLUTION: THE TABLES BELOW INDICATE SOME VALUES OF $f(x)$ AND $g(x)$. PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY CURVES TO GET THE REQUIRED GRAPHS AS INDICATED IN FIGURE 2.13 AND 2.14.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = \log_2 x$	−2	−1	0	1	2

x	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$
$g(x) = \log_{\frac{3}{2}} x$	−2	−1	0	1	2

A

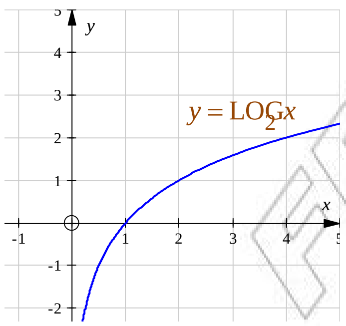


Figure 2.13 Graph of $f(x) = \log_2 x$

B

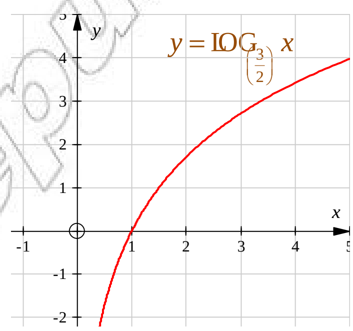


Figure 2.14 Graph of $g(x) = \log_{\left(\frac{3}{2}\right)} x$

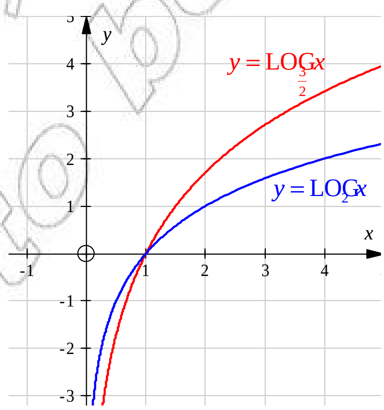
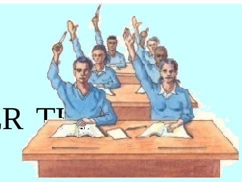


Figure 2.15 Graphs of $y = \log_2 x$ and $y = \log_{\left(\frac{3}{2}\right)} x$ drawn using the same coordinate axes

ACTIVITY 2.9



STUDY THE GRAPHS OF $f(x)=\log_2 x$ AND $g(x)=\log_{\left(\frac{3}{2}\right)} x$ TO ANSWER THE FOLLOWING QUESTIONS:

1 WHAT ARE THE DOMAINS OF f AND g?

2 FOR WHICH VALUES OF x IS $\log_2 x$ NEGATIVE? POSITIVE?

3 FOR WHICH VALUES OF x IS $\log_{\frac{3}{2}} x$ NEGATIVE? POSITIVE?

4 WHAT IS THE RANGE OF f AND g?

5 WHAT IS THE x-INTERCEPT?

6 DOES $\log_2 x$ INCREASE AS x INCREASES? WHAT ABOUT $\log_{\frac{3}{2}} x$?

7 DO THE GRAPHS CROSS THE y-AXIS?

8 WHAT IS THE ASYMPTOTE OF THE GRAPHS?

IN GENERAL, THE GRAPH OF $f(x)=\log_b x$, FOR ANY $b>1$ LOOKS LIKE THE ONE GIVEN BELOW.

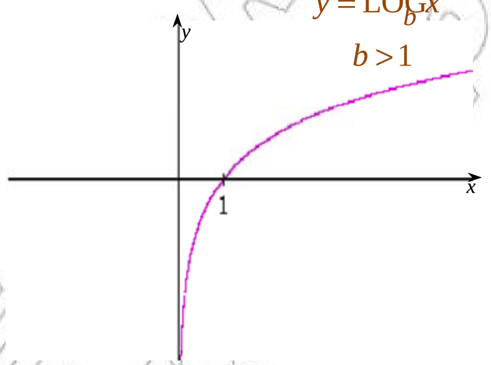


Figure 2.16 Graph of $y=\log_b x$

Basic properties

THE GRAPH OF $y=\log_b x$ ($b>1$) HAS THE FOLLOWING PROPERTIES.

1 THE DOMAIN IS THE SET OF ALL POSITIVE REAL NUMBERS.

2 THE RANGE IS THE SET OF ALL REAL NUMBERS.

3 THE GRAPH INCLUDES THE POINT (1, 0). I.E. THE x-INTERCEPT OF THE GRAPH IS 1.

4 THE FUNCTION INCREASES, AS x INCREASES.

5 THE y-AXIS IS A VERTICAL ASYMPTOTE OF THE GRAPH.

6 THE VALUES OF THE FUNCTION ARE NEGATIVE FOR $x<1$ AND THEY ARE POSITIVE FOR $x>1$.

YOU WILL NEXT DISCUSS WHAT THE GRAPH OF THE FUNCTION $y = \log_b x$ LOOKS LIKE WHEN $0 < b < 1$.

EXAMPLE 2 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

I DIFFERENT COORDINATE SYSTEMS; **II** THE SAME COORDINATE SYSTEM.

A $h(x) = \log_{\frac{1}{2}} x$ **B** $k(x) = \log_{\frac{2}{3}} x$

SOLUTION: CALCULATE THE VALUES OF THE GIVEN FUNCTIONS FOR SOME VALUES OF x AS SHOWN IN THE TABLES BELOW. THEN PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY SMOOTH CURVES TO GET THE REQUIRED GRAPHS AS INDICATED IN FIGURE 2.17 AND 2.18.

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$h(x) = \log_{\frac{1}{2}} x$	−3	−2	−1	0	1	2	3

x	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$
$k(x) = \log_{\frac{2}{3}} x$	−3	−2	−1	0	1	2	3

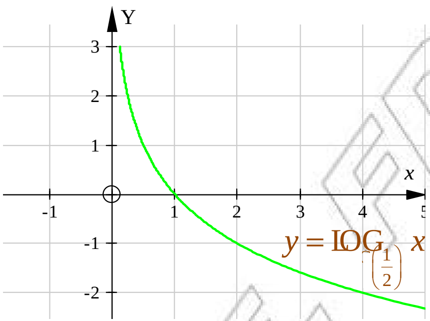


Figure 2.17 Graph of $h(x) = \log_{\frac{1}{2}} x$

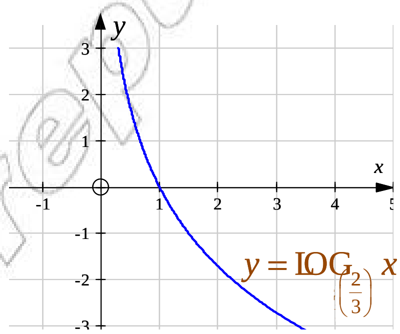


Figure 2.18 Graph of $k(x) = \log_{\frac{2}{3}} x$

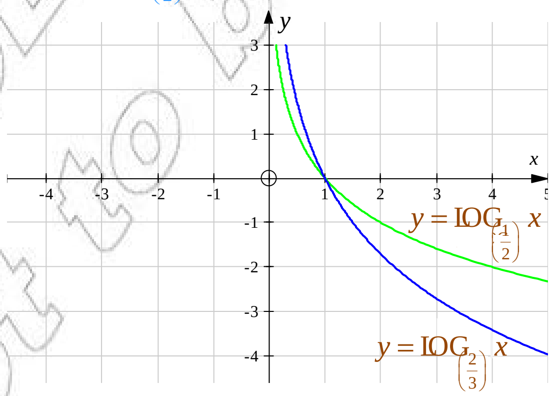


Figure 2.19 Graphs of $y = \log_{\frac{1}{2}} x$ and $y = \log_{\frac{2}{3}} x$ drawn using the same coordinate axes

IN GENERAL, THE GRAPH OF $f(x) = \log_b x$ FOR $0 < b < 1$ LOOKS LIKE THE ONE GIVEN BELOW.

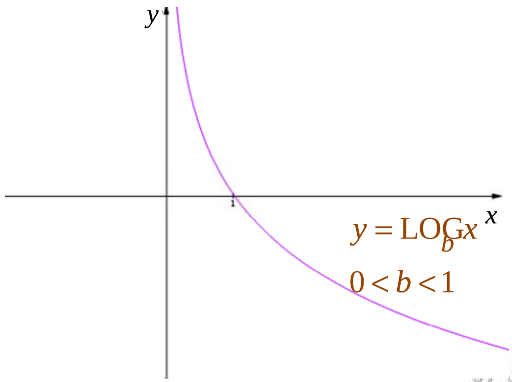


Figure 2.20

Basic properties

THE GRAPH OF $y = \log_b x$, ($0 < b < 1$) HAS THE FOLLOWING PROPERTIES.

1 THE DOMAIN IS THE SET OF ALL POSITIVE REAL NUMBERS.

2 THE RANGE IS THE SET OF ALL REAL NUMBERS.

3 THE GRAPH HAS x-INTERCEPT AT (1, 0) I.E. THE x-INTERCEPT IS 1.

4 THE FUNCTION DECREASES AS x INCREASES.

5 THE y-AXIS IS AN ASYMPTOTE OF THE GRAPH.

6 THE VALUES OF THE FUNCTION ARE POSITIVE WHEN $0 < x < 1$ AND THEY ARE NEGATIVE WHEN $x > 1$.

Exercise 2.8

1 DRAW THE GRAPHS OF:

A $h(x) = \log_3 x$ AND $g(x) = \log_{\left(\frac{1}{3}\right)} x$ USING THE SAME CO-ORDINATE SYSTEM.

B $k(x) = \log_{10} x$ AND $f(x) = \log_{\left(\frac{1}{10}\right)} x$ USING THE SAME CO-ORDINATE SYSTEM.

2 REFERRING TO THE FUNCTIONS IN QUESTION 1,

A WHAT ARE THE DOMAIN AND THE RANGE OF EACH FUNCTION?

B WHAT IS THE x-INTERCEPT OF EACH?

C WHICH FUNCTIONS ARE INCREASING AND WHICH ARE DECREASING?

D FIND THE ASYMPTOTES OF THE GRAPHS OF THE FUNCTIONS.

2.3.2 The Relationship Between the Functions $y = b^x$ and $y = \log_b x$ $(b > 0,\ b \neq 1)$

CONSIDER THE FOLLOWING TABLES OF VALUES, RESPECTIVELY CONSTRUCTED IN THE PREVIOUS SECTION FOR $y = 2^x$ AND $y = \log_2 x$.

x	–3	–2	–1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_2 x$	–3	–2	–1	0	1	2	3

ACTIVITY 2.10

REFER TO THE TABLES OF VALUES FOR $y = 2^x$ AND $y = \log_2 x$ TO ANSWER THE FOLLOWING QUESTIONS:

1. HOW ARE THE VALUES OF x AND y RELATED IN THE FUNCTIONS $y = 2^x$ AND $y = \log_2 x$?
2. SKETCH THE GRAPHS OF THE TWO FUNCTIONS USING THE SAME CO-ORDINATE SYSTEM.
3. FIND A RELATIONSHIP BETWEEN THE DOMAINS AND THE RANGES OF THE TWO FUNCTIONS.
4. DRAW THE LINE $y = x$ USING THE SAME CO-ORDINATE SYSTEM.
5. HOW ARE THE GRAPHS OF $y = 2^x$ AND $y = \log_2 x$ RELATED?
6. WHAT IS THE SIGNIFICANCE OF THE LINE $y = x$?

EXAMPLE 1 LET US CONSIDER THE FUNCTIONS $y = 10^x$ AND $y = \log_{10} x$.

THE TABLES OF VALUES FOR $y = 10^x$ AND $y = \log_{10} x$ FOR SOME INTEGRAL VALUES OF x ARE GIVEN BELOW:

x	–2	–1	0	1	2
$y = 10^x$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100

x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100
$y = \log_{10} x$	–2	–1	0	1	2

OBSERVE THAT:

THE VALUES OF x AND y ARE INTERCHANGED IN BOTH FUNCTIONS. THAT IS, THE DOMAIN OF $y = 10^x$ IS THE RANGE OF $y = \log_{10} x$ AND VICE VERSA.

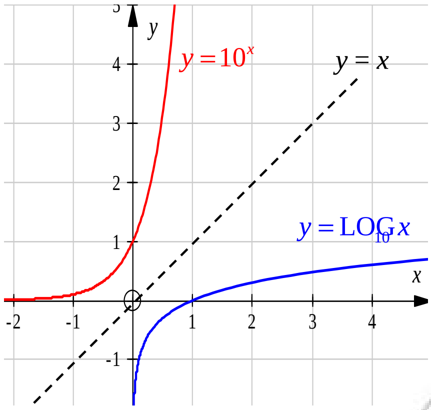


Figure 2.21

$y = 10^x$ IS OBTAINED BY REFLECTING $y = \log_{10} x$ ALONG THE LINE $y = x$.

IN SUCH CASES WE SAY ONE OF THE FUNCTIONS IS THE INVERSE OF THE OTHER.

IN GENERAL, THE RELATION BETWEEN THE FUNCTIONS $y = b^x$ AND $y = \log_b x$ ($b > 1$) IS SHOWN BELOW:

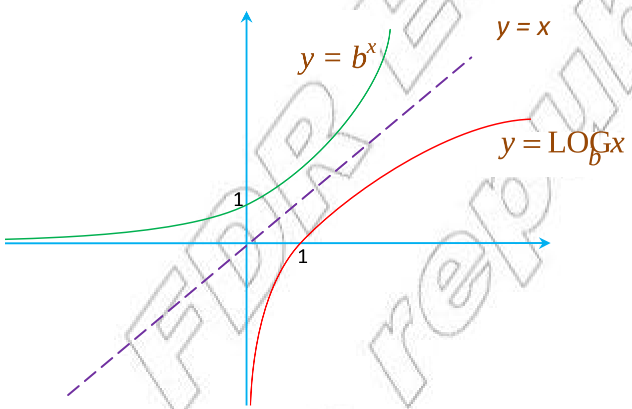


Figure 2.22

FROM THE GRAPHS ABOVE, WE OBSERVE THE FOLLOWING RELATIONSHIPS:

1 THE DOMAIN OF $y = b^x$ IS THE SET OF ALL REAL NUMBERS, WHICH IS THE SAME AS THE R OF $y = \log_b x$.

2 THE RANGE OF $y = b^x$ IS THE SET OF ALL POSITIVE REAL NUMBERS, WHICH IS THE SAME A DOMAIN OF $y = \log_b x$.

3 THE x-AXIS IS THE ASYMPTOTE OF $y = b^x$, WHEREAS THE y-AXIS IS THE ASYMPTOTE OF $y = \log_b x$.

4 $y = b^x$ HAS A y-INTERCEPT AT (0, 1) WHEREAS $y = \log_b x$ HAS AN x-INTERCEPT AT (1, 0).

DOMAIN OF $y = b^x$ IS EQUAL TO THE RANGE OF $y = \log_b x$.

RANGE OF $y = b^x$ IS EQUAL THE DOMAIN OF $y = \log_b x$.

THE FUNCTIONS $f(x) = b^x$ AND $g(x) = \log_b x$ ($b > 1$) ARE INVERSES OF EACH OTHER.

2.3.3 The Natural Logarithm

IF WE START WITH NATURAL EXPONENTIAL FUNCTION $y = e^x$ AND INTERCHANGE x AND y, WE OBTAIN $x = e^y$ WHICH IS THE SAME AS $y = \log_e x$.

$y = \log_e x$ IS THE MIRROR IMAGE OF $y = e^x$ ALONG THE LINE $y = x$.

Notation: $\ln x$ IS USED TO REPRESENT $\log_e x$.
$\ln x$ IS CALLED THE NATURAL LOGARITHM OF x.

THE GRAPHS OF $y = e^x$, $y = \ln x$ AND THE LINE $y = x$ ARE SHOWN BELOW:

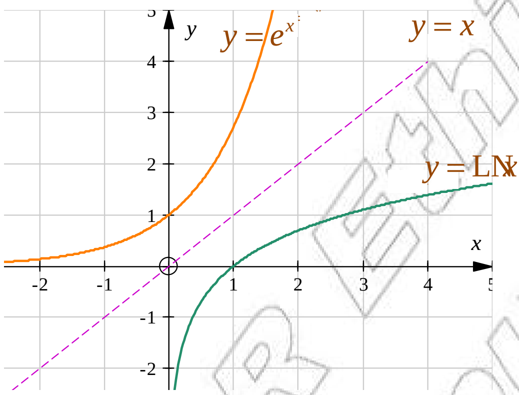


Figure 2.23

EXAMPLE 1 FIND:

A $\ln 1$ **B** $\ln e$ **C** $\ln e^2$ **D** $\ln \sqrt{e}$ **E** $\ln \frac{1}{e}$

SOLUTION:

A $\ln 1 = 0$ BECAUSE $e^0 = 1$

B $\ln e = 1$ BECAUSE $e^1 = e$

C $\ln e^2 = 2 \ln e = 2 \times 1 = 2$

D $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2} \ln e = \frac{1}{2}$

E $\ln \frac{1}{e} = \ln e^{-1} = -1 \ln e = -1$

Note: IN GENERAL, $\ln e^x = x$.

Exercise 2.9

1 SKETCH THE GRAPHS OF:

A $f(x) = 4^x$, $g(x) = \log_4 x$ AND $y = x$ USING THE SAME COORDINATE SYSTEM.

B $h(x) = \left(\frac{1}{4}\right)^x$ AND $k(x) = \log_{\left(\frac{1}{4}\right)} x$ USING THE SAME COORDINATE SYSTEM.

C HOW DO YOU COMPARE THE DOMAIN AND THE RANGE OF THE FUNCTIONS f AND g GIVEN IN QUESTION 1A?

D HOW DO YOU COMPARE THE DOMAIN AND THE RANGE OF THE FUNCTIONS h AND k GIVEN IN QUESTION 1B?

2 FIND:

A $\ln \sqrt[3]{e}$ **B** $\ln \frac{1}{e^2}$ **C** $\ln e^{3x}$ **D** $e^{\ln 3}$

3 SIMPLIFY:

A $\ln e^8$ **B** $\ln(e \times e)$ **C** $\ln(e^x \times e^y)$ **D** $\ln\left(\frac{e^x}{e^y}\right)$

2.4 EQUATIONS INVOLVING EXPONENTS AND LOGARITHMS

AN EXPONENTIAL EQUATION IS AN EQUATION WITH THE UNKNOWN IN THE EXPONENT.

EXAMPLES OF EXPONENTIAL EQUATIONS ARE:

$4^x = 8$ $4^x - 2^{x+1} - 8 = 0$

$2^{3x-2} = 5$ $9^{x^2+4x} = 3^{3x+7}$

A LOGARITHMIC EQUATION IS AN EQUATION THAT INVOLVES THE LOGARITHM OF AN UN

EXAMPLES OF LOGARITHMIC EQUATIONS ARE:

$4\log x = -5$ $\log_x(x - \ ) = 6$

$\log(x + 3) + \log x = 1$ $\log_2 x + \log_4(x + \ ) = 2$

2.4.1 Solving Exponential Equations

PROPERTIES OF EXPONENTS DISCUSSED IN THE PREVIOUS SECTIONS PLAY A MAJOR RO EXPONENTIAL EQUATIONS. READ CAREFULLY THROUGH THE PROPERTIES BELOW, TO MEMORY!

IF a AND b ARE POSITIVE NUMBERS, $a \neq 1$, $b \neq 1$, AND m AND n ARE REAL NUMBERS, THEN

1 $a^m \times a^n = a^{m+n}$ **2** $(a^m)^n = a^{mn}$

3 $(a \times b)^n = a^n \times b^n$ **4** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

5 $\frac{a^m}{a^n} = a^{m-n}$ **6** $a^{-n} = \frac{1}{a^n}$ AND $\frac{1}{a^{-n}} = a^n$

7 IT IS ALWAYS TRUE THAT FOR $k > 0$, $\left(\frac{a}{b}\right)^{-k} = \left(\frac{b}{a}\right)^k$

Additional properties:

Property of equality for exponential equations

FOR $b > 0$, $b \neq 1$, x AND y REAL NUMBERS,

1 $b^x = b^y$, IF AND ONLY IF $x = y$

2 $a^x = b^x$, $(x \neq 0)$, IF AND ONLY IF $a = b$

EXAMPLE 1 SOLVE FOR x.

A $3^x = 81$ **B** $2^x = \frac{1}{32}$ **C** $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$ **D** $4^x = \left(\frac{1}{2}\right)^{x-3}$

SOLUTION:

A $3^x = 81 = 3^4$ … *look for a common base*

$\Rightarrow x = 4$ … *property of equality of bases*

B $2^x = \frac{1}{2^5} = 2^{-5}$ … *look for a common base*

$\Rightarrow x = -5$ … *property of equality of bases*

C $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$

$\Rightarrow \left(\frac{2}{3}\right)^{2x+1} = \left(\frac{3}{2}\right)^{2x} = \left(\frac{2}{3}\right)^{-2x}$

$\Rightarrow 2x + 1 = -2x$

$\Rightarrow 2x + 2x = -1$

$\Rightarrow x = -\frac{1}{4}$

D $4^x = \left(\frac{1}{2}\right)^{x-3}$

$\Rightarrow 4^x = (2^{-1})^{x-3} = 2^{(-x+3)}$

$\Rightarrow (2^2)^x = 2^{(-x+3)}$

$\Rightarrow 2^{2x} = 2^{(-x+3)}$

$\Rightarrow 2x = -x + 3 \Rightarrow x = 1$

IF YOU CANNOT EASILY WRITE EACH SIDE OF AN EXPONENTIAL EQUATION USING THE S
CAN SOLVE THE EQUATION BY TAKING LOGARITHMS OF EACH SIDE.

EXAMPLE 2 SOLVE FOR x, BY TAKING THE LOGARITHM OF EACH SIDE:

A $4^x = 10$ **B** $2^{3x-2} = 5$ **C** $2^{2x} = 11$

SOLUTION:

A $4^x = 10$

LOG 4^x = LOG10 … *taking the logarithm of each side*

x LOG 4 = 1 … *since* LOG10 = 1, AND LOG $a^k = k$ LOG a

$x = \frac{1}{\text{LOG } 4} = \frac{1}{0.6021} = 1.6609$

B $2^{3x-2} = 5$

$\Rightarrow$ LOG $2^{(3x-2)}$ = LOG 5

$\Rightarrow$ $(3x-2)$LOG 2 = LOG 5

$\Rightarrow$ $3x-2 = \dfrac{\text{LOG}}{\text{LOG}}$

$\Rightarrow$ $3x = \dfrac{\text{LOG}}{\text{LOG}} + 2$

$\Rightarrow$ $x = \dfrac{1}{3}\left(\dfrac{\text{LOG}}{\text{LOG}} + 2\right) = 1.4408$

C $2^{2x} = 11$

$\Rightarrow$ LOG 2^{2x} = LOG 11

$\Rightarrow$ $2x$ LOG 2 = LOG 11

$\Rightarrow$ $2x = \dfrac{\text{LOG1}}{\text{LOG}}$

$\Rightarrow$ $x = \dfrac{1}{2}\left(\dfrac{\text{LOG 11}}{\text{LOG 2}}\right) = 1.730$

Exercise 2.10

1 SOLVE FOR x

A $5^x = 625$ **B** $2^{3-x} = 16$ **C** $4^{3x-8} = 2^{3x+9}$

D $\dfrac{1}{27} = \left(\dfrac{1}{9}\right)^{2X}$ **E** $3^{-x} = 81$ **F** $2^{x^2-2} = 4$

G $7^{x^2+x} = 49$ **H** $3^{6(x+2)} = 9^{x+2}$ **I** $3\left(\dfrac{27}{8}\right)^{\frac{2}{3}x-1} = 2\left(\dfrac{32}{243}\right)^{2x}$

2 SOLVE FOR x BY TAKING THE LOGARITHM OF EACH SIDE:

A $2^x = 15$ **B** $10^x = 14.3$ **C** $10^{3x+1} = 92$ **D** $1.05^x = 2$

E $6^{3x} = 5$ **F** $4^{2x} = 61$ **G** $10^{5x-2} = 348$ **H** $2^{-x} = 0.238$

2.4.2 Solving Logarithmic Equations

PROPERTIES OF LOGARITHMS DISCUSSED IN THE PREVIOUS SECTIONS PLAY A MAJOR R LOGARITHMIC EQUATIONS. REMEMBER THAT

IF a, b, c, x AND y ARE POSITIVE NUMBERS AND $a \neq 1, b \neq 1$, THEN

1 $\log_b xy = \log_b x + \log_b y$

2 $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3 FOR ANY REAL NUMBER k, $\log_b(x^k) = k\log_b x$

4 $\log_b b = 1$

5 $\log_b 1 = 0$

6 $\log_a x = \dfrac{\log_b x}{\log_b a}$ … *change of base law*

7 $b^{\log_b x} = x$

EXAMPLE 1 SOLVE EACH OF THE FOLLOWING, CHECKING THAT YOUR SOLUTIONS ARE VALID.

A $\log_2(x-3)=5$ **B** $\log_4(5x-1)=3$

C $\log(x+3)+\log x=1$ **D** $\log_3(x+1)-\log_3(x+3)=1$

E $\log 8x+\log(x-20)=3$

SOLUTION:

A $\log_2(x-3)=5 \Rightarrow 2^5 = x-3$ … *changing to exponential form*

HENCE, $32 = x-3$

THEREFORE, $x = 35$

Check!

FROM THE DEFINITION OF LOGARITHMS, WE KNOW THAT $\log_2(x-3)$ IS VALID ONLY WHEN $x-3>0$, I.E. WHEN $x>3$. SO $\{x \mid x>3\} = (3, \infty)$ IS KNOWN AS THE ***universe*** FOR $\log_2(x-3)$. SINCE $x = 35$ IS AN ELEMENT OF THE UNIVERSE, $x = 35$ IS THE SOLUTION OF THE GIVEN EQUATION.

A UNIVERSE IS THE LARGEST SET FOR WHICH THE GIVEN EXPRESSION IS DEFINED.

B $\log_4(5x-1)$ IS VALID WHEN $5x-1>0$

SO $x>\frac{1}{5}$. THEREFORE, THE UNIVERSE U $=\left(\frac{1}{5},\infty\right)$

$\log_4(5x-1)=3$

$\Rightarrow\ 5x-1=4^3$

$\Rightarrow\ 5x=64+1$

$\Rightarrow\ x=\frac{65}{5}=13$. SINCE $13\in\left(\frac{1}{5},\infty\right)$, $x = 13$ IS THE SOLUTION.

C REMEMBER THAT $\log(x+3)$ IS VALID FOR $x>-3$ AND $\log x$ IS VALID FOR $x>0$.

THEREFORE $\log(x+3)+\log x$ IS VALID FOR $x>0$. SO U = $(0, \infty)$.

NOW $\log(x+3)+\log x=1$

$\Rightarrow\ \log x(x+3)=1$ … *since* $\log x+\log y=\log xy$

$\Rightarrow\ x(x+3)=10^1$ … *changing to exponential form*

$\Rightarrow\ x^2+3x-10=0$

$\Rightarrow\ (x+5)(x-2)=0$

Thus, $x=-5$ OR $x=2$

BUT -5 IS NOT AN ELEMENT OF THE UNIVERSE.

SO, THE ONLY SOLUTION IS $x = 2$.

D $\log_3 (x + 1) - \log_3 (x + 3)$ IS VALID FOR $x + 1 > 0$ AND $x + 3 > 0$,

I.E. FOR $x > -1$ AND $x > -3$.

THEREFORE THE U $= (-1, \infty)$.

$$\log_3 (x + 1) - \log_3 (x + 3) = 1$$

$$\Rightarrow \quad \log_3 \left(\frac{x+1}{x+3}\right) = 1 \qquad \ldots \textit{since } \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\Rightarrow \quad \frac{x+1}{x+3} = 3^1$$

$$\Rightarrow \quad x + 1 = 3(x + 3) = 3x + 9$$

THEREFORE $2x = -8$ AND $x = -4$.

HOWEVER, -4 IS NOT IN THE UNIVERSE. HENCE, THERE IS NO x SATISFYING THE GIVEN EQUATION AND THE SOLUTION SET IS THE EMPTY SET.

E $\log 8x + \log (x - 20)$ IS VALID FOR $8x > 0$ AND $x - 20 > 0$; I.E. FOR $x > 0$ AND $x > 20$.

SO U $= (20, \infty)$.

NOW $\log 8x + \log (x - 20) = 3$

$$\Rightarrow \log 8x(x - 20) = 3 \qquad \ldots \log_b xy = \log_b x + \log_b y$$

$$\Rightarrow 8x(x - 20) = 10^3 = 1000$$

$$\Rightarrow 8x^2 - 160x = 1000$$

$$\Rightarrow 8x^2 - 160x - 1000 = 0$$

$$\Rightarrow 8(x^2 - 20x - 125) = 0$$

$$\Rightarrow x^2 - 20x - 125 = 0$$

$$\Rightarrow (x - 25)(x + 5) = 0$$

SO $x = 25$ OR $x = -5$. BUT $-5 \notin (20, \infty)$

SO THE ONLY SOLUTION IS $x = 25$.

Property of equality for logarithmic equations

IF b, x, AND y ARE POSITIVE NUMBERS, $b \neq 1$, THEN

$\log_b x = \log_b y$, IF AND ONLY IF $x = y$.

FOR INSTANCE, IF $\log_2 x = \log_2 7$, THEN $x = 7$. IF $x = 7$, THEN $\log_2 x = \log_2 7$.

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING FOR x.

A $\log 3x - \log(2 - x) = 0$

B $\log_5 (4x - 7) = \log_5 (x + 5)$

C $\log (x - 5) + \log (10 - x) = \log (x - 6) + \log (x - 1)$

SOLUTION:

A LOG $3x$ IS VALID WHEN $x > 0$ AND LOG$(2 - x)$ IS VALID WHEN $2 - x > 0$ I.E. $x < 2$.

SO U = (0, 2).

NOW LOG $3x$ – LOG$(2 - x) = 0$ GIVES

LOG $3x$ = LOG$(2 - x)$

HENCE, $3x = 2 - x$ … *property of equality*

$\Rightarrow \quad 3x + x = 2$

SO $x = \frac{1}{2}$ IS THE SOLUTION IN (0, 2).

B $\log_5(4x - 7)$ IS VALID WHEN $x > \frac{7}{4}$ AND $\log_5(x + 5)$ IS VALID WHEN $x > -5$.

SO U = $\left(\frac{7}{4}, \infty\right)$. NEXT $\log_5(4x - 7) = \log_5(x + 5)$ GIVES

$4x - 7 = x + 5 \quad \Rightarrow \quad 3x = 12$. SO $x = 4$ IS THE SOLUTION.

C THE TERM LOG$(x - 5)$ IS VALID WHEN $x > 5$, THE TERM LOG$(10 - x)$ IS VALID WHEN $x < 10$, THE TERM LOG$(x - 6)$ IS VALID WHEN $x > 6$, AND THE TERM LOG$(x - 1)$ IS VALID WHEN $x > 1$.

IF WE RESTRICT THE UNIVERSE TO THE SET OF ALL REAL NUMBERS x BETWEEN 6 AND 10 OR $6 < x < 10$, EVERY TERM IN THE EQUATION IS VALID.

THEREFORE (6 ,10) IS THE UNIVERSE.

LOG$(x - 5)$ + LOG$(10 - x)$ = LOG$(x - 6)$ + LOG$(x - 1)$

$\Rightarrow$ LOG$((x - 5)(10 - x))$ = LOG$((x - 6)(x - 1))$

$\Rightarrow (x - 5)(10 - x) = (x - 6)(x - 1)$

$\Rightarrow -x^2 + 15x - 50 = x^2 - 7x + 6$

$\Rightarrow 15x - 50 = 2x^2 - 7x + 6$ … *adding x^2 to both sides*

$\Rightarrow -50 = 2x^2 - 22x + 6$

$\Rightarrow 0 = 2x^2 - 22x + 56$

$\Rightarrow 0 = x^2 - 11x + 28$ … *dividing both sides by 2*

$\Rightarrow (x - 7)(x - 4) = 0$.

$\Rightarrow x = 7$ OR $x = 4$, BUT ONLY 7 IS IN THE UNIVERSE.

HENCE $x = 7$ IS THE SOLUTION.

Exercise 2.11

1 State the universe and solve each of the following for x:

- **a** $\log_3 (2x - 1) = 5$
- **b** $\log_{\sqrt{2}} x = -$
- **c** $\log_3 (x^2 - 2x) = 1$
- **d** $\log_2 (x^2 + 3x + 2) = 1$
- **e** $\log_2 (1 + \frac{1}{x}) = 3$
- **f** $\log_2 (x - 1) + \log_2 3 = 3$
- **g** $\log (x^2 - 121) - \log (x + 11) = 1$
- **h** $\log_3 (x + 4) - \log_3 (x - 1)$
- **i** $\log (6x + 5) - \log 3 = \log 2 - \log x$
- **j** $\log x - \log 3 = \log 4 - \log (x + 4)$
- **k** $\log_5 (x + 1) + \log_5 (x + 3) = 1$
- **l** $\log_2 2 + \log_2 (x + 2) - \log_2 (3x - 5) = 3$
- **m** $\log_x (x + 6) = 2$

2 Apply the property of equality for logarithmic functions to solve the following equations (check that your solutions are valid):

- **a** $\log_3 x + \log_3 \; = 5$
- **b** $\log_3 25 \;\; 2\log_3 x =$
- **c** $\log_5 x + \log_5 (x + \;) = 1 \;\; \log_5$
- **d** $\log 2^x - \log 16 = 0$
- **e** $\log_4 (3^{6(x+2)}) - \log_4 (9^{x+2}) = 0$
- **f** $\log_2 (x^2 - 9) - \log_2 (3 + x) = 2$

2.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

As mentioned at the start of this unit, exponential and logarithmic functions are used in describing and solving a wide variety of real-life problems. In this sect[ion] discuss some of their applications.

EXAMPLE 1 *Population Growth*

a Suppose that you are observing the behaviour of cell duplication in a laboratory. In an experiment, you started with one cell and the c[ells] every minute.

  i Write an equation to determine the number of cells after one hour.

  ii Determine how long it would take for the number of cells to r[each] 100,000.

b Ethiopia has a population of around 80 million and it is estimated that the population grows every year at an average growth rat[e] the population growth continues at the same rate;

  i What will be the population after

  - 10 years?
  - 20 years?

  ii How many years will it take the population to double? (Refer back to the opening problem)

SOLUTION AND EXPLANATION:

A **I** FIRST RECORD YOUR OBSERVATIONS BY MAKING A TABLE WITH TWO ROWS, ONE FOR THE TIME AND THE OTHER FOR THE NUMBER OF CELLS. THE NUMBE DEPENDS ON THE TIME.

FOR EXAMPLE, AT $t = 0$, THERE IS 1 CELL, AND THE CORRESPONDING POINT IS (0, 1).

AT $t = 1$, THERE ARE 2 CELLS, AND THE CORRESPONDING POINT IS (1, 2).

AT $t = 2$, THERE ARE 4 CELLS, AND THE CORRESPONDING POINT IS (2, 4).

AT $t = 3$, THERE ARE 8 CELLS, AND THE CORRESPONDING POINT IS (3, 8), ETC.

THIS RELATIONSHIP IS SUMMARIZED IN THE FOLLOWING TABLE:

Time (in min.) (t)	0	1	2	3	4	5	6
No. of cells (y)	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	$64 = 2^6$

THEREFORE, THE FORMULA TO ESTIMATE THE NUMBER OF CELLS AFTER t MINUTES IS GIVEN BY

$f(t) = 2^t$

DETERMINE THE NUMBER OF CELLS AFTER ONE HOUR:

CONVERT ONE HOUR TO MINUTES. (1 HR = 60 MIN)

SUBSTITUTE 60 FOR T IN THE EQUATION, $f(t) = 2^t$:

$f(60) = 2^{60} = 1.15 \times 10^{18} = 1,150,000,000,000,000,000$

SO THE NUMBER OF CELLS AFTER 1 HOUR WILL BE 1,150,000,000,000,000,000 = 1.15×10^{18}

II IN THIS EXAMPLE, YOU KNOW THE NUMBER OF CELLS AT THE BEGINNING OF THE EXPERIMENT (1) AND AT THE END OF THE EXPERIMENT (100,000), BUT YOU NOT KNOW THE TIME. SUBSTITUTE 100,000 FOR $f(t)$ IN THE EQUATION $f(t) = 2^t$:

$100,000 = 2^t$

TAKE THE NATURAL LOGARITHM OF BOTH SIDES:

$\text{LN}(100,000) = \text{LN}(2^t) \Rightarrow \text{LN}(100,000) = t\,\text{LN}(2)$

DIVIDE BOTH SIDES BY LN(2):

$$t = \frac{\text{LN}(100,000)}{\text{LN}(2)} \Rightarrow t = 16.60964 \text{ MINUTES}$$

IT WOULD TAKE ABOUT 16.6 MINUTES, FOR THE NUMBER OF CELLS TO REACH 1

B **I** LET P REPRESENT THE CURRENT POPULATION WHICH IS 80 MILLION = 8.0×10^7;

LET r REPRESENT THE ANNUAL GROWTH RATE WHICH IS 2.3%;

LET t REPRESENT THE TIME IN YEARS FROM NOW.

THE TOTAL POPULATION AFTER ONE YEAR:

$A_1 = 80 \text{ MILLION} + 2.3\% (80 \text{ MILLION}) = 8.0 \times 10^7 + 2.3\% (8.0 \times 10^7)$

$= 8.0 \times 10^7 (1 + 2.3\%)$

THE TOTAL POPULATION AFTER TWO YEARS:

$$A_2 = A_1 + 2.3\%\,(A_1) = A_1(1 + 2.3\%) = 8.0\times10^7\,(1 + 2.3\%)(1 + 2.3\%)$$
$$= 8.0\times10^7\,(1 + 2.3\%)^2$$

THE TOTAL POPULATION AFTER THREE YEARS:

$$A_3 = A_2 + 2.3\%\,(A_2) = A_2\,(1 + 2.3\%) = 8.0\times10^7\,(1 + 2.3\%)^2(1 + 2.3\%)$$
$$= 8.0\times10^7(1 + 2.3\%)^3$$

FROM THE ABOVE PATTERN WE CAN GENERALIZE:

THE TOTAL POPULATION AFTER t YEARS IS GIVEN BY THE FORMULA:

$$A_t = P\,(1 + r)^t$$

SO THE TOTAL POPULATION AFTER 10 YEARS WILL BE

$$A_{10} = 8.0\times10^7\,(1 + 2.3\%)^{10} = 100{,}426{,}036.81$$

THE TOTAL POPULATION AFTER TWENTY YEARS WILL BE

$$A_{20} = 8.0\times10^7\,[1 + 2.3\%\,]^{20} = 126{,}067{,}360.86$$

II WHEN WILL THE TOTAL POPULATION DOUBLE? (BE 160 MILLION)

THE TOTAL POPULATION AFTER t YEARS IS:

$$8.0\times10^7\,[1 + 2.3\%\,]^T = 160{,}000{,}000$$

$$\Rightarrow [1 + 2.3\%\,]^t = \frac{160{,}000{,}000}{80{,}000{,}000} = 2 \Rightarrow \log(1 + 2.3\%)^t = \log 2$$

$$\Rightarrow t\log(1 + 0.023) = 0.3010 \Rightarrow t\log(1.023) = 0.3010$$

THEREFORE, $t = \dfrac{0.3010}{\log(1.023)} \approx \dfrac{0.3010}{0.0099} \approx 30.40$

THEREFORE, THE CURRENT POPULATION IS EXPECTED TO DOUBLE IN ABOU

EXAMPLE 2 *Compound Interest*

IF BIRR 5000 IS INVESTED AT A RATE OF 6% COMPOUNDED QUARTERLY (4 TIMES A YEAR), THEN

A WHAT IS THE AMOUNT AT THE END OF 4 YEARS AND 10 YEARS?

B HOW LONG DOES IT TAKE TO DOUBLE THE INVESTMENT?

SOLUTION: WE USE THE FORMULA $A = p\left(1+\dfrac{r}{n}\right)^{nt}$

HERE $p = 5000$, $r = 6\% = 0.06$

$n = 4$ (COMPOUNDED 4 TIMES)

A TO FIND THE BALANCE AT THE END OF THE 4^{TH} YEAR.

$$A = p\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.06}{4}\right)^{4\times 4} = 5000\,(1+0.015)^{16}$$

$$= 5000\,(1.015)^{16} \cong 5000\,(1.2690) = \text{BIRR } 6345$$

THE BALANCE AT THE END OF THE 10^{TH} YEAR IS

$$A = p\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.06}{4}\right)^{4\times 10} = 5000\,(1+0.015)^{40} = 5000\,(1.015)^{40}$$

$$\cong 5000\,(1.8140) = \text{BIRR } 9070$$

B IF THE INVESTMENT IS TO BE DOUBLED, $2P = 2 \times 5000 = 10{,}000$

$$A = p\left(1+\frac{r}{n}\right)^{nt}$$

$$\Rightarrow \quad 10{,}000 = 5000\left(1+\frac{0.06}{4}\right)^{4t} = 5000\,(1+0.015)^{4T}$$

$$\Rightarrow \quad 10{,}000 = 5000\,(1.015)^{4T}$$

$2 = (1.015)^{4T}$ … *dividing both sides by* 5000

$\text{LOG}\,2 = \text{LOG}\,(1.015)^{4T} = 4T\,\text{LOG}\,(1.015)$

$$4t = \frac{\text{LOG}\,2}{\text{LOG}(1.015)} = \frac{0.3010}{0.0065} = 46.30769 \Rightarrow t = \frac{46.30769}{4} \approx 11.58 \text{ YEARS}$$

IT TAKES ABOUT 12 YEARS TO DOUBLE THE INVESTMENT.

EXAMPLE 3 *Chemistry* (REFER BACK TO ACTIVITY 2.8)

THE CONCENTRATION OF HYDROGEN IONS IN A SOLUTION IS DENOTED BY $[H^+]$ AND IS MEASURED IN MOLES PER LITRE. FOR EXAMPLE, $[H^+] = 0.0000501$ FOR BEER AND $[H^+] = 0.0004$ FOR WINE. CHEMISTS DEFINE THE PH OF A SOLUTION AS THE NUMBER $PH = -\text{LOG}[H^+]$. THE SOLUTION IS SAID TO BE AN ACID IF PH < 7 AND A BASE IF PH > 7. PURE WATER HAS A PH OF 7, WHICH MEANS IT IS NEUTRAL.

A IS BEER AN ACID OR A BASE? WHAT ABOUT WINE?

B WHAT IS THE HYDROGEN ION CONCENTRATION $[H^+]$ OF EGGS, IF THE PH OF EGGS IS 7.8?

SOLUTION:

A (TEST FOR BEER)

$PH = -\text{LOG}[H^+]$

$PH = -\text{LOG}[0.0000501] = -\text{LOG}[5.01 \times 10^{-5}] = -[\text{LOG}\,5.01 + (-5)] = -[0.6998 + (-5)] = 4.3$

SINCE PH = 4.3 < 7 BEER IS AN ACID.

(TEST FOR WINE)

$$PH = -\log[H^+] = -\log[0.0004] = -\log[4\times 10^{-4}] = -\log[4 + (-4)]$$

$$= -[0.6021+(-4)] \approx 3.4 \;\Rightarrow\; PH = 3.4 < 7.$$

SO WINE IS AN ACID.

B $PH = -\log[H^+] \Rightarrow -\log[H^+] = 7.8$

$\Rightarrow \log[H^+] = -7.8 \Rightarrow [H^+] = 10^{-7.8}$

$\Rightarrow [H^+] = 1.58\times 10^{-8}$

Group Work 2.6

NEWTON'S LAW OF COOLING STATES THAT AN OBJECT ... PROPORTIONAL TO THE DIFFERENCE BETWEEN THE TE... F THE OBJECT AND THE ROOM TEMPERATURE. THE TEMPERATURE OF THE OBJECT AT A TIME t IS GIVEN BY A FUNCTION

$$f(t) = ce^{rt} + a,$$

WHERE a = ROOM TEMPERATURE

c = INITIAL DIFFERENCE IN TEMPERATURE BETWEEN THE OBJECT AND THE RO

r = CONSTANT DETERMINED BY DATA IN THE PROBLEM

PROBLEM: SUPPOSE YOU MAKE YOURSELF A CUP OF TEA. INITIALLY THE TEA HAS A TEMPERATURE OF 95°C; 5 MINUTES LATER THE TEA HAS COOLED TO 65°C. WHEN WILL THE TEA REACH A DRINKABLE TEMPERATURE OF 40°C?

Hint: ASSUME THAT THE ROOM TEMPERATURE a = 22°C. FIRST SOLVE FOR r AND THEN FIND t APPLYING THE NATURAL LOGARITHM.

Exercise 2.12

1 SUPPOSE YOU ARE OBSERVING THE BEHAVIOUR OF CELL DUPLICATION IN A LABORATORY. IN ONE EXPERIMENT, YOU START WITH ONE CELL AND THE CELL POPULATION IS TRIPLING EV

A WRITE A FORMULA TO DETERMINE THE NUMBER OF CELLS AFTER t MINUTES.

B USE YOUR FORMULA TO CALCULATE THE NUMBER OF CELLS AFTER AN HOUR

C DETERMINE HOW LONG IT WOULD TAKE THE NUMBER OF CELLS TO REACH 100,000.

2 SUPPOSE IN AN EXPERIMENT YOU STARTED WITH 100,000 CELLS AND OBSERVED CELL POPULATION DECREASED BY ONE HALF EVERY MINUTE.

A WRITE A FORMULA FOR THE NUMBER OF CELLS AFTER t MINUTES.

B DETERMINE THE NUMBER OF CELLS AFTER 10 MINUTES.

C DETERMINE HOW LONG IT WOULD TAKE THE CELL POPULATION TO REACH 10.

3 A BIRR 1,000 DEPOSITS IS MADE AT A BANK THAT PAYS 12% INTEREST COMPOU MONTHLY. HOW MUCH WILL BE IN THE ACCOUNT AT THE END OF 10 YEARS?

4 IF YOU START A BIOLOGY EXPERIMENT WITH 5,000,000 CELLS AND 25% OF THE C DYING EVERY MINUTE, HOW LONG WILL IT BE BEFORE THERE ARE FEWER THAN 1,0

5 ***Learning curve:*** IN PSYCHOLOGICAL TESTS, IT IS FOUND THAT STUDENTS CAN MEMORIZE A LIST OF WORDS t HOURS AFTER, ACCORDING TO THE LEARNING CURVE $y = 50 - 50e^{-0.3t}$, WHERE y IS THE NUMBER OF WORDS A STUDENT CAN LEARN DURING THE t^{TH} HOUR OF STUDY. FIND HOW MANY WORDS A STUDENT WOULD BE EXPECTED TO LEARN IN THE NINTH HOUR OF

6 THE ENERGY RELEASED BY THE LARGEST EARTHQUAKE RECORDED, MEASURED IN JOULES, IS ABOUT 100 BILLION TIMES THE ENERGY RELEASED BY A SMALL EARTHQUAKE THAT FELT. IN 1935 THE CALIFORNIA SEISMOLOGIST ***Charles Richter*** DEVISED A LOGARITHMIC SCALE THAT BEARS HIS NAME AND IS STILL WIDELY USED. THE MAGNITUDE M ON THE RICHTER SCALE IS GIVEN AS FOLLOWS:

$$M = \frac{2}{3}\log\frac{E}{E_0} \quad \text{RICHTER SCALE}$$

WHERE E IS THE ENERGY RELEASED BY THE EARTHQUAKE MEASURED IN JOULES, AND E_0 IS THE ENERGY RELEASED BY A VERY SMALL REFERENCE EARTH QUAKE WHICH HAS BEEN STANDARDIZED TO BE $E_0 = 10^{4.40}$ JOULES.

QUESTION

AN EARTH QUAKE IN A CERTAIN TOWN X RELEASED APPROXIMATELY 5.96×10^{16} JOULES OF ENERGY. WHAT WAS ITS MAGNITUDE ON THE RICHTER SCALE? GIVE YOUR ANSWER DECIMAL PLACES.

7 ***Physics:*** THE BASIC UNIT OF SOUND MEASUREMENT IS CALLED A BELL, NAMED AFTER THE INVENTOR OF TELEPHONE, ALEXANDER GRAHAM BELL (1847-1922). THE LOUDEST SO HEALTHY PERSON CAN HEAR WITHOUT DAMAGE TO THE EARDRUM HAS AN INTENS (10^{12}) TIMES THAT OF THE SOFTEST OF SOUND A PERSON CAN HEAR. THE RELATIONSHIP OF LOUDNESS OF SOUND L AND INTENSITIES I AND $I_\circ$ IS GIVEN BY

$$L = 10\log\frac{I}{I_\circ},$$

WHERE L IS MEASURED IN DECIBELS, I_0 IS THE INTENSITY OF THE LEAST AUDIBLE SOUND TH AN AVERAGE HEALTHY PERSON CAN HEAR, WHICH IS TAKEN TO BE 10^{-12} WATT PER SQUARE METER, AND I IS THE INTENSITY OF THE SOUND IN QUESTION.

QUESTION FIND THE NUMBER OF DECIBELS:

A FROM AN ORDINARY CONVERSATION WITH SOUND INTENSITY $I = 3.2 \times 10^{-6}$ WATT PER SQUARE METER.

B FROM A ROCKMUSIC CONCERT WITH SOUND INTENSITY $I = 5.2 \times 10^{-3}$ WATT PER SQUARE CENTIMETRE.

Key Terms

- *antilogarithm*
- *base*
- *characteristics*
- *common logarithm*
- *exponent*
- *exponential equation*
- *exponential expression*
- *exponential function*
- *logarithm*
- *logarithm of a number*
- *logarithmic equation*
- *logarithmic expression*
- *logarithmic function*
- *mantissa*
- *natural logarithm*
- *power*

Summary

1 IF n IS A POSITIVE INTEGER, THEN a^n IS THE PRODUCT OF n FACTORS OF a.

I.E. $a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ FACTORS}}$

IN a^n, a IS CALLED THE ***base***, n IS CALLED THE ***exponent*** AND a^n IS THE n^{TH} ***power*** OF a.

2 **Laws of Exponents**

FOR a AND b POSITIVE AND r AND s REAL NUMBERS

A $a^r \times a^s = a^{r+s}$

B $\dfrac{a^r}{a^s} = a^{r-s}$

C $(a^r)^s = a^{rs}$

D $(a \times b)^s = a^s \times b^s$

E $\left(\dfrac{a}{b}\right)^s = \dfrac{a^s}{b^s}$

3 ANY NON – ZERO NUMBER RAISED TO ZERO IS 1, I.E. $a^0 = 1$, FOR $(a \neq 0)$

4 FOR $a \neq 0$ AND $n > 0$, $a^{-n} = \dfrac{1}{a^n}$.

5 FOR $a \neq 0$, $b \neq 0$ AND $n > 0$, $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

6 FOR ANY REAL NUMBER a AND ANY INTEGER $n > 1$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

$\sqrt[n]{a} \in \mathbb{R}$ IF $a \in \mathbb{R}$ AND n IS ODD; $\sqrt[n]{a} \notin \mathbb{R}$ IF $a < 0$ AND n IS EVEN.

7 If $a > 0$ and m, n are integers with $n > 1$, then $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

8 If x is an irrational number and $a > 0$, then a^x is a real number between a^{x_1} and a^{x_2} for all possible choices of rational numbers x_1 and x_2 such that $x_1 < x < x_2$.

9 For a fixed positive number $b \neq 1$, and for each $a \in \mathbb{R}$, $b^c = a$, if and only if $c = \log_b a$. ($c = \log_b a$ is read as "c is the logarithm of a to the base b")

10 Laws of logarithms

If b, x and y are positive numbers and $b \neq 1$, then

A $\log_b xy = \log_b x + \log_b y$

B $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

C For any real number k, $\log_b x^k = k \log_b x$

D $\log_b b = 1$

E $\log_b 1 = 0$

11 Logarithms to base 10 are called ***common logarithms***.

12 The characteristic of a common logarithm is the integer usually before the decimal point. The mantissa is a positive decimal less than 1.

13 If a, b, c are positive real numbers, $a \neq 1$, $b \neq 1$, then

A $\log_a c = \dfrac{\log_b c}{\log_b a}$ ("Change of base law") **B** $b^{\log_b c} = c$

14 $\log_e x = \ln x$ is called the ***natural logarithm*** of x.

15 The function $f(x) = b^x$, $b > 0$ and $b \neq 1$ defines an exponential function.

16 The function $f(x) = e^x$ is called the ***natural exponential function***.

17 All members of the family $f(x) = b^x$ ($b > 0$, $b \neq 1$) have graphs which

✓ pass through the point (0, 1)

✓ are above the x-axis for all values of x

✓ are asymptotic to the x-axis

✓ have domain the set of all real numbers.

✓ have range the set of all positive real numbers.

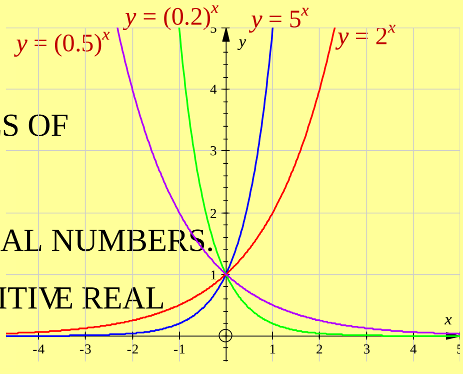


Figure 2.24

18 THE FUNCTION $f(x) = \text{LOG}_b\, x$, $b > 0$ AND $b \neq 1$ IS CALLED A LOGARITHMIC FUNCTION WITH BASE b.

19 THE FUNCTION $f(x) = \text{LOG}_e\, x = \text{LN}\, x$ IS CALLED THE ***natural logarithm*** OF x.

20 ALL MEMBERS OF THE FAMILY $y = \text{LOG}_b\, x$, $(b > 0, b \neq 1)$ HAVE GRAPHS WHICH

- ✓ PASS THROUGH THE POINT (1, 0)
- ✓ ARE ASYMPTOTIC TO THE y-AXIS
- ✓ HAVE DOMAIN THE SET OF ALL POSITIVE REAL NUMBERS
- ✓ HAVE RANGE THE SET OF REAL NUMBERS.

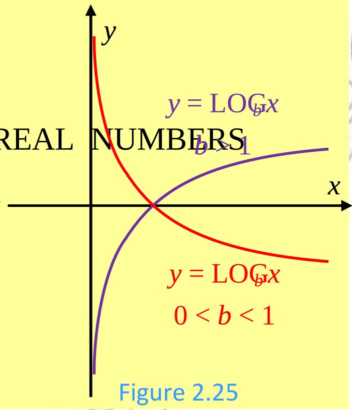


Figure 2.25

Review Exercises on Unit 2

1 WRITE THE SIMPLIFIED FORM OF EACH OF THE FOLLOWING EXPRESSIONS:

A 2^5 **B** -2^5 **C** 2^{-5} **D** -2^{-5}

E $\left(\frac{2}{3}\right)^2$ **F** $\left(\frac{2}{3}\right)^{-2}$ **G** $\frac{2^{-2}}{3^{-2}}$ **H** $\left(-\frac{2}{3}\right)^2$

2 USE THE LAWS OF EXPONENTS TO SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS:

A $2^5 \times 2^2$ **B** $\left(6^{\frac{1}{2}}\right)^2$ **C** $\frac{64^{\frac{3}{2}}}{8^{\frac{3}{2}}}$ **D** $a^{-3}b^{-3}$

E $(4n^5)^2$ **F** $\left(\frac{x}{2y}\right)^2$ **G** $\frac{d^{-4}}{d^{-2}}$ **H** $(x^{-3})^2$

I $E^{3x-1}\, E^{4-x}$ **J** $\frac{3^x}{3^{1-x}}$ **K** $\frac{5^{x-3}}{5^{x-4}}$ **L** $(2^x 3^y)^z$

3 CHANGE EACH LOGARITHMIC FORM TO AN EQUIVALENT EXPONENTIAL FORM:

A $\text{LOG}_3 81 = 4$ **B** $\text{LOG}_{25} 5 = \frac{1}{2}$

C $\text{LOG}_2 \frac{1}{4} = -2$ **D** $\text{LOG}_{\frac{1}{2}} \frac{1}{4} =$

4 FIND x IF:

A $\log_2 x = 5$ **B** $\log_4 16 = x$ **C** $\log_7 7 = x$

D $\log_x 16 = 2$ **E** $\log_8 x = \frac{1}{3}$ **F** $\log_{\frac{1}{3}} 9 = x$

G $\log_{49} \frac{1}{7} = x$ **H** $\log_x 1000 = \frac{3}{2}$

5 USE THE PROPERTIES OF LOGARITHMS TO WRITE EACH OF THE FOLLOWING EXPRESSIONS AS A SINGLE LOGARITHM:

A $\log_{10} 2 + \log_{10} 25$ **B** $\log_3 18 - \log_3 3$

C $3\log_3 5 - 2\log_3 7$ **D** $5\log_a x + 3\log_a y$

E $\log_a x^3 + \log_a \left(\frac{b}{\sqrt[3]{x}}\right)$ **F** $\ln x^3 - \ln\sqrt{x}$

6 USE THE TABLE OF COMMON LOGARITHMS TO FIND:

A LOG 4.21 **B** LOG 0.99 **C** LOG 8.2

D LOG 123 **E** LOG 0.34 **F** LOG 8.88

G LOG 0.00001 **H** LOG 500

7 FIND:

A ANTILOG 0.4183 **B** ANTILOG 0.3507 **C** ANTILOG 0.5428

D ANTILOG 0.8831 **E** ANTILOG 5.9736 **F** ANTILOG 1.7559

G ANTILOG(−10) **H** ANTILOG(−0.3)

8 STUDY THE FOLLOWING GRAPH (FIGURE 2.26) AND ANSWER THE QUESTIONS THAT FOLLOW:

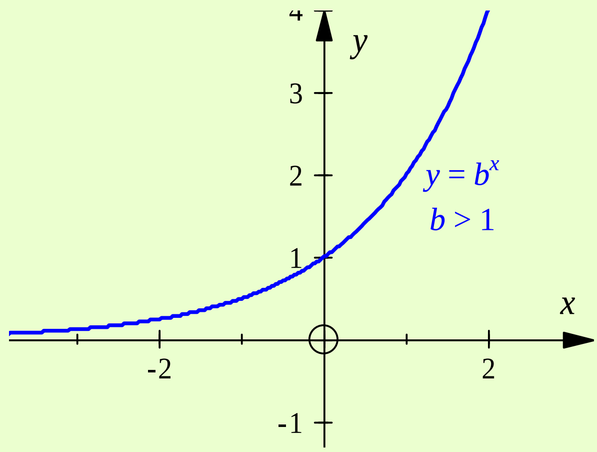


Figure 2.26

A GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.

B WHAT IS THE ASYMPTOTE OF THE GRAPH?

C IS THE FUNCTION INCREASING OR DECREASING?

D WHAT IS THE y-INTERCEPT?

E FOR WHICH VALUES OF x IS b^x GREATER THAN 1?

F WHAT CAN YOU SAY ABOUT THE VALUE OF b^x IF x IS NEGATIVE?

G FOR WHICH VALUES OF x IS b^x LESS THAN ZERO?

9 STUDY THE FOLLOWING GRAPH (FIGURE 2.27) AND ANSWER THE QUESTIONS GIVEN BELOW.

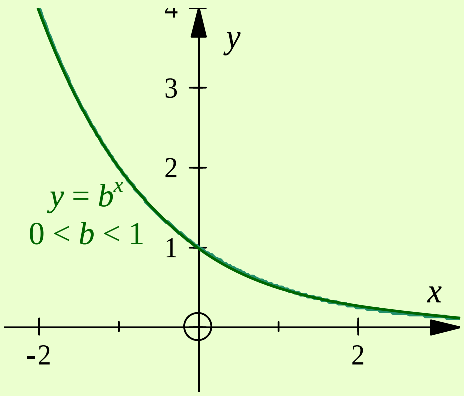


Figure 2.27

A GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.

B WHAT IS THE ASYMPTOTE OF THE GRAPH?

C IS THE FUNCTION INCREASING OR DECREASING?

D WHAT IS THE y-INTERCEPT?

E FOR WHICH VALUES OF x IS $b^x > 1$?

F WHAT IS THE VALUE OF b^x IF x IS POSITIVE?

G FOR WHICH VALUES OF x IS $b^x < 0$?

10 SKETCH THE FOLLOWING PAIRS OF FUNCTIONS USING THE SAME COORDINATE SYSTEM:

A $f(x) = 2^x - 3$ AND $g(x) = 2^x + 3$

B $f(x) = 3^x$ AND $g(x) = 3^x + 2$

C $f(x) = \left(\frac{3}{5}\right)^x$ AND $g(x) = \left(\frac{3}{5}\right)^{x+1}$

D $f(x) = 5^x$ AND $g(x) = \left(\frac{1}{5}\right)^x$

11 STUDY THE FOLLOWING GRAPH (FIGURE 2.28) AND ANSWER THE QUESTIONS THAT FOLLOW:

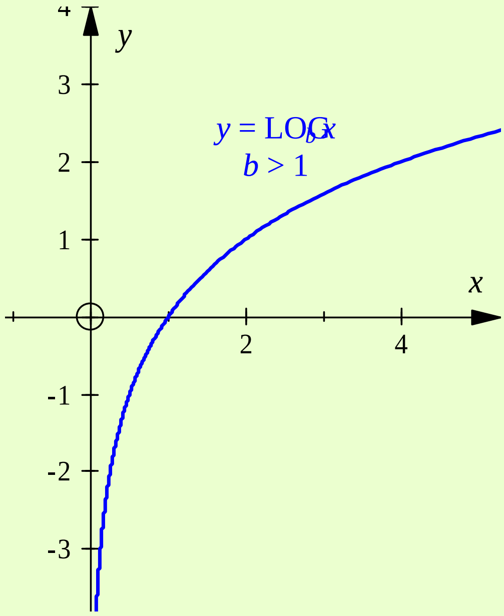


Figure 2.28

A GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.

B WHAT IS THE ASYMPTOTE OF THE GRAPH?

C IS THE FUNCTION INCREASING OR DECREASING?

D WHAT IS THE x-INTERCEPT?

E FOR WHICH VALUES OF x IS $\text{LOG}_b\, x > 0$?

F WHEN IS $\text{LOG}_b\, x < 0$?

12 SKETCH THE FOLLOWING PAIRS OF FUNCTIONS USING THE SAME COORDINATE SYSTEM:

A $f(x) = \text{LOG}_3\, x$ AND $g(x) = \text{LOG}_3(x - 2)$

B $f(x) = \text{LN}\, x$ AND $g(x) = \text{LN}(x + 2)$

C $f(x) = \text{LOG}_5\, x$ AND $g(x) = \text{LOG}_{\left(\frac{1}{5}\right)}\, x$

D $f(x) = 5^x$ AND $g(x) = \text{LOG}_5\, x$

13 STATE THE UNIVERSE FOR EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = \text{LOG}_3\, x$

B $g(x) = \text{LOG}_{\left(\frac{1}{3}\right)}(x + 3)$

C $f(x) = \text{LOG}_3(3 - x)$

D $g(x) = \text{LOG}_9(7x - 12)$

E $f(x) = \text{LOG}_2(3 - x) + \text{LOG}_2(3 + x)$

F $f(x) = \text{LOG}_2\left(x^2 - 2x\right)$

14 SOLVE EACH OF THE FOLLOWING EXPONENTIAL EQUATIONS:

A $3^x = 27$ **B** $2^{3-x} = 16$ **C** $5^{(4x-5)} = \frac{1}{25}$

D $4^{3x-8} = 2^{3x+9}$ **E** $36^{5x} = 6$ **F** $7^{x^2+x} = 49$

G $2^{6(x+2)} = 4^{x+2}$ **H** $2\left(\frac{243}{32}\right)^{2x} = 3\left(\frac{8}{27}\right)^{\left(\frac{2}{3}x-1\right)}$

15 SOLVE EACH OF THE FOLLOWING FOR x, CHECKING VALIDITY OF SOLUTIONS:

A $\mathrm{LOG}_3 x = 3$ **B** $\mathrm{LOG}_{16} x = \frac{3}{2}$

C $\mathrm{LOG}_x e^5 = 5$ **D** $\mathrm{LOG}\, 3x^2 - \mathrm{LOG}\, 9x = 2$

E $\mathrm{LOG}\, x - \mathrm{LOG}\, 3 = \mathrm{LOG}\, 4 - \mathrm{LOG}(x+4)$ **F** $\mathrm{LN}(x+3) - \mathrm{LN}\, x = 2\,\mathrm{LN}\, 2$

G $\mathrm{LN}(2x+1) - \mathrm{LN}(x-1) = \mathrm{LN}\, x$ **H** $\mathrm{LOG}(x^2-3) = 2\,\mathrm{LOG}(x-1)$

I $\mathrm{LOG}(4x)^5 = 5$ **J** $\mathrm{LOG}_2 x + \mathrm{LOG}_2 x^2 = 15$

K $\mathrm{LOG}_5(3+x) - \mathrm{LOG}_5 x = 2$

16 IF 2000 BIRR IS INVESTED AT 4% INTEREST, COMPOUNDED EVERY YEAR FOR 5 YEARS, WHAT IS THE AMOUNT REALIZED AT THE END OF 5 YEARS?

17 SUPPOSE THAT THE NUMBER OF BACTERIA IN A CERTAIN COLONY GROWS AT THE RATE OF 5% PER DAY. IF THERE ARE 1000 BACTERIA PRESENT INITIALLY, THEN W THE NUMBER OF BACTERIA PRESENT AFTER:

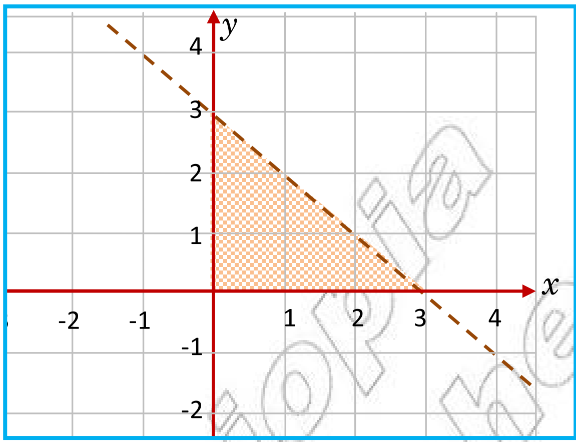
A 1 DAY? **B** 2 DAYS? **C** 3 DAYS? **D** 10 DAYS? **E** n DAYS?

18 THE POPULATION OF COUNTRY A IS 8.25×10^7 AND THAT OF COUNTRY B IS 1.1×10^8. IF THE ANNUAL GROWTH OF POPULATION OF COUNTRIES A AND B ARE 5.2% AN RESPECTIVELY, WHEN WILL COUNTRIES A AND B HAVE THE SAME POPULATION?

19 A CAR PURCHASED FOR 30,000 BIRR DEPRECIATES AT 5% PER ANNUM, THE DEPRECIATION BEING WORKED OUT ON THE VALUE OF THE CAR AT THE BEGINN YEAR. FIND ITS VALUE AFTER 10 YEARS.

Hint: IF V_0 IS THE VALUE OF A CERTAIN OBJECT AT A CERTAIN TIME, AND r% IS THE RATE OF DEPRECIATION PER YEAR, THEN THE VALUE V_t AT THE END OF t YEARS IS GIVEN BY: $V_t = V_0\left(1 - \frac{r}{100}\right)^t$, WHERE V_0 IS THE INITIAL VALUE.

Unit 3



SOLVING INEQUALITIES

Unit Outcomes:

After completing this unit, you should be able to:

- *know and apply methods and procedures in solving problems on inequalities involving absolute value.*
- *know and apply methods for solving systems of linear inequalities.*
- *apply different techniques for solving quadratic inequalities.*

Main Contents

3.1 Inequalities involving absolute value

3.2 Systems of linear inequalities in two variables

3.3 Quadratic inequalities

Key Terms

Summary

Review Exercises

INTRODUCTION

RECALL THAT OPEN STATEMENTS OF THE FORM $ax + b > 0$, $ax + b < 0$, $ax + b \leq 0$ AND $ax + b \geq 0$ FOR $a \neq 0$ ARE INEQUALITIES WITH SOLUTIONS USUALLY INVOLVING INTERVALS.

IN THIS UNIT, YOU WILL STUDY METHODS OF SOLVING INEQUALITIES INVOLVING ABS SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES AND QUADRATIC INEQUALITIES. LEARN ABOUT THE APPLICATIONS OF THESE METHODS IN SOLVING PRACTICAL PROB INEQUALITIES.

3.1 INEQUALITIES INVOLVING ABSOLUTE VALUE

THE METHODS FREQUENTLY USED FOR DESCRIBING SETS ARE THE COMPLETE LISTING METHOD, THE PARTIAL LISTING METHOD AND THE SET-BUILDER METHOD. SETS OF REAL NUMBERS C BE DESCRIBED BY USING THE SET-BUILDER METHOD OR INTERVALS (*sets of real numbers between any two given real numbers*).

Notation: FOR REAL NUMBERS a AND b WHERE $a < b$,

- ✓ (a, b) IS AN OPEN INTERVAL;
- ✓ $(a, b]$ AND $[a, b)$ ARE HALF CLOSED OR HALF OPEN INTERVALS; AND
- ✓ $[a, b]$ IS A CLOSED INTERVAL.

FOR EXAMPLE, (5, 9) IS THE SET OF REAL NUMBERS BETWEEN 5 AND 9 AND [5, 9] IS THE SE NUMBERS BETWEEN 5 AND 9 INCLUDING 5 AND 9.

THAT IS, $(5, 9) = \{x : 5 < x < 9 \text{ AND } x \in \mathbb{R}\}$

$[5, 9] = \{x : 5 \leq x \leq 9 \text{ AND } x \in \mathbb{R}\}$

IN GENERAL, IF a AND b ARE FIXED REAL NUMBERS WITH $a < b$, THEN

$[a, b] = \{x : a \leq x \leq b \text{ AND } x \in \mathbb{R}\}$ $\quad$ $(a, b) = \{x : a < x < b \text{ AND } x \in \mathbb{R}\}$

$(-\infty, a] = \{x : x \leq a \text{ AND } x \in \mathbb{R}\}$ $\quad$ $(a, \infty) = \{x : x > a \text{ AND } x \in \mathbb{R}\}$

Note: THE SYMBOL "∞" IS USED TO MEAN ***positive infinity*** AND "$-\infty$" IS USED TO MEAN ***negative infinity***.

GRAPHICALLY, $[a, b]$ IS

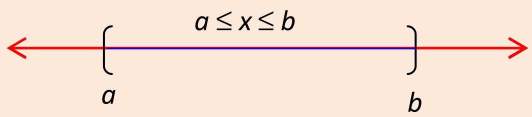


Figure 3.1

GRAPHICALLY, (a, b) IS

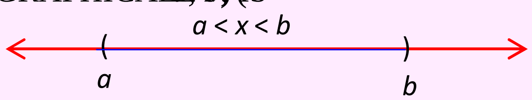


Figure 3.2

GRAPHICALLY, $(-\infty, a]$ IS

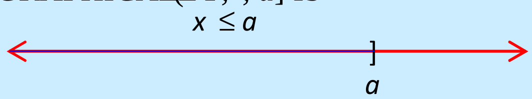


Figure 3.3

GRAPHICALLY, (a, ∞) IS

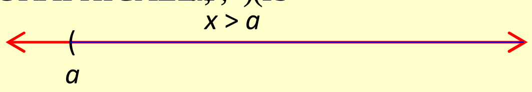


Figure 3.4

INTERVALS ARE COMMONLY USED TO EXPRESS THE SOLUTION SETS OF INEQUALITIES. FO
US FIND THE SOLUTION SET OF $2x + 4 \le 3x - 5$.

$2x + 4 \le 3x - 5$ IS EQUIVALENT TO $2x - 3x \le -5 - 4$ WHICH IS $-x \le -9$.

MULTIPLYING BOTH SIDES BY –1 GIVES $x \ge 9$. (REMEMBER THAT, WHEN YOU MULTIPLY OR DIVIDE BY A NEGATIVE NUMBER, THE INEQUALITY SIGN IS CHANGED.)

SO, THE SOLUTION SET IS $[9, \infty)$.

ACTIVITY 3.1

1 DISCUSS THE 3-METHODS OF DESCRIBING SETS: THE *complete listing method*, THE *partial listing method* AND THE *set-builder method*.

2 GIVE EXAMPLES FOR EACH OF THE METHODS USED FOR DESCRIBING A SET.

3 DESCRIBE EACH OF THE FOLLOWING SETS USING ANY ONE OF THE METHODS.

- **A** THE SET OF NUMBERS –3, –2, 1, 0, 2, 3.
- **B** THE SET OF ALL NEGATIVE MULTIPLES OF 2.
- **C** THE SET OF NATURAL NUMBERS GREATER THAN 50.6 AND LESS T

4 DESCRIBE EACH OF THE FOLLOWING SETS USING SET-BUILDER METHOD:

- **A** $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- **B** $\{0, 3, 6, 9, \ldots\}$
- **C** $[-3, 5)$
- **D** $[2, \infty)$

5 WRITE EACH OF THE FOLLOWING USING INTERVALS:

- **A** $\{x : x \in \mathbb{R} \setminus \{0\}\}$
- **B** $\{x : -1 \le x \le 2$ AND $x \in \mathbb{R}\}$
- **C** $\{x : 0.2 < x \le 0.8$ AND $x \in \mathbb{R}\}$
- **D** $\{x : x \in \mathbb{R}$ AND $x \ne -1\}$

6 FIND ALL VALUES OF x SATISFYING THE FOLLOWING INEQUALITIES:

- **A** $2x - 1 < 7$
- **B** $4 \le 1 - x < 5$

LOOK AT THE NUMBER LINE GIVEN BELOW.

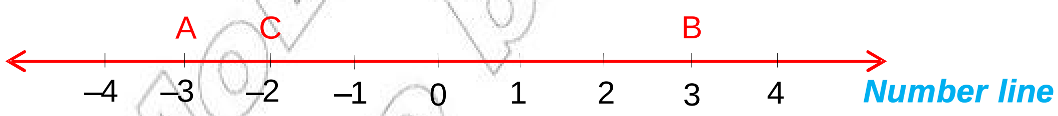


Figure 3.5

WHAT ARE THE COORDINATES OF POINTS A AND B, ON THE NUMBER LINE?

WHAT IS THE DISTANCE OF POINT A FROM THE ORIGIN? WHAT ABOUT B?

THE NUMBER THAT SHOWS ONLY THE DISTANCE FROM THE POINT CORRESPONDING TO
THE DIRECTION) IS CALLED THE ***absolute value***. FOR EXAMPLE, THE POINT C (WITH COORDINATE –2) IS 2 UNITS FROM THE POINT CORRESPONDING TO ZERO. THIS IS DENOTED BY $|-2| = 2$.

ON THE NUMBER LINE, $|x|$ IS THE DISTANCE BETWEEN THE POINT CORRESPONDING TO x AND NUMBE
THE POINT CORRESPONDING TO ZERO, REGARDLESS OF WHETHER THE POINT IS TO THE
THE POINT CORRESPONDING TO ZERO AS SHOWN IN FIGURE 3.6 BELOW.

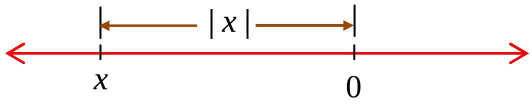


Figure 3.6

Definition 3.1

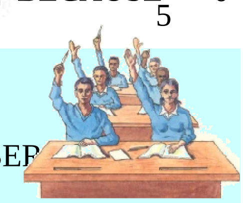
If x is a real number, then the absolute value of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

EXAMPLE 1

A $|25| = 25$ BECAUSE $25 > 0$ **B** $\left|-\frac{4}{5}\right| = -\left(-\frac{4}{5}\right) = \frac{4}{5}$ BECAUSE $-\frac{4}{5} < 0$

ACTIVITY 3.2



1 WHY IS IT ALWAYS TRUE THAT FOR ANY REAL NUMBER x, $|x| \geq 0$?

2 EVALUATE EACH OF THE FOLLOWING EXPRESSIONS:

A $|-3|$ **B** $|0|$ **C** $|-\sqrt{5}|$

D $|-3|\,|-2|$ **E** $|1-\sqrt{2}|$ **F** $|\sqrt{3}-\sqrt{5}|$

3 IF $x = -2$ AND $y = 3$, THEN EVALUATE EACH OF THE FOLLOWING:

A $|6x + y|$ **B** $|6x| + |y|$ **C** $|2x - 3y|$

4 VERIFY EACH OF THE FOLLOWING USING EXAMPLES:

A $|x - y| = |y - x|$ **B** $|2x-3y| = |3y-2x|$ **C** $\sqrt{x^2} = |x|$

D $|x|\,|y| = |xy|$ **E** $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

GEOMETRICALLY, THE EQUATION $|x| = 5$ MEANS THAT THE POINT WITH COORDINATE x IS 5 UNITS
AWAY FROM THE POINT CORRESPONDING TO ZERO, ON THE NUMBER LINE. OBVIOUSLY
LINE CONTAINS TWO POINTS THAT ARE 5 UNITS FROM THE POINT CORRESPONDING TO Z
TO THE LEFT AND THE OTHER TO THE RIGHT. SO, $|x| = 5$ HAS TWO SOLUTIONS, $x = 5$ AND $x = -5$.

Theorem 3.1 **Solutions of the equation $|x| = a$**

For any real number a, the equation $|x| = a$ has

I two solutions $x = a$ and $x = -a$, if $a > 0$;

II one solution, $x = 0$, if $a = 0$; and

III no solution, if $a < 0$.

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS:

A $|3x + 5| = 2$ **B** $\left|\frac{2}{3}x + 1\right| = 0$ **C** $|2x - 1| = -3$

SOLUTION:

A $|3x + 5| = 2$ IS EQUIVALENT TO $3x + 5 = -2$ OR $3x + 5 = 2$

$\Rightarrow 3x + 5 - 5 = -2 - 5$ OR $3x + 5 - 5 = 2 - 5$

$\Rightarrow 3x = -7$ OR $3x = -3$

$\Rightarrow x = -\frac{7}{3}$ OR $x = -1$

THEREFORE, $x = -\frac{7}{3}$ AND $x = -1$ ARE THE TWO SOLUTIONS.

B WE KNOW THAT $|x| = 0$ IF AND ONLY IF $x = 0$. THEREFORE, $\left|\frac{2}{3}x + 1\right| = 0$ IS EQUIVALENT TO $\frac{2}{3}x + 1 = 0$. HENCE, $\frac{2}{3}x = -1$

$\Rightarrow x = -\frac{3}{2}$ IS THE SOLUTION.

C SINCE $|x| \geq 0$ FOR ALL $x \in \mathbb{R}$, THE GIVEN EQUATION $|2x - 1| = -3$ HAS NO SOLUTION.

AS DISCUSSED ABOVE $|x| = 4$ MEANS $x = -4$ OR $x = 4$. HENCE, ON THE NUMBER LINE, THE POINT CORRESPONDING TO x IS 4 UNITS AWAY FROM THE POINT CORRESPONDING TO 0. WE SEE TH $|x| \leq 4$, THE DISTANCE BETWEEN THE POINT CORRESPONDING TO x AND THE POINT CORRESPONDING TO 0 IS LESS THAN 4 OR EQUAL TO 4. IT FOLLOWS THAT $|x| \leq 4$ IS EQUIVALENT TO $-4 \leq x \leq 4$.

WE HAVE THE FOLLOWING GENERALIZATION.

Theorem 3.2 **Solution of $|x| < a$ and $|x| \leq a$**

For any real number $a > 0$,

I the solution of the inequality $|x| < a$ is $-a < x < a$.

II the solution of the inequality $|x| \leq a$ is $-a \leq x \leq a$.

EXAMPLE 3 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE INEQUALITIES:

A $|2x - 5| < 3$ **B** $|3 - 5x| \leq 1$

SOLUTION:

A $|2x - 5| < 3$ IS EQUIVALENT TO $-3 < 2x - 5 < 3$,

$\Rightarrow -3 < 2x - 5$ AND $2x - 5 < 3$

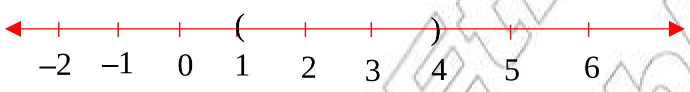
$\Rightarrow -3 + 5 < 2x - 5 + 5$ AND $2x - 5 + 5 < 3 + 5$

$\Rightarrow 2 < 2x$ AND $2x < 8$

$\Rightarrow 1 < x$ AND $x < 4$ THAT IS, $1 < x < 4$

THEREFORE, THE SOLUTION SET IS $\{x: 1 < x < 4\} = (1, 4)$

WE CAN REPRESENT THE SOLUTION SET ON THE NUMBER LINE AS FOLLOWS:



B $|3 - 5x| \leq 1$ IS EQUIVALENT TO $-1 \leq 3 - 5x \leq 1$

$\Rightarrow -1 \leq 3 - 5x$ AND $3 - 5x \leq 1$

$\Rightarrow -1 - 3 \leq 3 - 3 - 5x$ AND $3 - 3 - 5x \leq 1 - 3$

$\Rightarrow -4 \leq -5x$ AND $-5x \leq -2$

$\Rightarrow 5x \leq 4$ AND $2 \leq 5x$

$\Rightarrow x \leq \frac{4}{5}$ AND $x \geq \frac{2}{5}$ THAT IS, $\frac{2}{5} \leq x \leq \frac{4}{5}$

THEREFORE, THE SOLUTION SET IS $\left\{x: \frac{2}{5} \leq x \leq \frac{4}{5}\right\} = \left[\frac{2}{5}, \frac{4}{5}\right]$

Note: IN $|x| < a$, IF $a < 0$ THE INEQUALITY $|x| < a$ HAS NO SOLUTION.

Theorem 3.3 **Solution of $|x| > a$ and $|x| \geq a$**

For any real number a, if $a > 0$, then

I the solution of the inequality $|x| > a$ is $x < -a$ or $x > a$.

II the solution of the inequality $|x| \geq a$ is $x \leq -a$ or $x \geq a$.

EXAMPLE 4 SOLVE EACH OF THE FOLLOWING INEQUALITIES:

A $|5 + 2x| > 6$ **B** $\left|\frac{3}{5} - 2x\right| \geq 1$ **C** $|3 - x| > -2$

SOLUTION: ACCORDING TO THEOREM 3.5,

A $|5 + 2x| > 6$ IMPLIES $5 + 2x < -6$ OR $5 + 2x > 6$

$\Rightarrow 5 - 5 + 2x < -6 - 5$ OR $5 - 5 + 2x > 6 - 5$

$\Rightarrow 2x < -11$ OR $2x > 1$

$\Rightarrow x < \frac{-11}{2}$ OR $x > \frac{1}{2}$

THEREFORE, THE SOLUTION SET IS $\left\{x : x < -\frac{11}{2} \text{ OR } x > \frac{1}{2}\right\}$.

(TRY TO REPRESENT THIS SOLUTION ON THE NUMBER LINE)

B $\left|\frac{3}{5} - 2x\right| \geq 1$ IMPLIES $\frac{3}{5} - 2x \leq -1$ OR $\frac{3}{5} - 2x \geq 1$

HENCE, $\frac{3}{5} - 2x \leq -1$ OR $\frac{3}{5} - 2x \geq 1$ GIVES $\frac{3}{5} - \frac{3}{5} - 2x \leq -1 - \frac{3}{5}$ OR $\frac{3}{5} - \frac{3}{5} - 2x \geq 1 - \frac{3}{5}$

$\Rightarrow -2x \leq \frac{-8}{5}$ OR $-2x \geq \frac{2}{5}$

$\Rightarrow \frac{8}{5} \leq 2x$ OR $-\frac{2}{5} \geq 2x$

$\Rightarrow x \geq \frac{4}{5}$ OR $x \leq -\frac{1}{5}$

THEREFORE, THE SOLUTION SET IS $\left\{x : x \leq -\frac{1}{5} \text{ OR } x \geq \frac{4}{5}\right\}$.

C BY DEFINITION, $|3 - x| = |x - 3| \geq 0$. SO, $|3 - x| > -2$ IS TRUE FOR ALL REAL NUMBERS x

THEREFORE, THE SOLUTION SET IS $\mathbb{R}$.

Group Work 3.1

1 GIVEN THAT $0 < a < b$, EXPRESS THE FOLLOWING WITHOUT ABSOLUTE VALUE.

A $|a - b|$ **B** $|ab - a|$ **C** $\left|\frac{b}{a}\right|$

2 FOR ANY REAL NUMBER a, SHOW THAT

A $a \leq |a|$

Hint: IF $a \geq 0$, THEN $|a| = a$. SO, $a \leq |a|$.
IF $a < 0$, THEN $|a| > 0$. COMPARE a AND $|a|$

B $-|a| \leq a \leq |a|$

3 FOR ANY REAL NUMBERS x AND y, SHOW THAT

A $|x + y| \le |x| + |y|$

Hint: START FROM $|x + y|^2 = (x + y)^2$ AND EXPAND. THEN USE 2B ABOVE.

B $|x - y| \ge |x| - |y|$

4 SOLVE EACH OF THE FOLLOWING

A $\frac{3x - 1}{2} + x \le 7 + \frac{1}{2}x$ **B** $|-2| \ge 8 - |4x + 6|$

C $\left|\frac{1}{4}x - 2\right| > 1$ **D** $|2x - 1| < x + 3$

Exercise 3.1

1 SIMPLIFY AND WRITE EACH OF THE FOLLOWING USING INTERVALS:

A $\{ x : x \in \mathbb{R}$ AND $x \neq -2 \}$ **B** $\{ x : -1 \le x - 2 \le 2 \}$

C $\{ x : x + 3 > 2 \}$ **D** $\{ x : 5x - 9 \le 9 \}$

E $\{ x : 2x + 3 \ge -5x \}$ **F** $\{ x : 2x - 1 < x < 3 \}$

2 SOLVE EACH OF THE FOLLOWING INEQUALITIES:

A $2x - 5 \ge 3x$ **B** $3x + 1 < \frac{8x - 3}{2}$ **C** $\frac{1}{4}t + 2 > 3(5 - t)$

3 A NUMBER y IS 15 LARGER THAN A POSITIVE NUMBER x. IF THEIR SUM IS NOT MORE THAN 85, WHAT ARE THE POSSIBLE VALUES OF SUCH NUMBER y?

4 IF $x = -\frac{2}{3}$ AND $y = \frac{1}{5}$, THEN EVALUATE THE FOLLOWING:

A $|6x| + |5y|$ **B** $|3x| - |10y|$ **C** $|3x - 10y|$ **D** $\left|\frac{3x - 2y}{x + y}\right|$

5 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS:

A $|3x + 6| = 7$ **B** $|5x - 3| = 9$ **C** $|x - 6| = -6$

D $|7 - 2x| = 0$ **E** $|6 - 3x| + 5 = 14$ **F** $\left|\frac{3}{4}x + \frac{1}{8}\right| = \frac{1}{2}$

6 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE INEQUALITIES AND EXPRESS THEIR SOLUTION SETS IN INTERVALS:

A $|3 - 5x| \le 1$ **B** $|5x| - 2 < 8$ **C** $\left|\frac{2}{3}x - \frac{1}{9}\right| \ge \frac{1}{3}$

D $|6 - 2x| + 3 > 8$ **E** $|3x + 5| \le 0$ **F** $|x - 1| > -2$

7 FOR ANY REAL NUMBERS a, b AND c, SUCH THAT $a \neq 0$ AND $c > 0$, SOLVE EACH OF THE FOLLOWING INEQUALITIES:

A $|ax + b| < c$ **B** $|ax + b| \le c$ **C** $|ax + b| > c$ **D** $|ax + b| \ge c$

3.2 SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

RECALL THAT A FIRST DEGREE (LINEAR) EQUATION IN TWO VARIABLES HAS THE FORM

$$ax + by = c$$

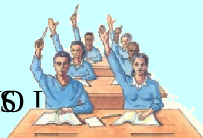
WHERE a AND b BOTH ARE NOT 0.

WHEN TWO OR MORE LINEAR EQUATIONS INVOLVE THE SAME VARIABLES, THEY ARE CA ***of linear equations***. AN ORDERED PAIR THAT SATISFIES ALL THE EQUATIONS OF A SYSTEM IS CALLED A ***solution of the system***. FOR INSTANCE

$$\begin{cases} 2x - y = 7 \\ x + 5y = -2 \end{cases}$$

IS A SYSTEM OF TWO LINEAR EQUATIONS. WHAT IS ITS SOLUTION?

ACTIVITY 3.3



1 WHAT CAN YOU SAY ABOUT THE SOLUTION SET OF TWO EQUATIONS IF THEIR GRAPHS DO NOT INTERSECT?

2 FIND THE SOLUTIONS OF EACH OF THE FOLLOWING SYSTEMS OF EQUATIONS GRAPHICALLY:

A $\begin{cases} x - y = -2 \\ x + y = 6 \end{cases}$ **B** $\begin{cases} x + y = 2 \\ 2x + 2y = 8 \end{cases}$ **C** $\begin{cases} x + 2y = 4 \\ 2x + 4y = 8 \end{cases}$

3 FIND THREE DIFFERENT ORDERED PAIRS WHICH BELONG TO R WHERE

$R = \{(x, y): y \le x + 1\}$.

4 DRAW THE GRAPH OF R GIVEN IN QUESTION 3 ABOVE.

5 DRAW THE GRAPHS OF EACH OF THE FOLLOWING RELATIONS:

A $R = \{(x, y): x \ge y$ AND $y \ge x - 1\}$ **B** $R = \{(x, y): y \le x + 1$ AND $y \ge 1 - x\}$.

6 SOLVE EACH OF THE FOLLOWING SYSTEMS OF INEQUALITIES AND WRITE YOUR ANSWER IN INTERVAL NOTATION:

A $\begin{cases} x \ge -1 \\ x \le 3 \\ y \ge 0 \end{cases}$ **B** $\begin{cases} x - y < 3 \\ x \ge 2 \end{cases}$

A SYSTEM OF TWO LINEAR EQUATIONS IN TWO VARIABLES OFTEN INVOLVES A PAIR OF IN THE PLANE. THE SOLUTION SET OF SUCH A SYSTEM OF EQUATIONS CAN BE DETERMI GRAPH AND IS THE SET OF ALL ORDERED PAIRS OF COORDINATES OF POINTS WHICH LIE

EXAMPLE 1 FIND THE SOLUTION SET OF THE SYSTEM OF EQUATIONS $\begin{cases} x - y = 3 \\ x + 2y = 0 \end{cases}$

SOLUTION: FIRST DRAW THE GRAPHS OF $x - y = 3$ AND $x + 2y = 0$ AS SHOWN BELOW.

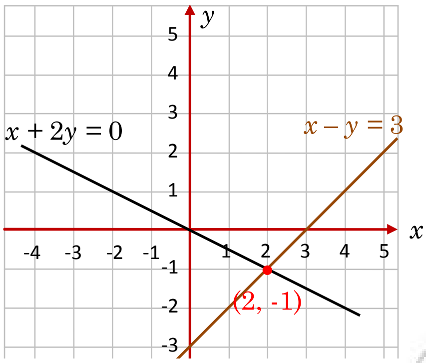


Figure 3.7

THE TWO LINES INTERSECT AT (2, –1).

THEREFORE, THE SOLUTION SET OF THE SYSTEM IS {(2, –1)}.

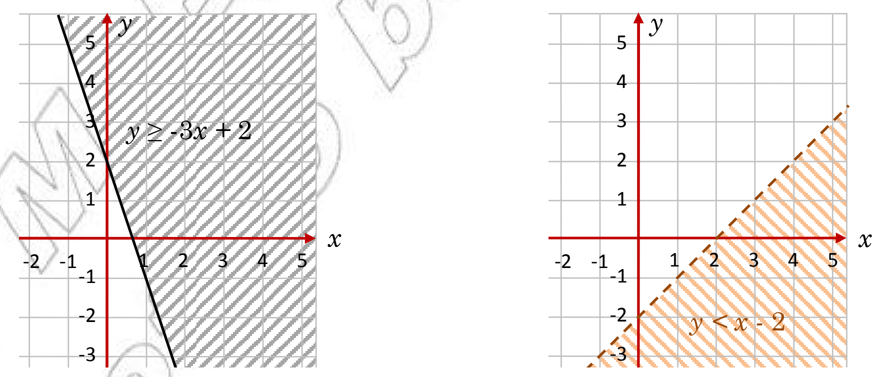
IN A SYSTEM OF EQUATIONS, IF "=" IS REPLACED BY "<", ">", "≤" OR "≥", THE SYSTEM BECOMES A SYSTEM OF LINEAR INEQUALITIES.

EXAMPLE 2 FIND THE SOLUTION OF THE FOLLOWING SYSTEM OF INEQUALITIES GRAPHICALLY:

$$\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$$

SOLUTION: FIRST DRAW THE GRAPH OF ONE OF THE BOUNDARY LINES, $y = -3x + 2$, CORRESPONDING TO THE FIRST INEQUALITY.

THE GRAPH OF $y \geq -3x + 2$ CONSISTS OF POINTS ON OR ABOVE THE LINE $y = -3x + 2$ AS SHOWN IN FIGURE 3.8A. THIS IS OBTAINED BY TAKING A TEST POINT, SAY (2, 0), AND CHECKING THAT $0 \geq -3(2) + 2 = -4$ IS TRUE. NEXT, DRAW THE GRAPH OF THE OTHER BOUNDARY LINE, $y = x - 2$, CORRESPONDING TO THE SECOND INEQUALITY. THE GRAPH OF $y < x - 2$ CONSISTS OF POINTS BELOW THE LINE $y = x - 2$. POINTS ON THE LINE ARE EXCLUDED AS SHOWN IN FIGURE 3.8B.



A B

Figure 3.8

THESE GRAPHS HAVE BEEN DRAWN USING DIFFERENT COORDINATE SYSTEMS IN O
THEM SEPARATELY. NOW, DRAW THEM USING THE SAME COORDINATE SYSTEM. THE
COORDINATE SYSTEM MARKED WITH BOTH TYPES OF SHADING IS THE SOLUTION SET
AS SHOWN IN FIGURE 3.9B.

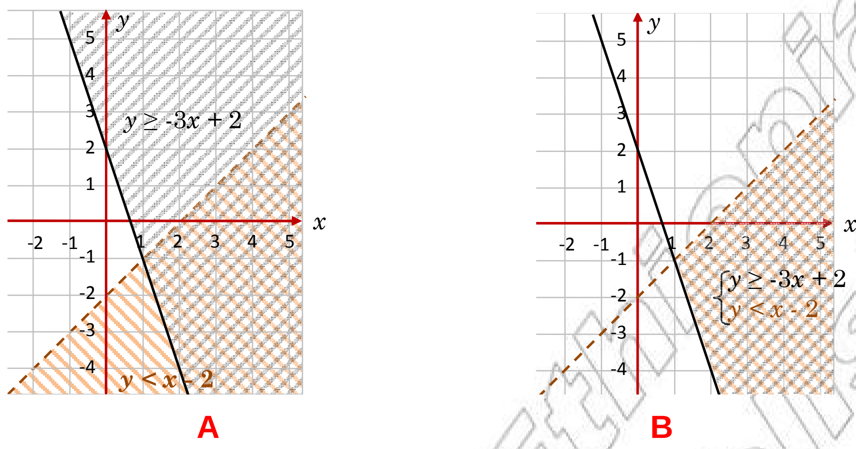


Figure 3.9

THE SOLUTION SET OF $\begin{cases} y \geq -3x+2 \\ y < x-2 \end{cases}$ IS SHOWN BY THE CROSS-SHADED REGION IN THE DIAGRAM.

SOLVING $\begin{cases} y = -3x+2 \\ y = x-2 \end{cases}$, WE GET $-3x + 2 = x - 2$

THEREFORE, $x = 1$ AND $y = -1$

SO, $x > 1$, $-3x + 2 \leq y < x - 2$

HENCE, THE SOLUTION SET OF THE SYSTEM IS EXPRESSED AS

$$\{(x, y): -3x + 2 \leq y < x - 2 \text{ AND } 1 < x < \infty\}$$

EXAMPLE 3 FIND THE SOLUTION OF EACH OF THE FOLLOWING SYSTEMS OF LINEAR INEQUALITIES, GRAPHICALLY:

A $\begin{cases} x + y < 3 \\ x \geq 0 \\ y \geq 0 \end{cases}$ **B** $\begin{cases} y + x > 0 \\ y - x \leq 1 \\ x \leq 2 \end{cases}$

SOLUTION:

A HERE, OUR OBJECTIVE IS TO DETERMINE THE SET OF POINTS WITH COORDINATES (x, y) THAT
SATISFY ALL THREE OF THESE CONDITIONS. TO DO SO, LET US DRAW EACH BO
AS SHOWN BELOW. THE POINTS SATISFYING THE CONDITION $x \geq 0$ ARE THOSE LYING TO
THE RIGHT OF THE y-AXIS AS SHOWN IN FIGURE 3.10A.

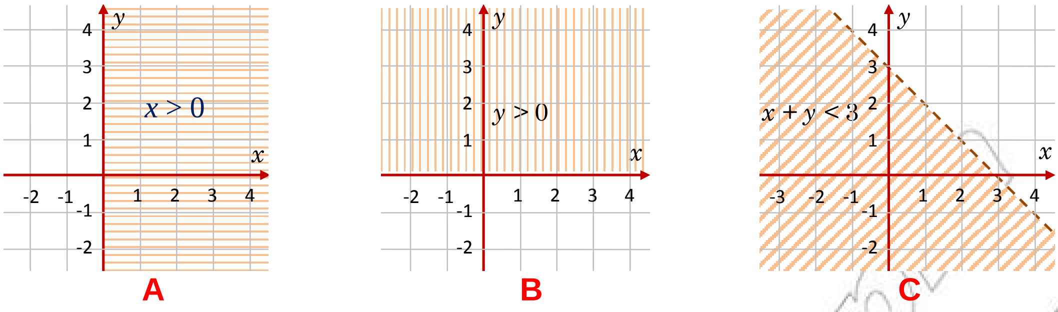


Figure 3.10

THE POINTS (x, y) WITH $y > 0$ ARE THE POINTS THAT LIE ABOVE THE x-AXIS AS SHOWN IN FIGURE 3.10B. THE POINTS (x, y) WITH $x + y < 3$ IS THE SET OF POINTS LYING BELOW THE LINE $x + y = 3$. POINTS ON THE LINE ARE EXCLUDED.

NOW, DRAW THE GRAPH OF THE THREE INEQUALITIES $x \geq 0$, $y \geq 0$ AND $x + y < 3$, USING THE SAME COORDINATE SYSTEM, TAKING ONLY THE INTERSECTION OF THE THREE REGI

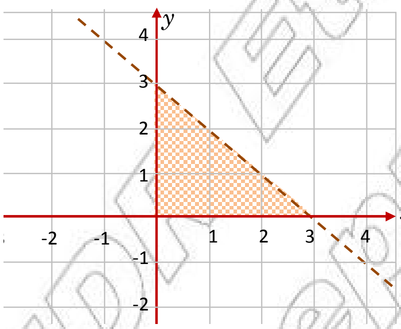


Figure 3.11

THE POINTS SATISFYING THE SYSTEM OF INEQUALITIES ARE THE POINTS THAT SATISFY ALL THE THREE INEQUALITIES. THE CORRESPONDING REGION IS THE TRIANGULAR REGION IN FIGURE 3.11. THAT IS, THE SET OF (x, y) SUCH THAT $x \in [0, 3)$ AND $y \in [0, 3 - x)$

B FIRST, DRAW THE GRAPH OF THE BOUNDARY LINE $y = -x$ (OR $y + x = 0$) FOR THE FIRST INEQUALITY. THE GRAPH OF $y + x > 0$ CONSISTS OF POINTS ABOVE THE LINE.

POINTS ON THE LINE ARE EXCLUDED AS SHOWN IN FIGURE 3.12A

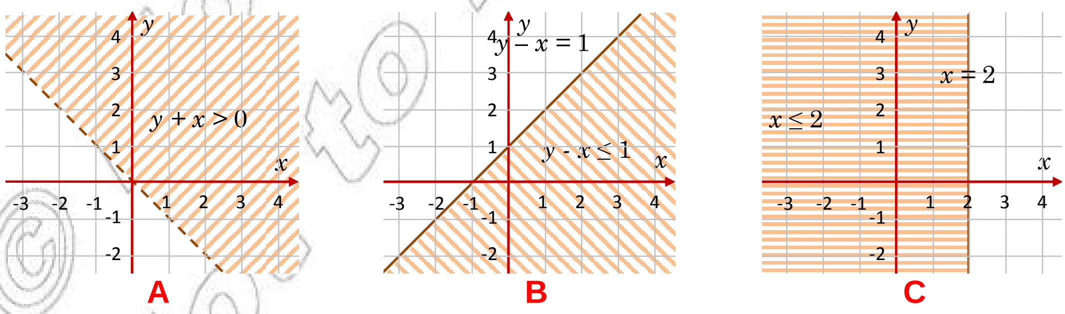
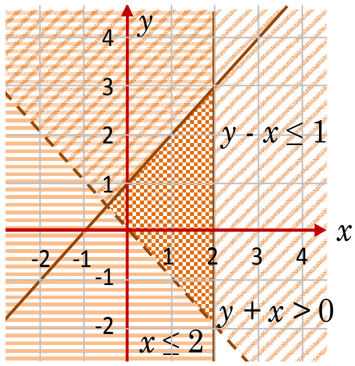


Figure 3.12

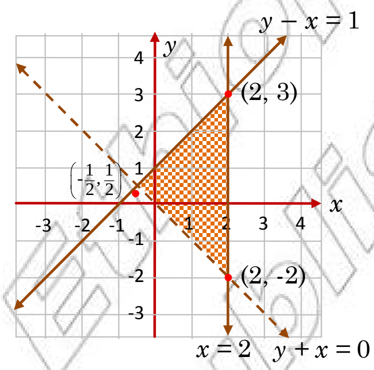
NEXT, DRAW THE GRAPH OF THE BOUNDARY LINE $y - x = 1$ FOR THE SECOND INEQUALITY. THE GRAPH OF $y - x \leq 1$ CONSISTS OF POINTS ON AND BELOW THE LINE $y - x = 1$ AS SHOWN IN FIGURE 3.12B.

FINALLY, DRAW THE GRAPH OF THE BOUNDARY LINE $x = 2$ FOR THE THIRD INEQUALITY. THE POINTS (x, y) SATISFYING THE CONDITION $x \leq 2$ ARE THOSE LYING ON AND TO THE LEFT OF THE LINE $x = 2$ AS SHOWN IN FIGURE 3.12C.

NOW, DRAW THE GRAPH OF THE THREE INEQUALITIES USING THE SAME CO… SYSTEM AS SHOWN IN FIGURE 3.13A.



A

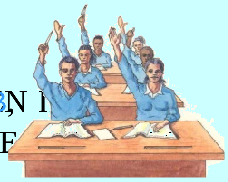


B

Figure 3.13

BECAUSE THERE ARE INFINITE SOLUTIONS TO THE SYSTEM, THEY CANNOT BE LISTED. BUT THE GRAPH IS EASY TO DESCRIBE. THE SOLUTION IS THE TRIANGULAR R… VERTICES $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(2, 3)$ AND $(2, -2)$, EXCEPT THOSE POINTS ON THE LINE $y + x = 0$, AS SHOWN IN FIGURE 3.13B.

ACTIVITY 3.4



1 BY OBSERVING THE GRAPH OF THE INEQUALITY GIVEN IN FIGURE 3.13B, NAME AT LEAST 10 ORDERED PAIRS THAT SATISFY THE…

2 IF R = {(x, y): $y + x > 0$, $y - x \leq 1$ AND $x \leq 2$}, WHAT IS THE DOMAIN AND RANGE OF R?

WE SHALL NOW CONSIDER AN EXAMPLE INVOLVING AN APPLICATION OF A SYSTEM… INEQUALITIES.

EXAMPLE 4 A FURNITURE COMPANY MAKES TABLES AND CHAIRS. TO PRODUCE A TABLE IT REQUIRES 2 HRS ON MACHINE A, AND 4 HRS ON MACHINE B. TO PRODUCE A C… IT REQUIRES 3 HRS ON MACHINE A AND 2 HRS ON MACHINE B. MACHINE A… OPERATE AT MOST 12 HRS A DAY AND MACHINE B CAN OPERATE AT MOST… DAY. IF THE COMPANY MAKES A PROFIT OF BIRR 12 ON A TABLE AND BIRR 1… CHAIR, HOW MANY OF EACH SHOULD BE PRODUCED TO MAXIMIZE ITS PROFI…

SOLUTION: LET x BE THE NUMBER OF TABLES TO BE PRODUCED AND y BE THE NUMBER OF CHAIRS TO BE PRODUCED.

THEN, IF A TABLE IS PRODUCED IN 2 HRS ON MACHINE A, x TABLES REQUIRE $2x$ HRS. SIMILARLY, y CHAIRS REQUIRE $3y$ HRS ON MACHINE A. ON MACHINE B, x TABLES REQUIRE $4x$ HRS AND y CHAIRS REQUIRE $2y$ HRS. SINCE MACHINES A AND B CAN OPERATE AT MOST 12 HRS AND 16 HRS, RESPECTIVELY, YOU HAVE THE FOLLOWING SYSTEM OF LINEAR INEQUALIT

FROM MACHINE A: $2x + 3y \le 12$

FROM MACHINE B: $4x + 2y \le 16$

ALSO, $x \ge 0$ AND $y \ge 0$ SINCE x AND y ARE NUMBERS OF TABLES AND CHAIRS.

HENCE, YOU OBTAIN A SYSTEM OF LINEAR INEQUALITIES GIVEN AS FOLLOWS:

$$\begin{cases} 2x+3y \le 12 \\ 4x+2y \le 16 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

SINCE THE INEQUALITIES INVOLVED IN THE SYSTEM ARE ALL LINEAR, THE BOUN GRAPH OF THE SYSTEM ARE STRAIGHT LINES. THE REGION CONTAINING THE SO SYSTEM IS THE QUADRILATERAL SHOWN BELOW.

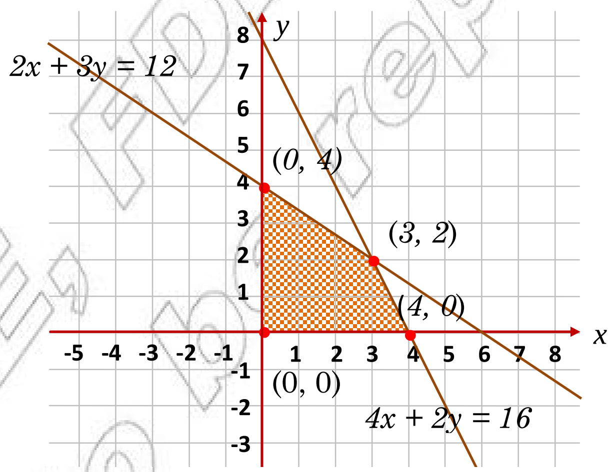


Figure 3.14

THE PROFIT MADE IS BIRR 12 ON A TABLE, SO BIRR $12x$ ON x TABLES AND BIRR 10 ON A CHAIR, SO BIRR $10y$ ON y CHAIRS. THE PROFIT FUNCTION P IS GIVEN BY $P = 12x + 10y$.

THE VALUES OF x AND y WHICH MAXIMIZE OR MINIMIZE THE PROFIT FUNCTION ON SUCH SYSTEM ARE USUALLY FOUND AT VERTICES OF THE SOLUTION REGION.

HENCE, FROM THE GRAPH, YOU HAVE THE COORDINATES OF EACH VERTEX AS FIGURE 3.14.

THE PROFIT: $P = 12x + 10y$ AT EACH VERTEX IS FOUND TO BE:

AT (0, 0), P = 12 (0) + 10 (0) = 0

AT (0, 4), P = 12 (0) + 10 (4) = 40

AT (3, 2), P = 12 (3) + 10 (2) = 56

AT (4, 0), P = 12 (4) + 10 (0) = 48

THEREFORE, THE PROFIT IS MAXIMUM AT THE VERTEX (3, 2), SO THE COMPANY PRODUCE 3 TABLES AND 2 CHAIRS PER DAY TO GET THE MAXIMUM PROFIT OF BIRR

Group Work 3.2

1 FIND THE SOLUTIONS OF EACH OF THE FOLLOWING SYSTEM OF LINEAR INEQUALITIES GRAPHICALLY:

A $\begin{cases} y + x \geq 0 \\ y - x \geq 0 \\ y \leq 3 \end{cases}$ **B** $\begin{cases} x + y < 1 \\ 2x - y > -1 \\ y - 3x \geq -3 \end{cases}$

2 LET $R = \{(x, y): y \geq x,\ y \geq -x \text{ AND } y \leq 3\}$ AND

$r = \{(x, y): x + y < 1,\ 2x - y > -1 \text{ AND } y - 3x \geq -3\}$

USING QUESTION 1 ABOVE, FIND THE DOMAIN AND RANGE OF THE RELATIONS R AND r

Exercise 3.2

1 DRAW THE GRAPHS OF EACH OF THE FOLLOWING RELATIONS:

A $R = \{(x, y) : x - y \geq 1 \text{ AND } 2x + y < 3\}$

B $R = \{(x, y) : x \leq y - 1 \text{ AND } y + 2x > 2\}$

C $R = \{(x, y) : x > y ;\ x > 0 \text{ AND } y + x < 1\}$

D $R = \{(x, y) : x + y \geq 0 ;\ y \geq 0 \text{ AND } x + y < 1\}$

2 SOLVE EACH OF THE FOLLOWING SYSTEM OF INEQUALITIES GRAPHICALLY:

A $\begin{cases} y \leq 2x + 3 \\ y - x \geq 0 \\ y > 0 \end{cases}$ **B** $\begin{cases} 3x + y < 5 \\ x > 0 \\ x + y < 6 \end{cases}$ **C** $\begin{cases} y \leq 1 - x \\ y > x + 2 \\ y > 0 \end{cases}$

D $\begin{cases} x \geq -1 \\ y \leq 2 \\ y \geq x - 1 \end{cases}$ **E** $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$ **F** $\begin{cases} x > 0 \\ y > 0 \\ x + y < 4 \end{cases}$

3 DESCRIBE EACH OF THE FOLLOWING SHADED REGIONS BY A SYSTEM OF LINEAR INEQUALITIES:

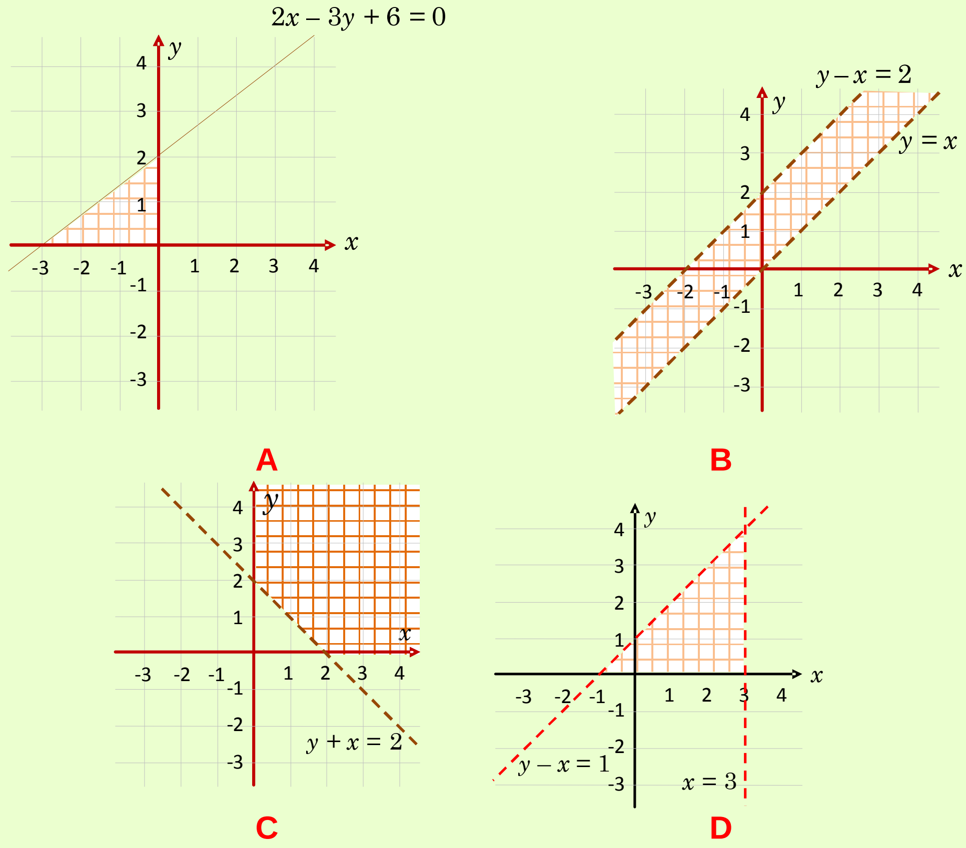


Figure 3.15

4 GIVE A PAIR OF LINEAR INEQUALITIES THAT DESCRIBES THE SET OF ALL POINTS IN THE FIRST QUADRANT.

5 GIVE A SYSTEM OF LINEAR INEQUALITIES WHOSE SOLUTION SET IS ALL THE POINTS INSIDE A RECTANGLE.

6 SUPPOSE THE SUM OF TWO POSITIVE NUMBERS x AND y IS LESS THAN 10 AND GREATER THAN 5. SHOW ALL POSSIBLE VALUES FOR x AND y GRAPHICALLY.

7 SUPPOSE A SHOE FACTORY PRODUCES BOTH LOW-GRADE AND HIGH-GRADE SHOES. THE FACTORY PRODUCES AT LEAST TWICE AS MANY LOW-GRADE AS HIGH-GRADE MAXIMUM POSSIBLE PRODUCTION IS 500 PAIRS OF SHOES. A DEALER CALLS FOR DE AT LEAST 100 HIGH-GRADE PAIRS OF SHOES PER DAY. SUPPOSE THE OPERATION PROFIT OF BIRR 2.00 PER A PAIR OF SHOES ON HIGH-GRADE SHOES AND BIRR 1.00 P OF SHOES ON LOW-GRADE SHOES. HOW MANY PAIRS OF SHOES OF EACH TYPE S PRODUCED FOR MAXIMUM PROFIT?

Hint: LET x DENOTE THE NUMBER OF HIGH-GRADE SHOES.
LET y DENOTE THE NUMBER OF LOW-GRADE SHOES.

3.3 QUADRATIC INEQUALITIES

IN UNIT 2 OF GRADE 9 MATHEMATICS, YOU HAVE LEARNT HOW TO SOLVE QUADRATIC EQUATIONS. (RECALL THAT EQUATIONS OF THE FORM $ax^2 + bx + c = 0$, $a \neq 0$ ARE QUADRATIC EQUATIONS.)

Can similar methods be used to solve quadratic inequalities?

Definition 3.2

An ***inequality*** that can be reduced to any one of the following forms:

$ax^2 + bx + c \leq 0$ or $ax^2 + bx + c < 0$,

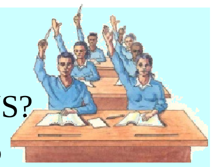
$ax^2 + bx + c \geq 0$ or $ax^2 + bx + c > 0$,

where a, b and c are constants and $a \neq 0$, is called a ***quadratic inequality***.

FOR EXAMPLE $x^2 - 3x + 2 < 0$, $x^2 + 1 \geq 0$, $x^2 + x \leq 0$ AND $x^2 - 4 > 0$ ARE ALL QUADRATIC INEQUALITIES.

THE FOLLOWING ACTIVITY WILL HELP YOU TO RECALL WHAT YOU HAVE LEARNED A EQUATIONS IN GRADE 9.

ACTIVITY 3.5



1 WHICH OF THE FOLLOWING ARE QUADRATIC EQUATIONS?

- **A** $x - 2 = x^2 + 2x$
- **B** $x^2 - 2x = x^2 + 3x + 6$
- **C** $2(x - 4) - (x - 2) = (x + 2)(x - 4)$
- **D** $x^3 - 3 = 1 + 4x + x^2$
- **E** $(x - 1)(x + 2) \geq 0$
- **F** $x(x - 1)(x + 1) = 0$.

2 WHICH OF THE FOLLOWING ARE QUADRATIC INEQUALITIES?

- **A** $2x^2 \leq 5x + x^2 - 3$
- **B** $2x^2 > 2x + x^2 + 8$
- **C** $x(1 - x) \leq (x + 2)(1 - x)$
- **D** $3x^2 + 5x + 6 > 0$
- **E** $5 - 2(x^2 + x) < 6x - 2x^2$
- **F** $(x - 2)(x + 1) \geq 2 - 2x$
- **G** $-1 > (x^2 + 1)(x + 2)$.

3 IF THE PRODUCT OF TWO REAL NUMBERS IS ZERO, WHAT CAN YOU SAY ABOUT THE TWO NUMBERS?

4 FACTORIZE EACH OF THE FOLLOWING IF POSSIBLE:

- **A** $x^2 + 6x$
- **B** $35x - 28x^2$
- **C** $\frac{1}{16} - 25x^2$
- **D** $4x^2 + 7x + 3$
- **E** $x^2 - x + 3$
- **F** $x^2 + 2x - 3$
- **G** $3x^2 - 11x - 4$
- **H** $x^2 + 4x + 4$.

5 GIVEN A QUADRATIC EQUATION $ax^2 + bx + c = 0$,

- **A** WHAT IS ITS DISCRIMINANT?
- **B** STATE WHAT MUST BE TRUE ABOUT THE DISCRIMINANT IF THE EQUATION HAS ONE REAL ROOT, TWO DISTINCT REAL ROOTS, AND NO REAL ROOT.

3.3.1 Solving Quadratic Inequalities Using Product Properties

SUPPOSE YOU WANT TO SOLVE THE QUADRATIC INEQUALITY

$(x – 2) (x + 3) > 0.$

CHECK THAT $x = 3$ MAKES THE STATEMENT TRUE WHILE $x = 1$ MAKES IT FALSE. HOW DO YOU FIND THE SOLUTION SET OF THE GIVEN INEQUALITY? OBSERVE THAT THE LEFT HAND SIDE O IS THE PRODUCT OF $x – 2$ AND $x + 3$. THE PRODUCT OF TWO REAL NUMBERS IS POSITIVE, IF AN ONLY IF EITHER BOTH ARE POSITIVE OR BOTH ARE NEGATIVE. THIS FACT CAN BE USE GIVEN INEQUALITY.

Product properties:

1 $m.n > 0$, if and only if

I $m > 0$ and $n > 0$ or **II** $m < 0$ and $n < 0$.

2 $m.n < 0$, if and only if

I $m > 0$ and $n < 0$ or **II** $m < 0$ and $n > 0$.

EXAMPLE 1 SOLVE EACH OF THE FOLLOWING INEQUALITIES:

A $(x + 1) (x – 3) > 0$ **B** $3x^2 – 2x \geq 0$

C $–2x^2 + 9x + 5 < 0$ **D** $x^2 – x – 2 \leq 0$

SOLUTION:

A BY PRODUCT PROPERTY 1, $(x + 1) (x – 3)$ IS POSITIVE IF EITHER BOTH THE FACTORS ARE POSITIVE OR BOTH ARE NEGATIVE.

NOW, CONSIDER CASE BY CASE AS FOLLOWS:

Case i WHEN BOTH THE FACTORS ARE POSITIVE

$x + 1 > 0$ AND $x – 3 > 0$

$x > – 1$ AND $x > 3$

THE INTERSECTION OF $x > –1$ AND $x > 3$ IS $x > 3$. THIS CAN BE ILLUSTRATED ON THE NUMBER LINE AS SHOWN IN FIGURE 3.16 BELOW.

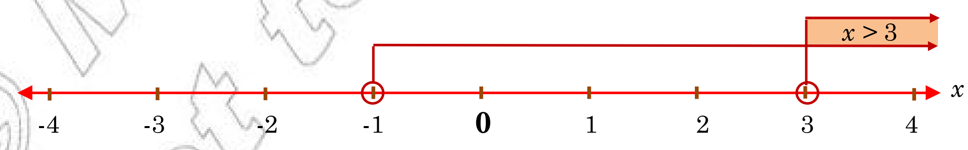


Figure 3.16

THE SOLUTION SET FOR THIS FIRST CASE IS $S_1 = \{x: x > 3\} = (3, \infty)$.

Case ii WHEN BOTH THE FACTORS ARE NEGATIVE

$x + 1 < 0$ AND $x - 3 < 0$

$x < -1$ AND $x < 3$

THE INTERSECTION OF $x < -1$ AND $x < 3$ IS $x < -1$.

THIS CAN BE ILLUSTRATED ON THE NUMBER LINE AS SHOWN BELOW IN FIGURE 3.17

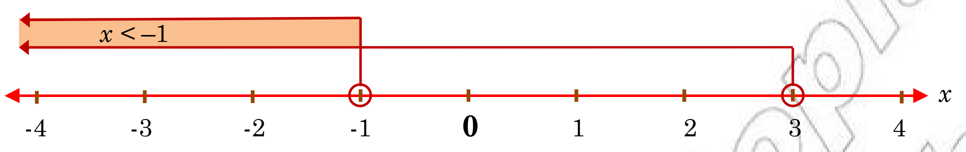


Figure 3.17

THE SOLUTION SET FOR THIS SECOND CASE IS $S_2 = \{x: x < -1\} = (-\infty, -1)$.

THEREFORE, THE SOLUTION SET OF $(x + 1)(x - 3) > 0$ IS:

$S_1 \cup S_2 = \{x: x < -1 \text{ OR } x > 3\} = (-\infty, -1) \cup (3, \infty)$

B FIRST, FACTORIZE $3x^2 - 2x$ AS $x(3x - 2)$

SO, $3x^2 - 2x \geq 0$ MEANS $x(3x - 2) \geq 0$ EQUIVALENTLY,

I $x \geq 0$ AND $3x - 2 \geq 0$ OR

II $x \leq 0$ AND $3x - 2 \leq 0$

Case i WHEN $x \geq 0$ AND $3x - 2 \geq 0$

$x \geq 0$ AND $x \geq \frac{2}{3}$

THE INTERSECTION OF $x \geq 0$ AND $x \geq \frac{2}{3}$ IS $x \geq \frac{2}{3}$. GRAPHICALLY,

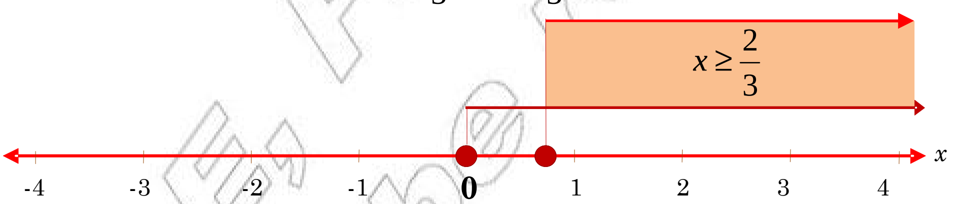


Figure 3.18

SO, $S_1 = \{ x : x \geq \frac{2}{3} \} = [\frac{2}{3}, \infty)$

Case ii WHEN $x \leq 0$ AND $3x - 2 \leq 0$ THAT IS $x \leq 0$ AND $x \leq \frac{2}{3}$

THE INTERSECTION OF $x \leq 0$ AND $x \leq \frac{2}{3}$ IS $x \leq 0$. GRAPHICALLY,

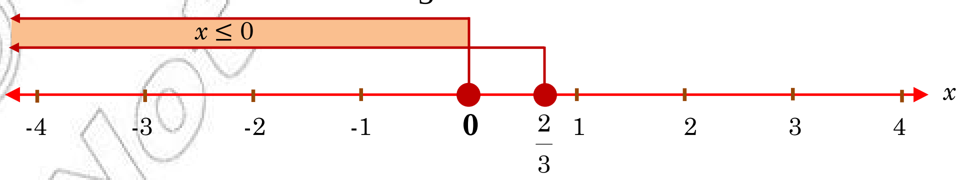


Figure 3.19

SO, $S_2 = \{x: x \leq 0\} = (-\infty, 0]$

THEREFORE, THE SOLUTION SET FOR $3x^2 - 2x \geq 0$ IS

$$S_1 \cup S_2 = \{x: x \leq 0 \text{ OR } x \geq \frac{2}{3}\} = (-\infty, 0] \cup [\frac{2}{3}, \infty)$$

C $\quad -2x^2 + 9x + 5 = (-2x - 1)(x - 5) < 0$

BY PRODUCT PROPERTY 2, $(-2x - 1)(x - 5)$ IS NEGATIVE IF ONE OF THE FACTORS IS NEGATIVE AND THE OTHER IS POSITIVE.

AS BEFORE, CONSIDER CASE BY CASE AS FOLLOWS:

Case i WHEN $-2x - 1 > 0$ AND $x - 5 < 0$

$$x < -\frac{1}{2} \text{ AND } x < 5$$

THE INTERSECTION OF $x < -\frac{1}{2}$ AND $x < 5$ IS $x < -\frac{1}{2}$. GRAPHICALLY,

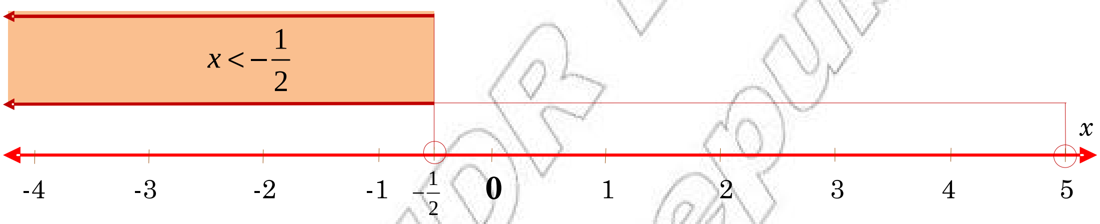


Figure 3.20

SO, $S_1 = \{x: x < -\frac{1}{2}\} = (-\infty, -\frac{1}{2})$

Case ii WHEN $-2x - 1 < 0$ AND $x - 5 > 0$

$$x > -\frac{1}{2} \text{ AND } x > 5$$

THE INTERSECTION OF $x > 5$ AND $x > -\frac{1}{2}$ IS $x > 5$. GRAPHICALLY,

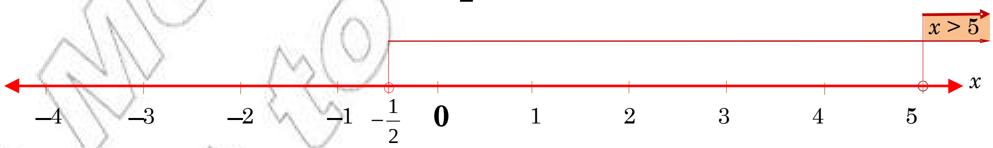


Figure 3.21

SO, $S_2 = \{x: x > 5\} = (5, \infty)$

THEREFORE, THE SOLUTION SET FOR $(-2x - 1)(x - 5) < 0$ IS

$$S_1 \cup S_2 = \{x: x < -\frac{1}{2} \text{ OR } x > 5\} = (-\infty, -\frac{1}{2}) \cup (5, \infty)$$

D $x^2 - x - 2 = (x + 1)(x - 2)$

SO, $x^2 - x - 2 \le 0$ MEANS $(x + 1)(x - 2) \le 0$

BY PRODUCT PROPERTY 2, $(x + 1)(x - 2)$ IS NEGATIVE IF ONE OF THE FACTORS IS NEGATIVE AND THE OTHER IS POSITIVE. TO SOLVE $(x + 1)(x - 2) \le 0$, CONSIDER CASE BY CASE AS FOLLOWS:

Case i $x + 1 \ge 0$ AND $x - 2 \le 0$

$x \ge -1$ AND $x \le 2$

THE INTERSECTION OF $x \ge -1$ AND $x \le 2$ IS $-1 \le x \le 2$. GRAPHICALLY,

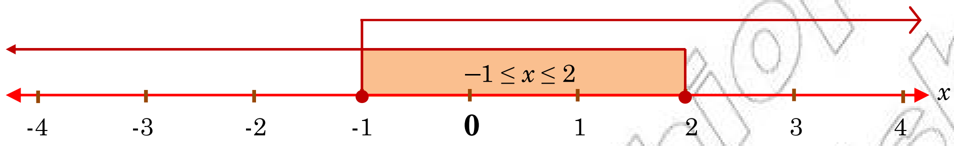


Figure 3.22

SO, $S_1 = \{x: -1 \le x \le 2\} = [-1, 2]$

Case ii $x + 1 \le 0$ AND $x - 2 \ge 0$

$x \le -1$ AND $x \ge 2$

THERE IS NO INTERSECTION OF $x \le -1$ AND $x \ge 2$. GRAPHICALLY,

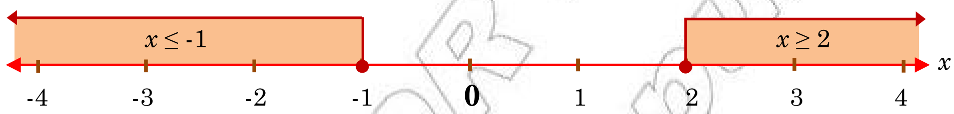


Figure 3.23

SO, $S_2 = \varnothing$

THEREFORE, THE SOLUTION SET FOR $x^2 - x - 2 \le 0$ IS

$S_1 \cup S_2 = \{x: -1 \le x \le 2\} \cup \varnothing = \{x: -1 \le x \le 2\} = [-1, 2]$

Exercise 3.3

1 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING THE PRODUCT PROPERTIES:

A $x(x + 5) > 0$ **B** $(x - 1)^2 \le 0$

C $(4 + x)(x - 4) > 0$ **D** $(5x - 3)(x + 7) < 0$

E $(1 + x)(3 - 2x) \ge 0$ **F** $(5 - x)(1 - \frac{1}{3}x) \le 0$

2 FACTORIZE AND SOLVE EACH OF THE FOLLOWING INEQUALITIES USING PRODUCT PROPERTIES:

A $x^2 + 5x + 4 < 0$ **B** $x^2 - 4 > 0$ **C** $x^2 + 5x + 6 \ge 0$

D $x^2 - 2x + 1 \le 0$ **E** $3x^2 + 4x + 1 \ge 0$ **F** $2x^2 - 7x + 3 < 0$

G $25x^2 - \frac{1}{16} < 0$ **H** $x^2 + 4x + 4 > 0$.

3 **A** FIND THE SOLUTION SET OF THE INEQUALITY $x^2 < 25$

B WHY IS $\{x \mid x < 5\}$ NOT THE SOLUTION SET OF $x^2 < 25$?

4 IF $x < y$, DOES IT FOLLOW THAT $x^2 < y^2$?

5 IF A BALL IS THROWN UPWARD FROM GROUND LEVEL WITH AN INITIAL VELOCITY OF 24 M/S, ITS HEIGHT H IN METRES AFTER T SECONDS IS GIVEN BY $h(t) = 24t - 16t^2$. WHEN WILL THE BALL BE AT A HEIGHT OF MORE THAN 8 METRES?

3.3.2 Solving Quadratic Inequalities Using the Sign Chart Method

SUPPOSE YOU NEED TO SOLVE THE QUADRATIC INEQUALITY

$$x^2 + 3x - 4 < 0.$$

CONSIDER HOW THE SIGN OF $x^2 + 3x - 4$ CHANGES AS YOU VARY THE VALUES OF THE UNKNOW AS x IS MOVED ALONG THE NUMBER LINE, THE QUANTITY $x^2 + 3x - 4$ IS SOMETIMES POSITIVE, SOMETIMES ZERO, AND SOMETIMES NEGATIVE. TO SOLVE THE INEQUALITY, YOU MUS VALUES OF x FOR WHICH $x^2 + 3x - 4$ IS NEGATIVE. INTERVALS WHERE $x^2 + 3x - 4$ IS POSITIVE ARE SEPARATED FROM INTERVALS WHERE IT IS NEGATIVE BY VALUES OF x FOR WHICH IT IS ZERO. TO LOCATE THESE VALUES, SOLVE THE EQUATION $x^2 + 3x - 4 = 0$.

FACTORIZE $x^2 + 3x - 4$ AND FIND THE TWO ROOTS (–4 AND 1). DIVIDE THE NUMBER LINE I THREE OPEN INTERVALS. THE EXPRESSION $x^2 + 3x - 4$ WILL HAVE THE SAME SIGN IN EACH OF THESE INTERVALS, $(-\infty, -4)$, $(-4, 1)$ AND $(1, \infty)$.

THE "SIGN CHART" METHOD ALLOWS YOU TO FIND THE SIGN OF $x^2 + 3x - 4$ IN EACH INTERVAL.

Step 1 FACTORIZE $x^2 + 3x - 4 = (x + 4)(x - 1)$

Step 2 DRAW A SIGN CHART, NOTING THE SIGN OF EACH FACTOR AND HENCE THE WHOLE EXPRESSION AS SHOWN BELOW.

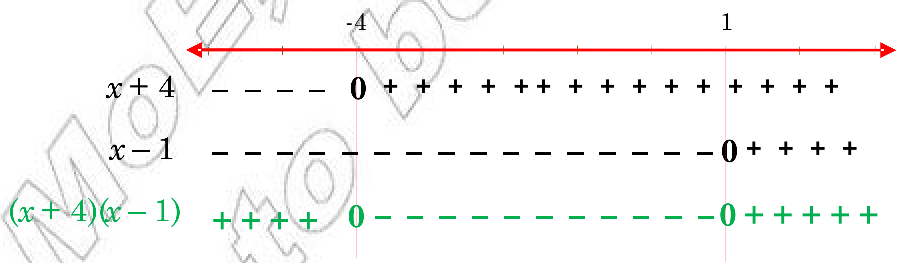


Figure 3.24

Step 3 READ THE SOLUTION FROM THE LAST LINE OF THE SIGN C

$x^2 + 3x - 4 < 0$ FOR $x \in (-4, 1)$

THEREFORE, THE SOLUTION SET IS THE INTERVAL $(-4, 1)$

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING THE SIGN CHART METHOD:

A $6 + x - x^2 \le 0$ **B** $2x^2 + 3x - 2 \ge 0$.

SOLUTION:

A FACTORIZE $6 + x - x^2$ SO THAT $6 + x - x^2 = (x + 2)(3 - x) \le 0$.

WE MAY IDENTIFY THE SIGN OF $x + 2$ AND $3 - x$ AS FOLLOWS.

$x + 2 < 0$ FOR EACH $x < -2$, $x + 2 = 0$ AT $x = -2$ AND $x + 2 > 0$ FOR EACH $x > -2$.

SIMILARLY, $3 - x > 0$ FOR EACH $x < 3$, $3 - x = 0$ AT $x = 3$ AND $3 - x < 0$ FOR EACH $x > 3$.

THEREFORE, THE ABOVE RESULTS ARE SHOWN IN THE SIGN CHART GIVEN BELOW IN FIGURE 3.25.

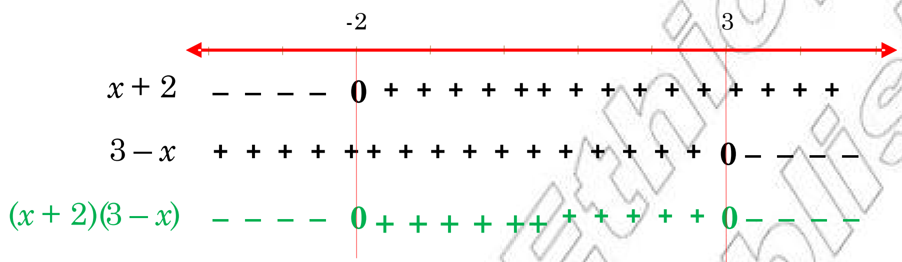


Figure 3.25

FROM THE SIGN CHART, YOU CAN IMMEDIATELY READ THE FOLLOWING:

I THE SOLUTION SET OF $(x + 2)(3 - x) < 0$ IS $\{x: x < -2$ OR $x > 3\} = (-\infty, -2) \cup (3, \infty)$.

II THE SOLUTION SET OF $(x + 2)(3 - x) > 0$ IS $\{x: -2 < x < 3\} = (-2, 3)$.

III THE SOLUTION SET OF $(x + 2)(3 - x) = 0$ IS $\{-2, 3\}$.

IV THE SOLUTION SET OF $(x + 2)(3 - x) \le 0$ IS $(-\infty, -2] \cup [3, \infty)$

THEREFORE, THE SOLUTION SET OF $6 + x - x^2 \le 0$ IS $(-\infty, -2] \cup [3, \infty)$.

B $2x^2 + 3x - 2 = (2x - 1)(x + 2) \ge 0$.

$2x - 1 < 0$ FOR EACH $x < \frac{1}{2}$, $2x - 1 = 0$ AT $x = \frac{1}{2}$, AND $2x - 1 > 0$ FOR EACH $x > \frac{1}{2}$.

SIMILARLY, $x + 2 < 0$ FOR EACH $x < -2$, $x + 2 = 0$ AT $x = -2$ AND $x + 2 > 0$ FOR EACH $x > -2$.

THE ABOVE RESULTS ARE SHOWN IN THE SIGN CHART GIVEN BELOW:

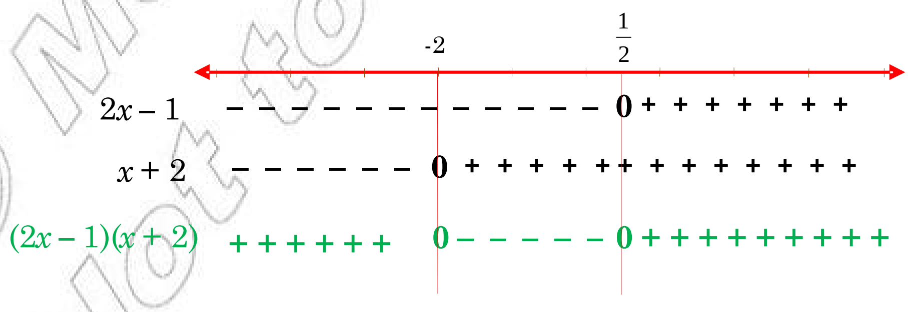


Figure 3.26

FROM THE SIGN CHART, YOU CAN CONCLUDE THAT

$(2x - 1)(x + 2) \geq 0$ FOR EACH $x \in (-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$ AND

$(2x - 1)(x + 2) < 0$ FOR EACH $x \in \left(-2, \frac{1}{2}\right)$.

THEREFORE, THE SOLUTION SET OF $2x^2 + 3x - 2 \geq 0$ IS $(-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$

EXAMPLE 3 FOR WHAT VALUE(S) OF k DOES THE QUADRATIC EQUATION $kx^2 - 2x + k = 0$ HAS

I ONLY ONE REAL ROOT? **II** TWO DISTINCT REAL ROOTS?

III NO REAL ROOTS?

SOLUTION: THE QUADRATIC EQUATION $kx^2 - 2x + k = 0$ IS EQUIVALENT TO THE QUADRATIC EQUATION $ax^2 + bx + c = 0$ WITH $a = k$, $b = -2$ AND $c = k$

THE GIVEN QUADRATIC EQUATION HAS

I ONLY ONE REAL ROOT WHEN $b^2 - 4ac = 0$

SO, $(-2)^2 - 4(k)(k) = 0$

$4 - 4k^2 = 0$ EQUIVALENTLY $(2 - 2k)(2 + 2k) = 0$

$2 - 2k = 0$ OR $2 + 2k = 0$

$k = 1$ OR $k = -1$

THEREFORE, $kx^2 - 2x + k = 0$ HAS ONLY ONE REAL ROOT IF EITHER $k = 1$ OR $k = -1$.

II TWO DISTINCT REAL ROOTS WHEN $b^2 - 4ac > 0$

IT FOLLOWS THAT, $4 - 4k^2 > 0$

$(2 - 2k)(2 + 2k) > 0 \Rightarrow 4(1 - k)(1 + k) > 0$

NOW, USE THE SIGN CHART SHOWN BELOW:

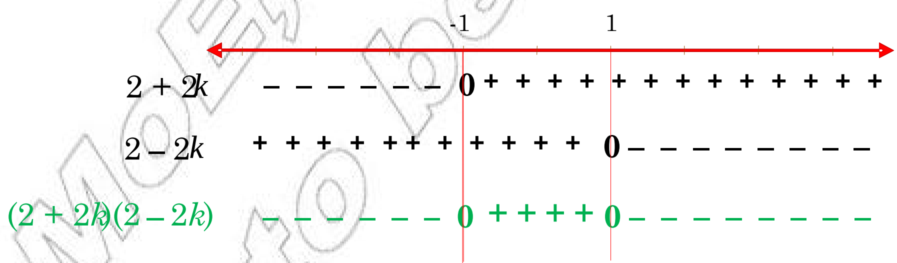


Figure 3.27

THEREFORE, FOR EACH $k \in (-1, 1)$, THE GIVEN QUADRATIC EQUATION HAS TWO DISTINCT RE ROOTS.

III $kx^2 - 2x + k = 0$ HAS NO REAL ROOT FOR EACH $k \in (-\infty, -1) \cup (1, \infty)$ WHERE $B^2 - 4AC < 0$

What do you do if $ax^2 + bx + c,\ a \neq 0$ is not factorizable into linear factors?

THAT IS, THERE ARE NO REAL NUMBERS x_1 AND x_2 SUCH THAT $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

IN THIS CASE, EITHER $ax^2 + bx + c > 0$ FOR ALL VALUES OF x OR $ax^2 + bx + c < 0$ FOR ALL VALUES OF x.

AS A RESULT, THE SOLUTION SET OF $ax^2 + bx + c > 0$ OR $ax^2 + bx + c \geq 0$ IS EITHER $(-\infty, \infty)$ OR { }.

TAKE A TEST POINT AND SUBSTITUTE, IN ORDER TO DECIDE WHICH IS THE CASE.

EXAMPLE 4 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES:

A $x^2 - 2x + 5 \geq 0$ **B** $-3x^2 + x - 1 \geq 0$.

SOLUTION:

A FOR $x^2 - 2x + 5 \geq 0$

$a = 1,\ b = -2,\ c = 5$ AND $b^2 - 4ac = (-2)^2 - 4(1)(5) = -16 < 0$.

HENCE, $x^2 - 2x + 5$ CANNOT BE FACTORIZED.

TAKE A TEST POINT, SAY $x = 0$. THEN, $0^2 - 2(0) + 5 = 5 > 0$

SO, $x^2 - 2x + 5 > 0$ FOR ALL $x \in (-\infty, \infty)$

THEREFORE, THE SOLUTION SET S = $(-\infty, \infty)$

B FOR $-3x^2 + x - 1 \geq 0$

$b^2 - 4ac = (1)^2 - 4(-3)(-1) = 1 - 12 = -11 < 0$

HENCE, $-3x^2 + x - 1$ CANNOT BE FACTORIZED. TAKE A TEST POINT, SAY $x = 0$.

$-3(0)^2 + 0 - 1 = -1 < 0$. HENCE, $-3x^2 + x - 1 \geq 0$ IS FALSE.

THEREFORE, S = { }

Group Work 3.3

1 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING

I PRODUCT PROPERTIES **II** SIGN CHARTS:

A $x^2 - \frac{2}{3}x < 0$ **B** $2x^2 + 5x > 3$

C $(x-1)^2 \geq 2x^2 - 2x$ **D** $(2x-1)(x+1) \leq x(x-3) + 4$

2 WHAT MUST BE THE VALUE(S) OF k SO THAT $(k - 4)x^2 + 2kx - 1 = 0$ HAS

I TWO DISTINCT REAL ROOTS? **II** ONE REAL ROOT? **III** NO REAL ROOTS?

3 A MANUFACTURER DETERMINES THAT THE PROFIT OBTAINED FROM x UNITS OF A CERTAIN ITEM IN BIRR IS $P(x) = 10x - 0.002x^2$

A HOW MANY UNITS MUST BE PRODUCED TO SECURE PROFIT?

B IN THE PROCESS OF PRODUCTION, AT HOW MANY UNITS WILL THERE BE NO PROFIT AND NO LOSS?

Exercise 3.4

1 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING SIGN CHARTS

A $x(x + 5) > 0$ **B** $(x - 3)^2 \geq 0$

C $(4 + x)(4 - x) < 0$ **D** $\left(1 + \frac{x}{3}\right)(5 - x) < 0$

E $3 - x - 2x^2 > 0$ **F** $-6x^2 + 2 \leq x$

G $2x^2 \geq -3 - 5x$ **H** $4x^2 - x - 8 < 3x^2 - 4x + 2$

I $-x^2 + 3x < 4.$

2 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING THE PRODUCT PROPERTIES OR SIGN CHARTS:

A $x^2 + x - 12 > 0$ **B** $x^2 - 6x + 9 > 0$ **C** $x^2 - 3x - 4 \leq 0$

D $5x - x^2 < 6$ **E** $x^2 + 2x < -1$ **F** $x - 1 \leq x^2 + 2$

3 FOR WHAT VALUE(S) OF k DOES EACH OF THE FOLLOWING QUADRATIC EQUATIONS HAVE

I ONE REAL ROOT? **II** TWO DISTINCT REAL ROOTS? **III** NO REAL ROOT?

A $(k + 2)x^2 - (k + 2)x - 1 = 0$

B $x^2 + (5 - k)x + 9 = 0$

4 FOR WHAT VALUE (S) OF k IS

A $kx^2 + 6x + 1 > 0$ FOR EACH REAL NUMBER x?

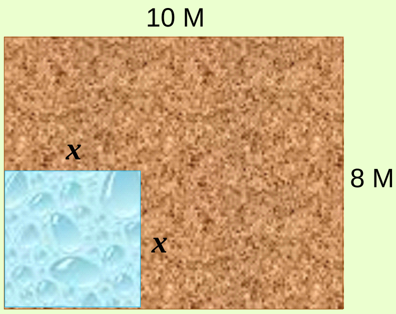
B $x^2 - 9x + k < 0$ ONLY FOR $x \in (2, 11)$?

5 A ROCKET IS FIRED STRAIGHT UPWARD FROM GROUND LEVEL WITH AN INITIAL VELOCITY OF 480 KM/HR. AFTER t SECONDS, ITS DISTANCE ABOVE THE GROUND LEVEL IS GIVEN BY

$480t - 16t^2$.

FOR WHAT TIME INTERVAL IS THE ROCKET MORE THAN 3200 KM ABOVE GROUND LEVE

6 A FARMER HAS 8M BY 10M PLOT OF LAND. HE NEEDS TO CONSTRUCT A WATER RESERVOIR AT ONE CORNER OF THE PLOT WITH EQUAL LENGTH AND WIDTH AS SHOWN BELOW.



FOR WHAT VALUES OF x IS THE AREA OF THE REMAINING PART LESS THAN THE AREA NEE THE RESERVOIR?

3.3.3 Solving Quadratic Inequalities Graphically

IN ORDER TO USE GRAPHS TO SOLVE QUADRATIC INEQUALITIES, IT IS NECESSARY TO U
NATURE OF QUADRATIC FUNCTIONS AND THEIR GRAPHS.

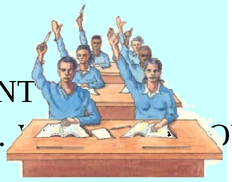
I IF $a > 0$, THEN THE GRAPH OF THE QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$ IS AN ***upward parabola***.

II IF $a < 0$, THEN THE GRAPH OF THE QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$ IS A ***downward parabola***.

ACTIVITY 3.6



1 FOR A QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$, FIND THE POINT
WHICH THE GRAPH TURNS UPWARD OR DOWNWARD. OU CALL
THIS TURNING POINT?

2 SKETCH THE GRAPH AND FIND THE TURNING POINT OF:

A $f(x) = x^2 - 1$ **B** $f(x) = 4 - x^2$

3 WHAT IS THE CONDITION FOR THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ TO HAVE A MAXIMUM VALUE? WHEN WILL IT HAVE A MINIMUM VALUE?

4 WHAT IS THE VALUE OF x AT WHICH THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ ATTAINS ITS MAXIMUM OR MINIMUM VALUE?

THE GRAPH OF A QUADRATIC FUNCTION HAS BOTH ITS ENDS GOING UPWARD OR
DEPENDING ON WHETHER a IS POSITIVE OR NEGATIVE. FROM DIFFERENT GRAPHS YOU CAN OB
THAT THE GRAPH OF A QUADRATIC FUNCTION

$$f(x) = ax^2 + bx + c$$

I CROSSES THE x-AXIS TWICE, IF $b^2 - 4ac > 0$.

II TOUCHES THE x-AXIS AT A POINT, IF $b^2 - 4ac = 0$.

III DOES NOT TOUCH THE x-AXIS AT ALL, IF $b^2 - 4ac < 0$.

TO SOLVE A QUADRATIC INEQUALITY GRAPHICALLY, FIND THE VALUES OF x FOR WHICH THE PART OF THE GRAPH OF THE CORRESPONDING QUADRATIC FUNCTION IS ABOVE THE x-AXIS, BELOW THE x-AXIS OR ON THE x-AXIS. CONSIDER THE FOLLOWING EXAMPLES.

EXAMPLE 5 SOLVE THE QUADRATIC INEQUALITY $x^2 - 3x + 2 < 0$, GRAPHICALLY.

SOLUTION: BEGIN BY DRAWING THE GRAPH OF $f(x) = x^2 - 3x + 2$. SOME VALUES FOR x AND $f(x)$ ARE GIVEN IN THE TABLE BELOW AND THE CORRESPONDING GR
GIVEN IN FIGURE 3.28. COMPLETE THE TABLE FIRST.

x	−3	−2	−1	0	1	2	3
$f(x)$		12		2		0	

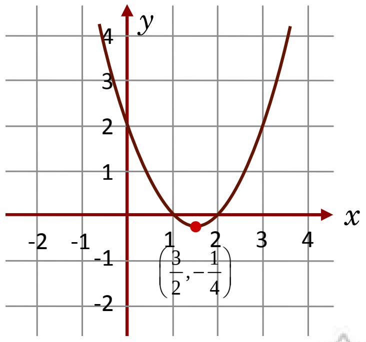


Figure 3.28 *Graph of* $f(x) = x^2 - 3x + 2$

FROM THE GRAPH, $f(x) = 0$ WHEN $x = 1$ AND WHEN $x = 2$. ON THE OTHER HAND, $f(x) > 0$ WHEN $x < 1$ AND WHEN $x > 2$ AND $f(x) < 0$ WHEN x LIES BETWEEN 1 AND 2.

THIS INEQUALITY COULD BE TESTED BY SETTING $x = \frac{3}{2}$, GIVING $f\left(\frac{3}{2}\right) = -\frac{1}{4}$. SO $f\left(\frac{3}{2}\right) < 0$.

IT FOLLOWS THAT THE SOLUTION SET OF $x^2 - 3x + 2 < 0$ CONSISTS OF ALL REAL NUMBERS GREATER THAN 1 AND LESS THAN 2. THAT IS, $S = \{x : 1 < x < 2\} = (1, 2)$.

EXAMPLE 6 SOLVE THE INEQUALITY $x^2 + 4x + 5 > 0$, GRAPHICALLY.

SOLUTION: MAKE A TABLE OF VALUES AND COMPLETE THE TABLE FOR SOME VALUES OF x AND $f(x)$ AS IN THE TABLE BELOW AND SKETCH THE CORRESPONDING GRAPH.

x	−3	−2	−1	0	1	2	3
$f(x)$	2		2		10		

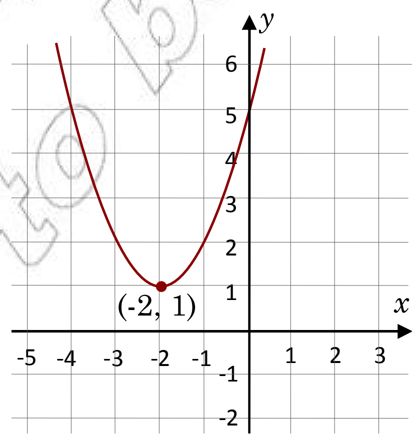


Figure 3.29 Graph of $f(x) = x^2 + 4x + 5$

AS SHOWN IN THE FIGURE 3.29 ABOVE, THE GRAPH OF $f(x) = x^2 + 4x + 5$ DOES NOT CROSS THE x-AXIS BUT LIES ABOVE THE x-AXIS. THUS, THE SOLUTION SET OF THIS INEQUALITY CONSISTS OF ALL REAL NUMBERS. SO, S.S = $(-\infty, \infty)$.

NOTE THAT, IF YOU USE THE PROCESS OF COMPLETING THE SQUARE, YOU OBTAIN

$$x^2 + 4x + 5 > 0 \Rightarrow x^2 + 4x > -5$$

$$x^2 + 4x + 4 > -5 + 4$$

$$(x + 2)^2 > -1$$

SINCE THE SQUARE OF ANY REAL NUMBERS IS NON-NEGATIVE, $(x + 2)^2 > -1$ IS TRUE FOR ALL REAL NUMBERS x.

BASED ON THE ABOVE INFORMATION, COULD YOU SHOW THAT THE SOLUTION SET INEQUALITY $x^2 + 4x + 5 < 0$ IS THE EMPTY SET? WHY?

EXAMPLE 7 SOLVE THE INEQUALITY $-x^2 + 2x + 3 < 0$, GRAPHICALLY.

SOLUTION: MAKE A TABLE OF SELECTED VALUES FOR x AND $f(x)$. THE GRAPH PASSES THROUGH (0, 3) AND (–1, 0) AS SHOWN IN FIGURE 3.30.

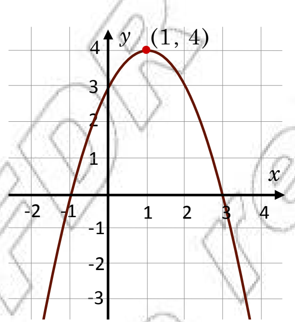


Figure 3.30 Graph of $f(x) = -x^2 + 2x + 3$

THE GRAPH OF $f(x) = 2x - x^2 + 3$ CROSSES THE x-AXIS AT $x = -1$ AND $x = 3$. SO, THE SOLUTION SET OF THIS INEQUALITY IS

S.S = $\{x \mid x < -1 \text{ OR } x > 3\}$.

IF THE QUADRATIC EQUATION $ax^2 + bx + c = 0$, $a \neq 0$ HAS DISCRIMINANT $b^2 - 4ac < 0$, THEN THE EQUATION HAS NO REAL ROOTS. MOREOVER,

I THE SOLUTION SET OF $ax^2 + bx + c \geq 0$ IS THE SET OF ALL REAL NUMBERS IF $a > 0$ AND IS EMPTY SET IF $a < 0$.

II THE SOLUTION SET OF $ax^2 + bx + c \leq 0$ IS THE SET OF ALL REAL NUMBERS IF $a < 0$ AND IS EMPTY SET IF $a > 0$.

Exercise 3.5

1 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES, GRAPHICALLY:

A $x^2 + 6x + 5 \geq 0$ **B** $x^2 + 6x + 5 < 0$

C $x^2 + 8x + 16 < 0$ **D** $x^2 + 2x + 3 \geq 0$

E $3x - x^2 + 2 < 0$ **F** $4x^2 - x \leq 3x^2 + 2$

G $x(x - 2) < 0$ **H** $(x + 1)(x - 2) > 0$

I $3x^2 + 4x + 1 > 0$ **J** $x^2 + 3x + 3 < 0$

K $3x^2 + 22x + 35 \geq 0$ **L** $6x^2 + 1 \geq 5x.$

2 SUPPOSE THE SOLUTION SET OF $x^2 - kx + 2 > 0$ CONSISTS OF THE SET OF ALL REAL NUMBERS. FIND ALL POSSIBLE VALUES OF k.

Key Terms

absolute value	***linear inequality***	***quadratic equation***
closed intervals	***open downward***	***quadratic function***
complete listing	***open intervals***	***quadratic inequality***
discriminant	***open upward***	***sign chart***
infinity	***partial listing***	***solution set***
linear equation	***product property***	

Summary

1 THE OPEN INTERVAL (a, b) WITH END-POINTS a AND b IS THE SET OF ALL REAL NUMBERS x SUCH THAT $a < x < b$.

2 THE CLOSED INTERVAL $[a, b]$ WITH END-POINTS a AND b IS THE SET OF ALL REAL NUMBERS x SUCH THAT $a \leq x \leq b$.

3 THE HALF-OPEN INTERVAL OR HALF-CLOSED INTERVAL $[a, b)$ WITH END-POINTS a AND b IS THE SET OF ALL REAL NUMBERS x SUCH THAT $a \leq x < b$.

4 IF x IS A REAL NUMBER, THEN THE ABSOLUTE VALUE OF x, $|x|$ IS DEFINED BY

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

5 FOR ANY POSITIVE REAL NUMBER a, THE SOLUTION SET OF:

- **i** THE EQUATION $|x| = a$ IS $x = a$ OR $x = -a$;
- **ii** THE INEQUALITY $|x| < a$ IS $-a < x < a$ AND
- **iii** THE INEQUALITY $|x| > a$ IS $x < -a$ OR $x > a$.

6 WHEN TWO OR MORE LINEAR EQUATIONS INVOLVE THE SAME VARIABLES, THEY ARE CALLED A ***system of linear equations***.

7 AN INEQUALITY THAT CAN BE REDUCED TO EITHER $ax^2 + bx + c \le 0$, $ax^2 + bx + c < 0$, $ax^2 + bx + c \ge 0$ OR $ax^2 + bx + c > 0$, WHERE a, b AND c ARE CONSTANTS AND $a \neq 0$, IS CALLED A ***quadratic inequality***.

8 GIVEN ANY QUADRATIC EQUATION $ax^2 + bx + c = 0$,

- **i** IF $b^2 - 4ac > 0$, IT HAS TWO DISTINCT REAL ROOTS.
- **ii** IF $b^2 - 4ac = 0$, IT HAS ONLY ONE REAL ROOT.
- **iii** IF $b^2 - 4ac < 0$, IT HAS NO REAL ROOT.

9 WHEN THE DISCRIMINANT $b^2 - 4ac < 0$, THEN

- **i** THE SOLUTION SET OF $ax^2 + bx + c > 0$ IS THE SET OF ALL REAL NUMBERS, IF $a > 0$ AND EMPTY SET IF $a < 0$.
- **ii** THE SOLUTION SET OF $ax^2 + bx + c < 0$ IS THE SET OF ALL REAL NUMBERS, IF $a < 0$ AND EMPTY SET IF $a > 0$.

10 PRODUCT PROPERTY:

- **i** $mn > 0$, IF AND ONLY IF $m > 0$ AND $n > 0$ OR $m < 0$ AND $n < 0$.
- **ii** $mn < 0$, IF AND ONLY IF $m > 0$ AND $n < 0$ OR $m < 0$ AND $n > 0$.

Review Exercises on Unit 3

1 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING THE PRODUCT PROPERTIES:

A	$(x + 1)(x - 3) < 0$	**B**	$\left(\frac{2}{3}x + 3\right)(x - 1) < 0$
C	$(x - \sqrt{3})(x + \sqrt{2}) > 0$	**D**	$x^2 > x$
E	$x^2 + 5x + 4 \ge 0$	**F**	$(x - 2)^2 \le 2 - x$
G	$1 - 2x \ge (1 + x)^2$	**H**	$3x^2 - 6x + 5 < x^2 - 2x + 3.$

2 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING SIGN CHARTS:

- **A** $(1 - x)(5 - x) > 0$ **B** $x^2 \leq 9$ **C** $(x + 2)^2 < 25$
- **D** $1 - x \geq 2x^2$ **E** $6t^2 + 1 < 5t$ **F** $2t^2 + 3t \leq 5.$

3 SOLVE EACH OF THE FOLLOWING INEQUALITIES GRAPHICALLY:

- **A** $x^2 - x + 1 > 0$ **B** $x^2 > x + 6$ **C** $x^2 - 4x - 1 > 0$
- **D** $x^2 + 25 \geq 10x$ **E** $x^2 + 32 \geq 12x + 6$ **F** $x(6x - 13) > -6$
- **G** $x(10 - 3x) < 8$ **H** $(x - 3)^2 \leq 1$

4 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING ANY CONVENIENT METHOD:

- **A** $2x^2 < x + 2$ **B** $-2x^2 + 6x + 15 \leq 0$
- **C** $\frac{1}{2}x^2 + \frac{25}{2} \geq 5x$ **D** $6x^2 - x + 3 < 5x^2 + 5x - 5$
- **E** $x(10x + 19) \leq 15$ **F** $(x + 2)^2 > (3x + 1)^2.$

5 WHAT MUST THE VALUE(S) OF k BE SO THAT:

- **A** $kx^2 - 10x - 5 \leq 0$ FOR ALL x?
- **B** $2x^2 + (k - 3)x + k - 5 = 0$ HAS ONE REAL ROOT? TWO REAL ROOTS? NO REAL ROOT?

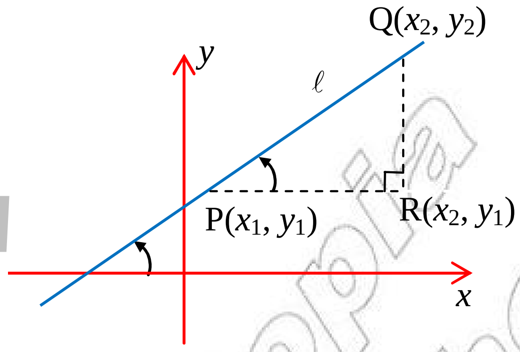
6 THE SUM OF A NON-NEGATIVE NUMBER AND ITS SQUARE IS LESS THAN 12. WHAT COULD THE NUMBER BE?

7 THE SUM OF A NUMBER x AND TWICE ANOTHER IS 20. IF THE PRODUCT OF THESE NUMBERS IS NOT MORE THAN 48, WHAT ARE ALL POSSIBLE VALUES OF x?

8 THE PROFIT OF A CERTAIN COMPANY IS GIVEN BY $Y(x) = 10,000 + 350x - \frac{1}{2}x^2$

WHERE x IS THE AMOUNT (BIRR IN TENS) SPENT ON ADVERTISING. WHAT AMOUNT GIVES A PROFIT OF MORE THAN BIRR 40,000?

Unit 4



COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *apply the distance formula to find the distance between any two given points in the coordinate plane.*
- *formulate and apply the section formula to find a point that divides a given line segment in a given ratio.*
- *write different forms of equations of a line and understand related terms.*
- *describe parallel or perpendicular lines in terms of their slopes.*

Main Contents

4.1 Distance between two points

4.2 Division of a line segment

4.3 Equation of a line

4.4 Parallel and perpendicular lines

Key Terms

Summary

Review Exercises

INTRODUCTION

IN UNIT 3, YOU HAVE SEEN AN IMPORTANT CONNECTION BETWEEN ALGEBRA AND GEOME THE GREAT DISCOVERIES OF 17TH CENTURY MATHEMATICS WAS THE SUBJECT OF ***analytic geometry***. IT IS OFTEN REFERRED TO AS CARTESIAN GEOMETRY, WHICH IS THE METHOD OF STUDYING GEOMETRY BY MEANS OF A COORDINATE SYSTEM AND ASSOCIATED ALG

IN ANALYTIC GEOMETRY, WE DESCRIBE PROPERTIES OF GEOMETRIC FIGURES SUCH AS CIRCLES, ETC., IN TERMS OF ORDERED PAIRS AND EQUATIONS.

4.1 DISTANCE BETWEEN TWO POINTS

IN GRADE 9, YOU HAVE DISCUSSED THE NUMBER LINE AND YOU HAVE SEEN THAT THERE TO-ONE CORRESPONDENCE BETWEEN THE SET OF REAL NUMBERS AND THE SET OF NUMBER LINE. YOU HAVE ALSO SEEN HOW TO LOCATE A POINT IN THE COORDINATE P REMEMBER THE FACT THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN THE THE PLANE AND THE SET OF ALL ORDERED PAIRS OF REAL NUMBERS?

THE FOLLOWING ACTIVITY WILL HELP YOU TO REVIEW THE FACTS YOU DISCUSSED IN GRADE 9.

ACTIVITY 4.1

1 CONSIDER THE NUMBER LINE GIVEN IN FIGURE 4.1.

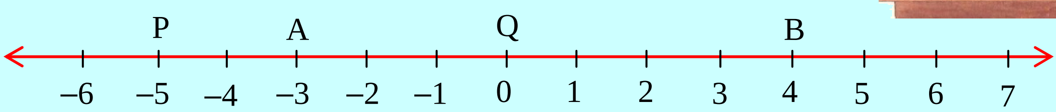


Figure 4.1

A FIND THE COORDINATES OF POINTS P, Q, A AND B.

B FIND THE DISTANCE BETWEEN POINTS

I P AND Q **II** Q AND B **III** P AND B

2 ON A NUMBER LINE, THE TWO POINTS P AND Q HAVE COORDINATES x_1 AND x_2.

A FIND THE DISTANCE BETWEEN P AND Q (OR PQ).

B FIND THE DISTANCE BETWEEN Q AND P.

C DISCUSS THE RELATIONSHIP BETWEEN YOUR ANSWERS IN A AND B.

D DISCUSS THE RELATIONSHIP BETWEEN $|x_2 - x_1|$ AND $|x_1 - x_2|$.

3 HOW DO YOU PLOT THE COORDINATES OF POINTS IN THE COORDINATE PLANE?

4 WHAT ARE THE COORDINATES OF THE ORIGIN OF THE xy-PLANE?

5 DRAW A COORDINATE PLANE AND PLOT THE FOLLOWING POINTS.

P (3,–4), Q (–3,–2), R (–2, 0), S (4, 0), T (2, 3), U (–4, 5) AND V (0, 0).

6 THE POSITION OF EACH POINT ON THE COORDINATE PLANE IS DETERMINED BY ITS PAIR OF NUMBERS.

- **A** WHAT IS THE x-COORDINATE OF A POINT ON THE y-AXIS?
- **B** WHAT IS THE y-COORDINATE OF A POINT ON THE x-AXIS?

7 LET P (2, 3) AND Q (2, 8) BE POINTS ON THE COORDINATE PLANE.

- **A** PLOT THE POINTS P AND Q.
- **B** IS THE LINE THROUGH POINTS P AND Q VERTICAL OR HORIZONTAL?
- **C** WHAT IS THE DISTANCE BETWEEN P AND Q?

8 LET R (–2, 4) AND T (5, 4) BE POINTS ON THE COORDINATE PLANE.

- **A** PLOT THE POINTS T AND R.
- **B** IS THE LINE THROUGH R AND T VERTICAL OR HORIZONTAL?
- **C** WHAT IS THE DISTANCE BETWEEN POINTS R AND T?

Distance between points in a plane

SUPPOSE P(x_1, y_1) AND Q(x_2, y_2) ARE TWO DISTINCT POINTS ON THE xy-COORDINATE PLANE. WE CAN FIND THE DISTANCE BETWEEN THE TWO POINTS P AND Q BY CONSIDERING THREE CASES.

Case i WHEN P AND Q ARE ON A LINE PARALLEL TO THE x-AXIS (THAT IS, $\overline{PQ}$ IS A HORIZONTAL SEGMENT) AS IN FIGURE 4.2.

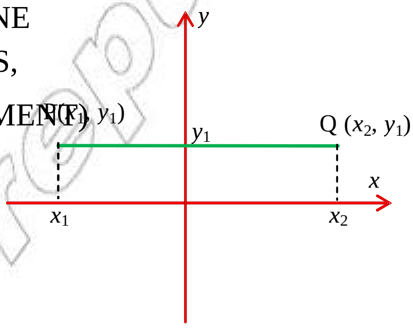


Figure 4.2

SINCE THE TWO POINTS P AND Q HAVE THE SAME y-COORDINATE (***ordinate***), THE DISTANCE BETWEEN P AND Q IS

$$PQ = |x_2 - x_1|$$

Case ii WHEN P AND Q ARE ON A LINE PARALLEL TO THE y-AXIS (THAT IS, $\overline{PQ}$ IS A VERTICAL SEGMENT) AS IN FIGURE 4.3.

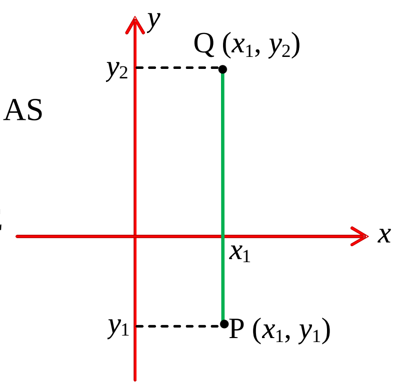


Figure 4.3

SINCE THE TWO POINTS HAVE THE SAME x-COORDINATE (***abscissa***), THE DISTANCE BETWEEN P AND Q IS

$$PQ = |y_2 - y_1|$$

Case iii WHEN $\overline{PQ}$ IS NEITHER VERTICAL NOR HORIZONTAL (THE GENERAL CASE).

TO FIND THE DISTANCE BETWEEN THE POINTS P AND Q, DRAW A LINE PASSING THROUGH P PARALLEL TO THE x-AXIS AND DRAW A LINE PASSING THROUGH Q PARALLEL TO THE y-AXIS.

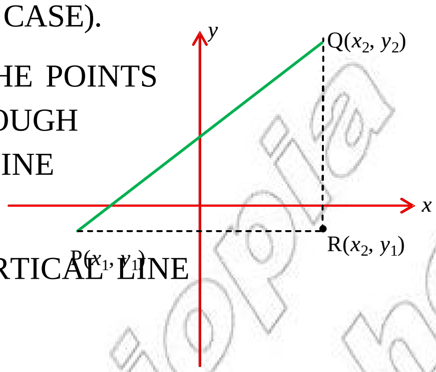


Figure 4.4

THE HORIZONTAL LINE AND THE VERTICAL LINE INTERSECT AT $R(x_2, y_1)$.

USING CASE I AND CASE II, WE HAVE

$$PR = |x_2 - x_1| \text{ AND } RQ = |y_2 - y_1|$$

SINCE $\triangle PRQ$ IS A RIGHT ANGLED TRIANGLE AT R, YOU CAN USE ***Pythagoras' Theorem*** TO FIND THE DISTANCE BETWEEN POINTS P AND Q AS FOLLOWS:

$$PQ^2 = PR^2 + RQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

THEREFORE, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

THE RADICAL HAS POSITIVE SIGN (WHY?).

IN GENERAL, THE DISTANCE d BETWEEN ANY TWO POINTS $P(x_1, y_1)$ AND $Q(x_2, y_2)$ IS GIVEN BY

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

THIS IS CALLED THE ***distance formula***.

EXAMPLE 1 FIND THE DISTANCE BETWEEN THE GIVEN POINTS.

A A $(1, \sqrt{2})$ AND B $(1, -\sqrt{2})$

B P $\left(\frac{17}{4}, -2\right)$ AND Q $\left(\frac{1}{4}, -2\right)$

C R $\left(-\sqrt{2}, -1\right)$ AND S $\left(\sqrt{2}, -\sqrt{2}\right)$

D A $(a, -b)$ AND B $(-b, a)$

SOLUTION:

A $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(1-1)^2 + (-\sqrt{2} - \sqrt{2})^2}$

$= \sqrt{(0)^2 + (-2\sqrt{2})^2} = 2\sqrt{2}.$

OR, MORE SIMPLY

$AB = |y_2 - y_1| = |-\sqrt{2} - \sqrt{2}|$

$= 2\sqrt{2}$ UNIT

B $PQ = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{\left(\frac{1}{4} - \frac{17}{4}\right)^2 + (-2-(-2))^2}$

$= \sqrt{\left(\frac{-16}{4}\right)^2 + (0)^2} = \sqrt{(-4)^2} = \sqrt{16} = 4$ UNIT

OR, MORE SIMPLY

$PQ = |x_2 - x_1| = \left|\frac{1}{4} - \frac{17}{4}\right|$

$= 4$ UNITS

C $RS = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\sqrt{2} - (-\sqrt{2})\right)^2 + \left(-\sqrt{2} - (-1)\right)^2}$

$= \sqrt{\left(2\sqrt{2}\right)^2 + \left(1 - \sqrt{2}\right)^2} = \sqrt{11 - 2\sqrt{2}}$

D $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-b-a)^2 + (a-(-b))^2}$

$= \sqrt{(b+a)^2 + (a+b)^2} = \sqrt{2(a+b)^2} = \sqrt{2}\,|a+b|$ UNIT

Exercise 4.1

1 IN EACH OF THE FOLLOWING, FIND THE DISTANCE BETWEEN THE TWO GIVEN POINT

A A (1, –5) AND B (7, 3)

B C $\left(-2, \frac{1}{2}\right)$ AND D $\left(\frac{1}{2}, 2\right)$

C E$(\sqrt{2}, 1)$ AND F$(-\sqrt{6}, \sqrt{3})$

D G $(a, -b)$ AND H $(-a, b)$

E THE ORIGIN AND K$\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$

F L$\left(\sqrt{2}, 1\right)$ AND M$\left(1, \sqrt{2}\right)$

G P$\left(\sqrt{2}, \sqrt{3}\right)$ AND Q$\left(\sqrt{2}\ 2, \sqrt{2}\ 3\right)$

H R$\left(\sqrt{2}a, c\right)$ AND T$\left(\sqrt{2}b, c\right)$

2 USING THE DISTANCE FORMULA, SHOW THAT THE DISTANCE BETWEEN P AND Q IS:

A $|x_2 - x_1|$, WHEN $\overline{PQ}$ IS HORIZONTAL **B** $|y_2 - y_1|$, WHEN $\overline{PQ}$ IS VERTICAL.

3 LET A (3, –7) AND B (–1, 4) BE TWO ADJACENT VERTICES OF A SQUARE. CALCULATE T OF THE SQUARE.

4 P (3, 5) AND Q (1, –3) ARE TWO OPPOSITE VERTICES OF A SQUARE. FIND ITS AREA.

5 SHOW THAT THE PLANE FIGURE WITH VERTICES:

A A (5, –1), B (2, 3) AND C (1, 1) IS A RIGHT ANGLED TRIANGLE.

B A (–4, 3), B (4, –3) AND C $(3\sqrt{3}, 4\sqrt{3})$ IS AN EQUILATERAL TRIANGLE.

C A (2, 3), B (6, 8) , C (7, –1) IS AN ISOSCELES TRIANGLE.

6 AN EQUILATERAL TRIANGLE HAS TWO VERTICES AT A (–4, 0) AND B (4, 0). WHAT CO COORDINATES OF THE THIRD VERTEX BE?

7 WHAT ARE THE POSSIBLE VALUES OF b IF THE POINT A(b, 4) IS 10 UNITS AWAY FROM B (0, –2)?

4.2 DIVISION OF A LINE SEGMENT

RECALL THAT, A LINE SEGMENT PASSING THROUGH TWO POINTS A AND B IS ***horizontal*** IF THE TWO POINTS HAVE THE SAME ***y*-coordinate**. I.E., A LINE SEGMENT WHOSE END-POINTS ARE A (x_1, y) AND B (x_2, y) IS A HORIZONTAL LINE SEGMENT AS SHOWN IN FIGURE 4.5.

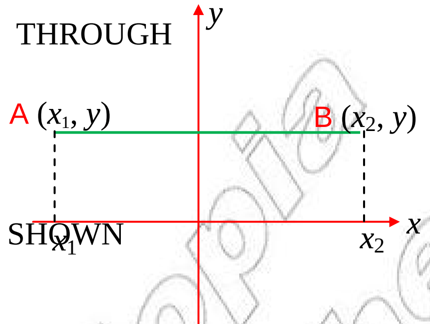
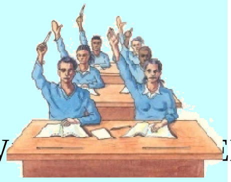


Figure 4.5

What is the mid-point of $\overline{AB}$?

ACTIVITY 4.2



1 DEFINE THE RATIO OF TWO QUANTITIES.

2 WHAT IS MEANT BY THE RATIO OF THE LENGTH OF TWO LINE SEGMENTS?

3 IN FIGURE 4.6, FIND THE RATIO OF THE LENGTH OF $\overline{AP}$ TO $\overline{PB}$.

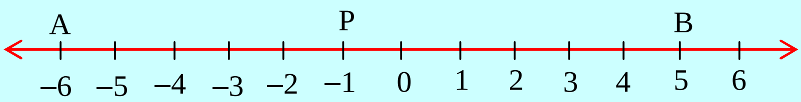


Figure 4.6

4 WHAT IS MEANT BY A POINT THAT DIVIDES A LINE SEGMENT INTERNALLY?

5 PLOT THE FOLLOWING POINTS ON THE COORDINATE PLANE AND FIND THE MID-POINT OF THE SEGMENT JOINING THE POINTS.

A A (2, –1) AND B (2, 5) **B** C (–3, 3) AND D (3, 3) **C** E (2, 0) AND F (–2, 4).

CONSIDER THE HORIZONTAL LINE SEGMENT WITH END-POINTS A (x_1, y) AND B (x_2, y) AS SHOWN IN FIGURE 4.7. IN TERMS OF THE COORDINATES OF A AND B, DETERMINE THE COORDINATES OF THE POINT P (x_O, y_O) THAT DIVIDES $\overline{AB}$ INTERNALLY IN THE RATIO $m:n$.

CLEARLY, THE RATIO OF THE LENGTH OF SEGMENT AP TO THE SEGMENT PB IS GIVEN BY $\frac{AP}{PB}$.

THE DISTANCE BETWEEN A AND P IS $AP = x_O - x_1$.

THE DISTANCE BETWEEN P AND B IS $PB = x_2 - x_O$.

THEREFORE, $\frac{AP}{PB} = \frac{m}{n}$ I.E., $\frac{x_o - x_1}{x_2 - x_o} = \frac{m}{n}$.

SOLVING THIS EQUATION FOR x_O

$\Rightarrow n(x_O - x_1) = m(x_2 - x_O)$

$\Rightarrow nx_O - nx_1 = mx_2 - mx_O$

$\Rightarrow nx_O + mx_O = nx_1 + mx_2$

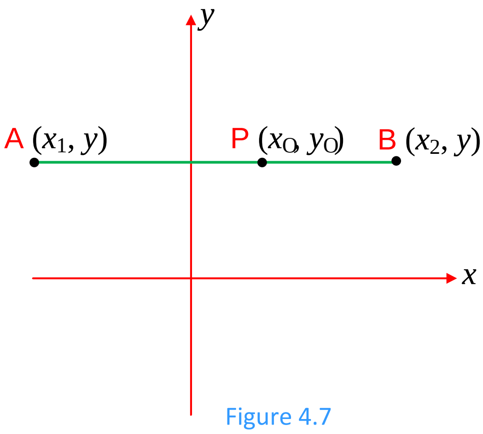


Figure 4.7

$$\Rightarrow x_O(n+m) = nx_1 + mx_2$$

$$\Rightarrow x_O = \frac{nx_1 + mx_2}{n+m}$$

SINCE $\overline{AB}$ IS PARALLEL TO THE x-AXIS ($\overline{AB}$ IS A HORIZONTAL LINE SEGMENT) AND OBVIOUSLY, $y_O = y$, THEREFORE, THE POINT $P(x_O, y_O)$ IS $\left(\frac{nx_1 + mx_2}{n+m}, y\right)$.

GIVEN A LINE SEGMENT PQ WITH END POINT COORDINATES $P(x_1, y_1)$ AND $Q(x_2, y_2)$, LET US FIND THE COORDINATES OF THE POINT R DIVIDING THE LINE SEGMENT PQ INTERNALLY IN THE RATIO $m : n$

I.E., $\frac{PR}{RQ} = \frac{m}{n}$, WHERE m AND n ARE GIVEN POSITIVE REAL NUMBERS.

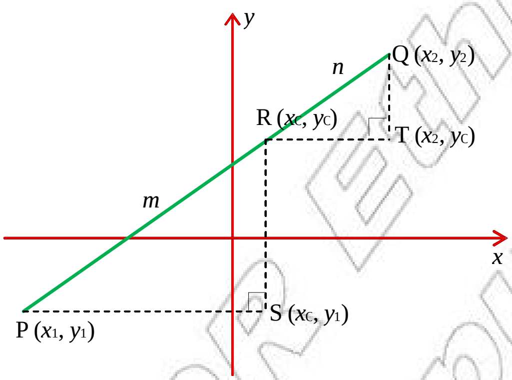


Figure 4.8

LET THE COORDINATES OF R BE (x_O, y_O). ASSUME THAT $x_1 \neq x_2$ AND $y_1 \neq y_2$.

IF YOU DRAW LINES THROUGH THE POINTS P, Q AND R PARALLEL TO THE AXES AS SHOWN IN FIGURE 4.8, THE POINTS S AND T HAVE THE COORDINATES (x_O, y_1) AND (x_2, y_O), RESPECTIVELY.

$PS = x_O - x_1$, $RT = x_2 - x_O$, $SR = y_O - y_1$ AND $TQ = y_2 - y_O$

SINCE TRIANGLES PSR AND RTQ ARE SIMILAR (WHY?),

$$\frac{PS}{RT} = \frac{PR}{RQ} \text{ AND } \frac{SR}{TQ} = \frac{PR}{RQ}$$

$$\frac{x_o - x_1}{x_2 - x_o} = \frac{m}{n} \text{ AND } \frac{y_o - y_1}{y_2 - y_o} = \frac{m}{n}$$

SOLVING FOR x_O AND y_O

$\Rightarrow \quad n(x_O - x_1) = m(x_2 - x_O)$ AND $n(y_O - y_1) = m(y_2 - y_O)$

$\Rightarrow \quad nx_O - nx_1 = mx_2 - mx_O$ AND $ny_O - ny_1 = my_2 - my_O$

$\Rightarrow \quad nx_O + mx_O = nx_1 + mx_2$ AND $ny_O + my_O = ny_1 + my_2$

$\Rightarrow \quad x_O(n+m) = nx_1 + mx_2$ AND $y_O(n+m) = ny_1 + my_2$

$$\Rightarrow \quad x_O = \frac{nx_1 + mx_2}{n+m} \text{ AND } y_o = \frac{ny_1 + my_2}{n+m}$$

THE POINT R(x_O, y_O) DIVIDING THE LINE SEGMENT PQ INTERNALLY IN THE RATIO $m : n$ IS GIVEN BY

$$R\,(x_O, y_O) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$$

THIS IS CALLED THE ***section formula***.

EXAMPLE 1 FIND THE COORDINATES OF THE POINT R THAT DIVIDES THE LINE SEGMENT WITH END- POINTS A (6, 2) AND B (1, – 4) IN THE RATIO 2:3.

SOLUTION: PUT (x_1, y_1) = (6, 2), (x_2, y_2) = (1, –4), m = 2 AND n = 3. USING THE SECTION FORMULA, YOU HAVE

$$R\,(x_O, y_O) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right) = \left(\frac{3\times 6 + 2\times 1}{3+2}, \frac{3\times 2 + 2\times(-4)}{3+2}\right)$$

$$= \left(\frac{18+2}{5}, \frac{6-8}{5}\right) = \left(4, -\frac{2}{5}\right)$$

THEREFORE, R IS $\left(4, -\frac{2}{5}\right)$.

EXAMPLE 2 A LINE SEGMENT HAS END-POINTS (–2, –3) AND (7, 12) AND IT IS DIVIDED INT THREE EQUAL PARTS. FIND THE COORDINATES OF THE POINTS THAT TRISECT SEGMENT.

SOLUTION: THE FIRST POINT DIVIDES THE LINE SEGMENT IN THE RATIO 1:2, AND HENCE

$$x_O = \frac{nx_1 + mx_2}{n+m} \text{ AND } y_o = \frac{ny_1 + my_2}{n+m}$$

$$\text{SO, } x_O = \frac{2\times(-2) + 1\times 7}{1+2} \text{ AND } y_O = \frac{2\times(-3) + 1\times 12}{1+2}$$

$$\Rightarrow x_O = \frac{-4+7}{3} \text{ AND } y_O = \frac{-6+12}{3} \Rightarrow x_O = 1 \text{ AND } y_O =$$

THEREFORE, THE FIRST POINT IS (1, 2).

THE SECOND POINT DIVIDES THE LINE SEGMENT IN THE RATIO 2:1. THUS,

$$x_O = \frac{nx_1 + mx_2}{n+m} \text{ AND } y_o = \frac{ny_1 + my_2}{n+m}$$

$$\text{SO, } x_o = \frac{1\times(-2) + 2\times 7}{1+2} \text{ AND } y_O = \frac{1\times(-3) + 2\times 12}{1+2}$$

$$\Rightarrow x_O = \frac{-2+14}{3} \text{ AND } y_O = \frac{-3+24}{3}$$

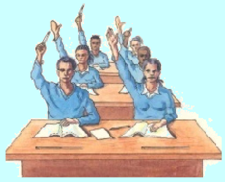
$$\Rightarrow x_O = 4 \text{ AND } y_O = 7.$$

THEREFORE, THE SECOND POINT IS (4, 7).

The mid-point formula

A POINT THAT DIVIDES A LINE SEGMENT INTO TWO EQUAL PARTS IS THE MID-POINT OF T

ACTIVITY 4.3



1 CONSIDER THE POINTS P (2, 1) AND Q (12, 1).

- **A** FIND THE DISTANCE BETWEEN *P* AND *Q*.
- **B** IF R IS A POINT WITH COORDINATES (7, 1),
  - **I** FIND *PR*. **II** FIND *RQ*.
  - **III** IS *PR* EQUAL TO *RQ*? **IV** WHAT IS THE MID-POINT OF *PQ*?
- **C** DIVIDE $\overline{PQ}$ IN THE RATIO 1:1.

2 FIND THE COORDINATES OF THE MID-POINT OF EACH OF THE FOLLOWING LINE SEG END-POINTS:

- **I** P (x_1, y_1) AND Q (x_1, y_2). **II** R (x_1, y_1) AND S (x_2, y_1).

WHICH OF THE ABOVE SEGMENTS ARE HORIZONTAL?

LET P (x_1, y_1) AND Q (x_2, y_2) BE THE END-POINTS OF $\overline{PQ}$.

IF *PR* = *RQ* (THE CASE WHERE *n* = *m*), THEN R IS THE MID-POINT OF THE LINE SEGMENT *PQ*.

NOW LET US DERIVE THE MID-POINT FORMULA.

$$R\ (x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$$

$$= \left(\frac{nx_1 + nx_2}{n+n}, \frac{ny_1 + ny_2}{n+n}\right) = \left(\frac{n\ (x_1 + x_2)}{2n}, \frac{n\ (y_1 + y_2)}{2n}\right) (\text{AS } m = n)$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

THIS IS THE FORMULA USED TO FIND THE ***mid-point*** OF THE LINE SEGMENT *PQ* WHOSE END POINTS ARE P (x_1, y_1) AND Q (x_2, y_2).

THE ***mid-point*** OF THE LINE SEGMENT JOINING THE POINTS (x_1, y_1) AND (x_2, y_2) IS GIVEN BY

$$M\ (x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

EXAMPLE 3 FIND THE COORDINATES OF THE MID-POINT OF THE LINE SEGMENT WHOSE END POINTS ARE:

- **A** P (–3, 2) AND Q (5, –4)
- **B** P $(3 - \sqrt{2}, 3 + \sqrt{2})$ AND Q $(1 + \sqrt{2}, 3 - \sqrt{2})$.

SOLUTION:

A $M(x_O, y_O) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$x_O = \frac{x_1+x_2}{2}$ AND $y_o = \frac{y_1+y_2}{2}$

$x_O = \frac{-3+5}{2} = 1$ AND $y_O = \frac{2-4}{2} = -$

THEREFORE, $M(x_O, y_O) = (1, -1)$.

B $M(x_O, y_O) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$x_O = \frac{x_1+x_2}{2}$ AND $y_o = \frac{y_1+y_2}{2}$

$x_O = \frac{3-\sqrt{2}+1+\sqrt{2}}{2}$ AND $y_O = \frac{3+\sqrt{2}+3-\sqrt{2}}{2}$

$x_O = \frac{4}{2} = 2$ AND $y_O = \frac{6}{2} = 3$

THEREFORE, $M(x_O, y_O) = (2, 3)$.

Group Work 4.1

1 A LINE SEGMENT HAS END-POINTS P (–3, 1) AND Q (5

A WHAT IS THE LENGTH OF THE LINE SEGMENT?

B FIND THE COORDINATES OF THE MID-POINT OF THE SEGMENT.

2 A LINE SEGMENT HAS ONE END-POINT AT A (4, 3). IF ITS MID-POINT IS AT M (1, –1), WHERE IS THE OTHER END-POINT?

3 FIND THE POINTS THAT DIVIDE THE LINE SEGMENT WITH END-POINTS AT P (4, –3) AND Q (–6, 7) INTO THREE EQUAL PARTS.

4 LET A (–2, –1), B (6, –1), C (6, 3) AND D (–2, 3) BE VERTICES OF A RECTANGLE. SUPPOSE *P, Q, R* AND *S* ARE MID-POINTS OF THE SIDES OF THE RECTANGLE.

I WHAT IS THE AREA OF RECTANGLE *ABCD*?

II WHAT IS THE AREA OF QUADRILATERAL *PQRS*?

III GIVE THE RATIO OF THE AREAS IN **I** AND **II**.

Exercise 4.2

1 FIND THE COORDINATES OF THE MID-POINT OF THE LINE SEGMENTS JOINING THE PO

- **A** A (1, 4) AND B (−2, 2)
- **B** (a, b) AND THE ORIGIN
- **C** M (p, q) AND N (q, p)
- **D** $\left(1\frac{1}{2}, -1\right)$ AND $\left(-\frac{5}{2}, 1\right)$
- **E** $E(1+\sqrt{2}, \sqrt{2})$ AND $F(2-\sqrt{2}, \sqrt{8})$
- **F** $G\left(\sqrt{5}, 1-\sqrt{3}\right)$ AND $H\left(-3\sqrt{5}, 1+\sqrt{3}\right)$

2 THE MID-POINT OF A LINE SEGMENT IS M (−3, 2). ONE END-POINT OF THE SEGMENT IS P (1, −3). FIND THE COORDINATES OF THE OTHER END-POINT.

3 FIND THE COORDINATES OF THE POINT R THAT DIVIDES THE LINE SEGMENT JOINING THE POINT A (1, 3) AND B (−4, −3) IN THE RATIO 2:3.

4 A LINE SEGMENT HAS END-POINTS P (−1, 5) AND Q (5, 2). FIND THE COORDINATES OF THE POINTS THAT TRISECT THE SEGMENT.

5 FIND THE MID-POINTS OF THE SIDES OF THE TRIANGLE WITH VERTICES A (−1, 3), B (4, 6) AND C (3, −1).

4.3 EQUATION OF A LINE

4.3.1 Gradient (slope) of a Line

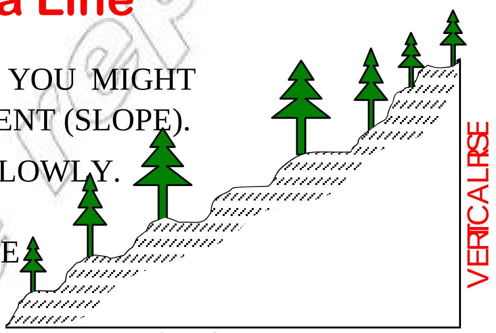


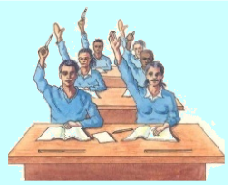
Figure 4.9

FROM YOUR EVERYDAY EXPERIENCE, YOU MIGHT BE FAMILIAR WITH THE IDEA OF GRADIENT (SLOPE).

A ***hill*** MAY BE ***steep*** OR MAY RISE VERY SLOWLY. THE NUMBER THAT DESCRIBES THE ***steepness*** OF A HILL IS CALLED THE ***gradient (slope)*** OF THE HILL.

WE MEASURE THE GRADIENT OF A HILL BY THE RATIO OF THE ***vertical rise*** TO THE ***horizontal run.***

ACTIVITY 4.4



GIVEN POINTS P (1, 2), Q (−1, −4), R (0, −1) AND S (3, 8)

- **A** FIND THE VALUE OF $\frac{y_2 - y_1}{x_2 - x_1}$ TAKING
  - **I** P AND Q
  - **II** P AND R
  - **III** Q AND R
  - **IV** R AND S
- **B** ARE THE VALUES OBTAINED IN A EQUAL? WHAT DO YOU CALL THESE VALUES?

IN COORDINATE GEOMETRY, THE GRADIENT OF A NON-VERTICAL STRAIGHT LINE IS THE RATIO OF "CHANGE IN *y*-COORDINATES" TO THE CORRESPONDING "CHANGE IN *x*-COORDINATES". THAT IS, THE SLOPE OF A LINE THROUGH *P* AND *Q* IS THE RATIO OF THE VERTICAL DISTANCE FROM *R* TO *Q* TO THE HORIZONTAL DISTANCE FROM *P* TO *Q*

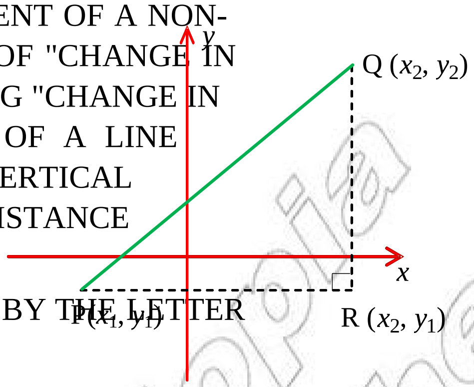


Figure 4.10

IF WE DENOTE THE GRADIENT OF A LINE BY THE LETTER *m*, THEN

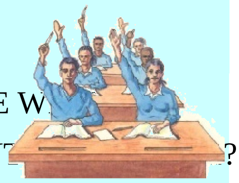
$$m = \frac{change\ in\ y\text{-}coordinates}{change\ in\ x\text{-}coordinates} = \frac{y_2 - y_1}{x_2 - x_1};\ x_1 \neq x_2$$

Definition 4.1

If (x_1, y_1) and (x_2, y_2) are points on a line with $x_1 \neq x_2$, then the ***gradient*** of the line, denoted by m, is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

ACTIVITY 4.5



1 IF A (x_1, y_1) AND B (x_2, y_2) ARE DISTINCT POINTS ON A LINE WHERE $x_1 = x_2$, THEN WHAT CAN BE SAID ABOUT THE GRADIENT OF THE LINE? IS THE LINE VERTICAL OR HORIZONTAL?

2 WHAT IS THE GRADIENT OF ANY HORIZONTAL LINE?

3 CONSIDER THE LINE WITH EQUATION $f(x) = 3x - 1$. TAKE THREE DISTINCT POINTS P_1, P_2 AND P_3 ON THE LINE.

A FIND THE GRADIENT USING P_1 AND P_2. **B** FIND THE GRADIENT USING P_1 AND P_3.

C WHAT DO YOU OBSERVE FROM A AND B?

4 LET P_1, P_2, P_3 AND P_4 BE POINTS ON A NON-VERTICAL STRAIGHT LINE $y = ax + b$ WITH COORDINATES (x_1, y_1), (x_2, y_2), (x_3, y_3) AND (x_4, y_4) RESPECTIVELY. FIND:

A THE GRADIENT OF THE LINE TAKING P_1 AND P_2

B THE GRADIENT OF THE LINE TAKING P_3 AND P_4

C ARE THE RATIOS $\frac{y_2 - y_1}{x_2 - x_1}$ AND $\frac{y_4 - y_3}{x_4 - x_3}$ EQUAL?

D COULD YOU CONCLUDE THE GRADIENT OF A LINE DOES NOT DEPEND ON OF POINTS ON THE LINE?

EXAMPLE 1 FIND THE GRADIENT OF THE LINE PASSING THROUGH EACH OF THE FOLLOWING PAIRS OF POINTS:

A P (–7, 2) AND Q (4, 3) **B** A $\left(\sqrt{2}, 1\right)$ AND B $\left(-\sqrt{2}, -3\right)$

C P (2, –3) AND Q (5,–3) **D** A $\left(-\frac{1}{2}, -2\right)$ AND B $\left(-\frac{1}{2}, 2\right)$.

SOLUTION:

A $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{4-(-7)} = \frac{1}{11}$

B $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{-\sqrt{2}-\sqrt{2}} = \frac{-4}{-2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

C $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-3)}{5-2} = \frac{-3+3}{3} = \frac{0}{3} = 0$

SO, $m = 0$. IS THE LINE HORIZONTAL? WHAT IS ITS EQUATION?

D $x_1 = -\frac{1}{2}$ AND $x_2 = -\frac{1}{2}$

THE LINE IS VERTICAL. SO IT HAS NO MEASURABLE GRADIENT.

THE EQUATION OF THE LINE IS $x = x_1 = x_2 = -\frac{1}{2}$ OR SIMPLY $x = -\frac{1}{2}$

Note: GRADIENT FOR A VERTICAL LINE IS NOT DEFINED.

EXAMPLE 2 CHECK THAT THE LINE ℓ_1 THROUGH P (0, 1) AND Q (–1, 4) AND ℓ_2 THROUGH R $\left(\frac{2}{3}, 0\right)$ AND T (1, –1) HAVE SAME GRADIENTS. ARE THE LINES PARALLEL?

SOLUTION: FOR ℓ_1, $m_1 = \frac{4-1}{-1-0} = \frac{3}{-1} = -3$. FOR ℓ_2, $m_2 = \frac{-1-0}{1-\frac{2}{3}} = \frac{-1}{\frac{1}{3}} = -3$.

HERE, $m_1 = m_2$. DRAW THE LINES AND SEE THAT ℓ_1 IS PARALLEL TO ℓ_2.

Exercise 4.3

1 FIND THE GRADIENTS OF THE LINES PASSING THROUGH THE FOLLOWING POINTS:

A A (4 , 3) AND B (8, 11) **B** P (3, 7) AND Q (1, 9)

C C ($\sqrt{2}$,–9) AND D ($2\sqrt{2}$, –7) **D** R (–5, –2) AND S (7, –8)

E E (5, 8) AND F (–2, 8) **F** H (1, 7) AND K (1, –6)

G R (1, b) AND S (b, a) , $b \neq 1$.

2 A (2, −3), B (7, 5) AND C (−2, 9) ARE THE VERTICES OF TRIANGLE ABC. FIND THE GRADIENT OF EACH OF THE SIDES OF THE TRIANGLE.

3 GIVEN THREE POINTS P (−1, −5), Q (1, −2) AND R (5, 4), FIND THE GRADIENT OF $\overline{PQ}$ AND $\overline{QR}$. WHAT DO YOU CONCLUDE FROM YOUR RESULT?

4 USE GRADIENTS TO SHOW THAT THE POINTS $P_1(-4, 6)$, $P_2(-1, 12)$ AND $P_3(-7, 0)$ ARE COLLINEAR, I.E., ALL LIE ON THE SAME STRAIGHT LINE.

5 SHOW THAT THE LINE PASSING THROUGH THE POINTS A (3, −2) AND B $\left(0, \frac{3}{2}\right)$, ALSO PASSES THROUGH THE POINT (−6, −9).

4.3.2 Slope of a Line in Terms of Angle of Inclination

THE ANGLE MEASURED FROM THE POSITIVE x-AXIS TO A LINE, IN ANTICLOCKWISE DIRECTION, IS CALLED THE ***inclination of the line*** OR THE ANGLE OF INCLINATION OF THE LINE.
THIS ANGLE IS ALWAYS LESS THAN 180°.

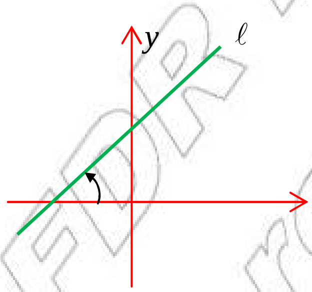


Figure 4.11

Group Work 4.2

CONSIDER THE RIGHT ANGLED TRIANGLE OAB IN FIGURE 4.12.

1 HOW LONG IS THE HYPOTENUSE $\overline{OB}$?
2 WHAT IS TANGENT OF ANGLE BOA?
3 WHAT IS MEASURE OF ANGLE BOA?
4 WHAT IS THE ANGLE OF INCLINATION OF LINE ℓ?
5 WHAT IS THE TANGENT OF THE ANGLE OF INCLINATION?
6 BY FINDING THE COORDINATES OF B, CALCULATE THE SLOPE OF LINE ℓ.

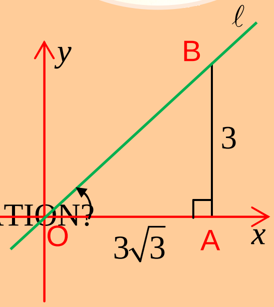


Figure 4.12

7 WHAT RELATIONSHIP DO YOU SEE BETWEEN YOUR ANSWERS FOR QUESTIONS 5 AND 6 ABOVE?

THE ABOVE GROUP WORK WILL HELP YOU TO UNDERSTAND THE RELATIONSHIP BETWEEN SLOPE AND ANGLE OF INCLINATION.

FOR A NON-VERTICAL LINE, THE ***tangent*** OF THIS ANGLE IS THE ***slope*** OF THE LINE. OBSERVE THE FOLLOWING.

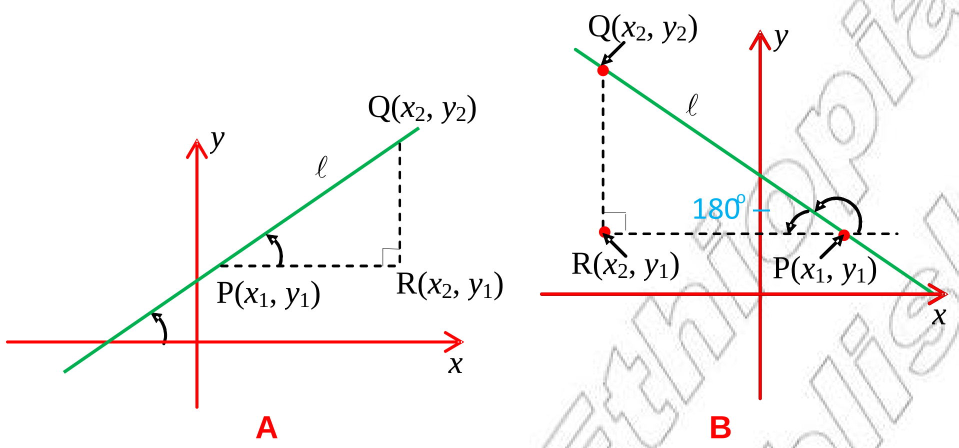


Figure 4.13

IN FIGURE 4.13A ABOVE, AS $y_2 - y_1$ REPRESENTS THE DISTANCE RQ AND $x_2 - x_1$ REPRESENTS THE DISTANCE PR, THE SLOPE OF THE STRAIGHT LINE ℓ IS ACTUALLY REPRESENTED BY THE RATIO

$$m = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(m\angle RPQ)$$

$$\therefore m = \tan\theta$$

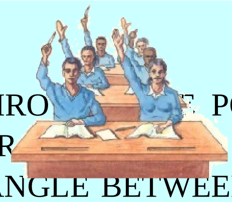
A LINE MAKING AN ACUTE ANGLE OF INCLINATION WITH THE POSITIVE DIRECTION OF THE x-AXIS HAS POSITIVE SLOPE.

SIMILARLY, A LINE WITH OBTUSE ANGLE OF INCLINATION, (SEE FIGURE 4.13B), HAS NEGATIVE SLOPE.

$$\text{SLOPE OF } \ell = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_1 - x_2} = -\frac{y_2 - y_1}{x_2 - x_1} = -\tan(180° - \theta) = -(-\tan\theta) = \tan\theta$$

(*In Unit 5, this will be clarified*)

ACTIVITY 4.6



1 HOW WOULD YOU DESCRIBE THE LINE PASSING THROUGH THE POINTS WITH COORDINATES (x_1, y_1) AND (x_1, y_2)? IS IT PERPENDICULAR TO THE x-AXIS OR THE y-AXIS? WHAT IS THE TANGENT OF THE ANGLE BETWEEN THIS LINE AND THE x-AXIS?

2 SUPPOSE A LINE PASSES THROUGH THE POINTS WITH COORDINATES (x_1, y_1) AND (x_2, y_1). FIND THE TANGENT OF THE ANGLE FORMED BY THE LINE AND THE x-AXIS. WHAT IS THE SLOPE OF THIS LINE?

3 WHAT IS THE ANGLE OF INCLINATION OF THE LINE $y = x$, AND THE LINE $y = -x$?

IN GENERAL, THE SLOPE OF A LINE MAY BE EXPRESSED IN TERMS OF THE COORDINATES (x_1, y_1) AND (x_2, y_2) ON THE LINE AS FOLLOWS:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \text{TAN}\,\theta,\ x_2 \neq x_1$$

WHERE θ IS THE ANTICLOCKWISE ANGLE BETWEEN THE POSITIVE x-AXIS AND THE LINE.

EXAMPLE 3 FIND THE SLOPE OF A LINE, IF ITS INCLINATION IS:

A 60° **B** 135°

SOLUTION:

A SLOPE : $m = \text{TAN}\,\theta = \text{TAN } 60^{\circ} = \sqrt{3}$

B SLOPE : $m = \text{TAN}\,\theta = \text{TAN } 135^{\circ} = \text{TAN } (180^{\circ} - 45^{\circ}) = -\text{TAN } 45^{\circ} = -1$

Note: IF θ IS AN OBTUSE ANGLE, THEN $\text{TAN}\,\theta = -\text{TAN}(180^{\circ} - \theta)$.

EXAMPLE 4 FIND THE ANGLE OF INCLINATION OF THE LINE

A CONTAINING THE POINTS A(3, –3) AND B(–1, 1)

B CONTAINING THE POINTS C(0, 5) AND D(4, 5).

SOLUTION:

A $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1-(-3)}{-1-3} = -1$. SO $\text{TAN}\,\theta = -1$ AND HENCE $\theta = 135^{\circ}$.

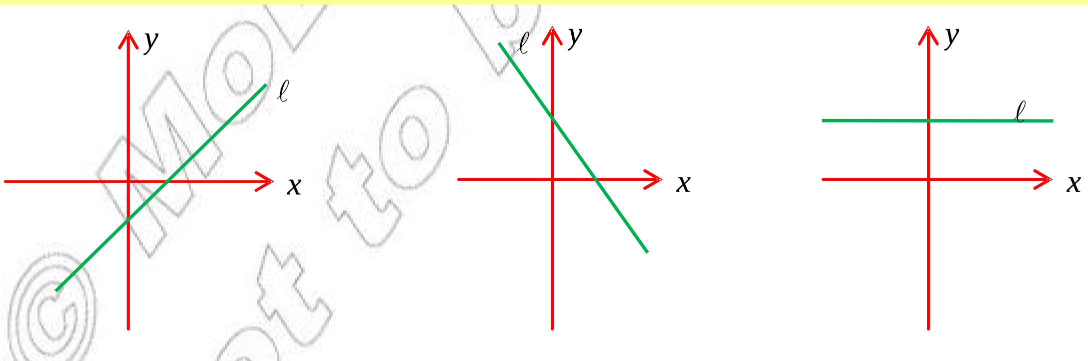
B $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-5}{4-0} = 0$, $\text{TAN}\,\theta = 0$. SO, $\theta = 0^{\circ}$.

Note: LET m BE THE SLOPE OF A NON-VERTICAL LINE.

I IF $m > 0$, THEN THE LINE RISES FROM LEFT TO RIGHT AS SHOWN IN FIGURE 4.14A.

II IF $m < 0$, THEN THE LINE FALLS FROM LEFT TO RIGHT AS SHOWN IN FIGURE 4.14B.

III IF $m = 0$, THEN THE LINE IS HORIZONTAL AS IN FIGURE 4.14C.



a b c

Figure 4.14

Exercise 4.4

1 FIND THE SLOPE OF THE LINE WHOSE ANGLE OF INCLINATION IS:

A 30^{O} B 75^{O} C 150^{O} D 90^{O} E 0^{O}.

2 FIND THE ANGLE OF INCLINATION OF THE LINE IF ITS SLOPE IS:

A $-\sqrt{3}$ B $\frac{-\sqrt{3}}{3}$ C 1 D $\frac{1}{\sqrt{3}}$ E 0.

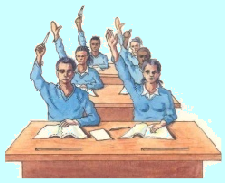
3 THE POINTS A(–2, 0), B (0, 2) AND C (2, 0) ARE VERTICES OF A TRIANGLE. FIND THE MEASURE OF THE THREE ANGLES OF TRIANGLE ABC. WHAT TYPE OF TRIANGLE IS IT?

4.3.3 Different Forms of Equations of a Line

FROM EUCLIDEAN GEOMETRY, YOU MAY RECALL THAT THERE IS A UNIQUE LINE PAS… TWO DISTINCT POINTS. THE EQUATION OF A LINE IS AN EQUATION IN x AND y WHICH IS SATISFIED BY THE COORDINATES OF EVERY POINT ON THE LINE AND IS NOT SATISFIED BY THE COORDINATES OF ANY POINT NOT ON THE LINE.

THE EQUATION OF A STRAIGHT LINE CAN BE EXPRESSED IN DIFFERENT FORMS. SOME … THE POINT-SLOPE FORM, THE SLOPE-INTERCEPT FORM AND THE TWO-POINT FORM.

ACTIVITY 4.7



1 SHOW THAT THE GRAPH OF THE EQUATION $x = 2$ CONTAINS POINTS

A (2, 0), B (2, –1), C (2, 2) AND D$\left(2, \frac{1}{3}\right)$.

2 CONSIDER THE GRAPH OF THE STRAIGHT LINE $y = x + 1$. DETERMINE WHICH OF THE FOLLOWING POINTS LIE ON THE LINE.

A (3, –1), B (–1, 0), C$\left(\frac{-1}{2}, \frac{3}{2}\right)$, D(0, 1), E$\left(\frac{-1}{2}, 1\right)$, F(–2, –1) AND G(–1, 2)

3 WHICH OF THE FOLLOWING POINTS LIE ON THE LINE $y = -5x + 4$?

A (–1, 9), B (–2, 12), C(0, 4), D$\left(\frac{2}{5}, 2\right)$, E (3, –10).

4 WHAT DO YOU CALL THE NUMBER b IF A LINE INTERSECTS THE y-AXIS AT POINT P (0, b)?

5 CONSIDER THE GRAPH OF THE STRAIGHT LINE $y = mx + b$. FIND ITS y-INTERCEPT AND x-INTERCEPT.

6 GIVE THE EQUATIONS OF THE LINES THROUGH THE POINTS:

A P(–1, 3) AND Q(4, 3) B R (–1, 1) AND S(1, –1).

The point-slope form of equation of a line

WE NORMALLY USE THIS FORM OF THE EQUATION OF A LINE IF THE SLOPE *m* OF THE LINE AND THE COORDINATES OF A POINT ON IT ARE GIVEN.

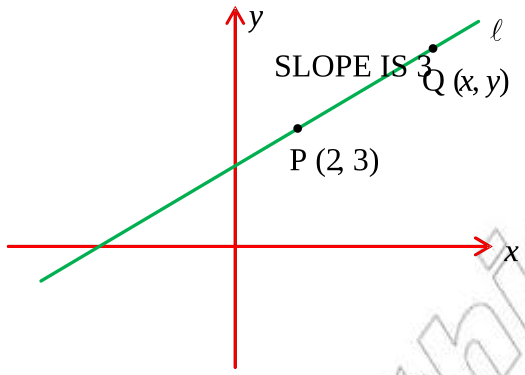


Figure 4.15

SUPPOSE YOU ARE ASKED TO FIND THE EQUATION OF THE STRAIGHT LINE WITH SLOPE 3 THROUGH THE POINT WITH COORDINATE (2, 3).

TAKE *P* TO BE THE POINT (2, 3) AND LET *Q* (x, y) BE ANY OTHER POINT ON THE LINE AS SHOWN IN FIGURE 4.15. WHAT IS THE SLOPE OF THE STRAIGHT LINE JOINING THE POINTS WITH COORD (x_1, y_1) AND (x_2, y_2)?

WHAT IS THE SLOPE OF $\overleftrightarrow{PQ}$? YOU ARE GIVEN THAT THE SLOPE OF THIS LINE IS 3. IF YOU HA ANSWERED CORRECTLY, YOU SHOULD OBTAIN

$$y = 3x - 3;$$

WHICH IS THE REQUIRED EQUATION OF THE STRAIGHT LINE.

IN GENERAL, SUPPOSE YOU WANT TO FIND THE EQUATION OF THE STRAIGHT LINE THROUGH THE POINT WITH COORDINATES (x_1, y_1) AND WHICH HAS SLOPE *m*. AGAIN, LET THE POINT WITH GIVEN COORDINATES (x_1, y_1) BE *A*. TAKE ANY OTHER POINT ON THE LINE, SAY *B*, WITH COORDINATES (x, y) AS SHOWN IN FIGURE 4.16.

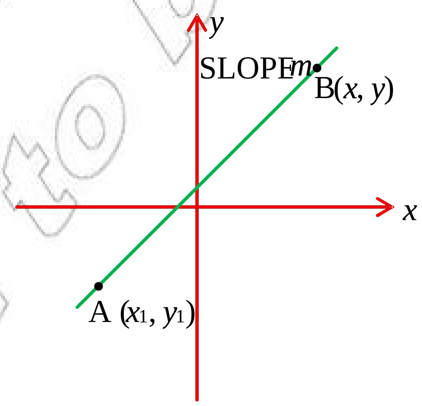


Figure 4.16

THEN THE SLOPE OF $\overleftrightarrow{AB}$ IS $\dfrac{y-y_1}{x-x_1}$

$\Rightarrow y - y_1 = m(x - x_1)$ WHICH IS THE SAME AS $y = y_1 + m(x - x_1)$.

THIS EQUATION IS CALLED THE ***point-slope form*** of the equation of a line

EXAMPLE 5 FIND THE EQUATION OF THE STRAIGHT LINE WITH SLOPE $\frac{-3}{2}$ AND WHICH PASSES THROUGH THE POINT (–3, 2).

SOLUTION: ASSUME THAT THE POINT (x, y) IS ANY POINT ON THE LINE OTHER THAN (–3, 2).

THUS, USING THE EQUATION $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 2 = \frac{-3}{2}(x + 3)$$

$$\Rightarrow y = -\frac{3}{2}x - \frac{5}{2} \quad \text{OR } 2y + 3x + 5 = 0.$$

The slope-intercept form of equation of a line

CONSIDER THE EQUATION $y = mx + b$. WHEN $x = 0$, $y = b$. ALSO, WHEN $x = 1$, $y = m + b$ AS SHOWN IN FIGURE 4.17.

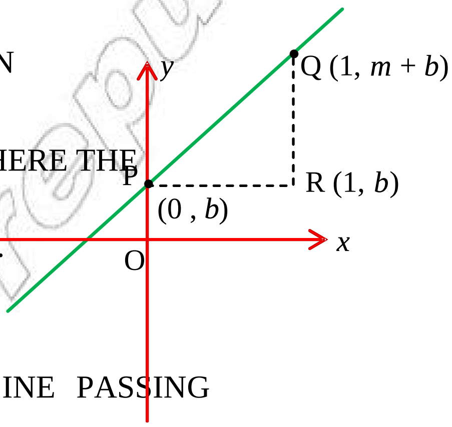


Figure 4.17

YOU CAN SEE THAT P $(0, b)$ IS THE POINT WHERE THE LINE WITH EQUATION $y = mx + b$ CROSSES THE y-AXIS. (b IS CALLED THE ***y-intercept*** OF THE LINE). LET Q BE $(1, m + b)$.

USING THE COORDINATES OF P AND Q, SHOW THAT THE SLOPE OF THE STRAIGHT LINE PASSING THROUGH P AND Q IS m

WRITING THE EQUATION OF THIS LINE THROUGH THE POINT $(0, b)$ WITH SLOPE m, USING THE POINT-SLOPE FORM, GIVES

$$y - b = m(x - 0) \Rightarrow y = mx + b$$

WHERE m IS SLOPE OF THE LINE AND b IS y-INTERCEPT OF THE LINE.

THIS EQUATION IS CALLED THE ***slope-intercept form*** OF THE EQUATION OF A LINE.

Note: THE SLOPE-INTERCEPT FORM OF EQUATION OF A LINE ENABLES US TO FIND THE THE y-INTERCEPT, ONCE THE EQUATION IS GIVEN.

EXAMPLE 6 FIND THE EQUATION OF THE LINE WITH SLOPE $-\frac{2}{3}$ AND y-INTERCEPT 3.

SOLUTION: HERE, $m = \frac{-2}{3}$ AND THE y-INTERCEPT IS 3.

THEREFORE, THE EQUATION OF THE LINE IS $y = -\frac{2}{3}x + 3$.

The two-point form of equation of a line

FINALLY, LET US LOOK AT THE SITUATION WHERE THE SLOPE OF A LINE IS NOT GIVEN BUT TWO POINTS ON THE LINE ARE GIVEN.

CONSIDER A STRAIGHT LINE WHICH PASSES THROUGH THE POINTS P(x_1, y_1) AND Q(x_2, y_2). IF R (x, y) IS ANY POINT ON THE LINE OTHER THAN P(x_1, y_1) AND Q(x_2, y_2), THEN THE SLOPE OF $\overleftrightarrow{PR}$ IS

$$m = \frac{y - y_1}{x - x_1},\ x \neq x_1$$

AND THE SLOPE OF $\overleftrightarrow{PQ}$ IS

$$m = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2$$

BUT THE SLOPE OF $\overleftrightarrow{PR}$ = THE SLOPE OF $\overleftrightarrow{PQ}$

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

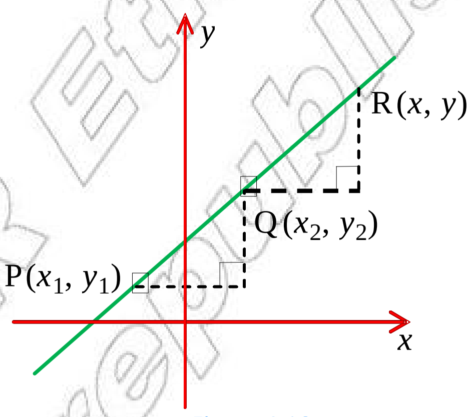


Figure 4.18

THIS EQUATION IS CALLED THE ***two-point form*** of the equation of a line.

EXAMPLE 7 FIND THE EQUATION OF THE LINE PASSING THROUGH THE POINTS P (–1, 5) AND Q (3, 13).

SOLUTION: TAKING (–1, 5) AS (x_1, y_1) AND (3, 13) AS (x_2, y_2), USE THE TWO-POINT FORM TO GET THE EQUATION OF THE LINE TO BE

$$y - 5 = \frac{13-5}{3+1}(x+1) = 2x + 2 \text{ WHICH IMPLIES } y = 2x + 7$$

The general equation of a line

A FIRST DEGREE (LINEAR) EQUATION IN x AND y IS AN EQUATION OF THE FORM;

$$Ax + By + C = 0$$

WHERE A, B AND C ARE FIXED REAL NUMBERS SUCH THAT $A \neq 0$ OR $B \neq 0$.

ALL THE DIFFERENT FORMS OF EQUATIONS OF LINES DISCUSSED ABOVE CAN BE EXPRESS

$$Ax + By + C = 0$$

CONVERSELY, ONE CAN SHOW THAT ANY LINEAR EQUATION IN x AND y IS THE EQUATION OF A LINE. SUPPOSE A LINEAR EQUATION IN x AND y IS GIVEN AS

$$Ax + By + C = 0.$$

IF $B \neq 0$, THEN THE EQUATION MAY BE SOLVED FOR y AS FOLLOWS:

$$Ax + By + C = 0$$

$$By = -Ax - C$$

$$y = \frac{-A}{B}x - \frac{C}{B}$$

THIS EQUATION IS OF THE FORM $y = mx + b$, AND THEREFORE REPRESENTS A STRAIGHT LINE WITH SLOPE $m = -\frac{A}{B}$ AND y-INTERCEPT $b = -\frac{C}{B}$.

WHAT WILL BE THE EQUATION $Ax + By + C = 0$, IF $B = 0$ AND $A \neq 0$?

EXAMPLE 8 FIND THE SLOPE AND y-INTERCEPT OF THE LINE WHOSE GENERAL EQUATION IS $3x - 6y - 4 = 0$.

SOLUTION: SOLVING FOR y, THE EQUATION $3x - 6y - 4 = 0$ GIVES,

$$-6y = -3x + 4 \Rightarrow y = \frac{-3x}{-6} + \frac{4}{-6} = \frac{1}{2}x - \frac{2}{3}$$

SO, THE SLOPE IS $m = \frac{1}{2}$ AND THE y-INTERCEPT IS $b = \frac{-2}{3}$.

EXAMPLE 9 WHAT IS THE EQUATION OF THE LINE PASSING THROUGH (–2, 0) AND (0, 5)?

SOLUTION: USING TWO-POINT FORM:

$$y - 0 = \frac{5-0}{0-(-2)}(x+2)$$

WHICH GIVES US, $5x - 2y + 10 = 0$ AS THE EQUATION OF THE LINE.

Exercise 4.5

1 FIND THE EQUATION OF THE LINE PASSING THROUGH THE GIVEN POINTS.

A A $(-2, -4)$ AND B $(-1, 5)$ **B** C $(2, -4)$ AND D $(-1, 5)$

C E (3, 7) AND F (8, 7) **D** G (1, 1) AND H $(1+\sqrt{2}, 1-\sqrt{2})$

E P $(-1, 0)$ AND THE ORIGIN **F** Q $(4, -1)$ AND R $(4, -4)$

G M (,) AND N (3, -5) **H** T $\left(1\frac{1}{2}, -\frac{5}{2}\right)$ AND S $\left(-\frac{3}{2}, 1\right)$.

2 FIND THE EQUATION OF THE LINE WITH SLOPE m, PASSING THROUGH THE GIVEN POINT P.

A $m = \frac{3}{2}$; P (0 , −6) **B** $m = 0$; P $\left(\frac{\ }{2}, \frac{-}{4}\right)$

C $m = 1\frac{2}{3}$; P (1, 1) **D** $m = -\ $; P (0 , 0)

E $m = \sqrt{2}$; P$\left(\sqrt{2}, -\sqrt{2}\right)$ **F** $m = -1$; P $\left(\frac{1}{3}, \frac{3}{2}\right)$.

3 FIND THE EQUATION OF THE LINE WITH SLOPE m AND y-INTERCEPT b.

A $m = 0.1$; $b = 0$ **B** $m = -\sqrt{2}$; $b = -1$ **C** $m = \ $; $b = 2$

D $m = 1\frac{1}{3}$; $b = \frac{-5}{3}$ **E** $m = \frac{-1}{4}$; $b = 5$ **F** $m = \frac{2}{3}$; $b = 1.5$

4 SUPPOSE A LINE HAS x-INTERCEPT a AND y-INTERCEPT b, FOR $a, b \neq 0$; SHOW THAT THE EQUATION OF THE LINE IS $\frac{x}{a} + \frac{y}{b} = 1$.

5 FOR EACH OF THE FOLLOWING EQUATIONS, FIND THE SLOPE AND y-INTERCEPT.

A $\frac{3}{5}x - \frac{4}{5}y + 8 = 0$ **B** $-y + 2 = 0$ **C** $2x - 3y + 5 = 0$

D $x + \frac{1}{2}y - 2 = 0$ **E** $y + 2 = 2(x - 3y + 1)$.

6 A LINE PASSES THROUGH THE POINTS A (5, –1) AND B (–3, 3). FIND:

A THE POINT-SLOPE FORM OF THE EQUATION OF THE LINE.

B THE SLOPE-INTERCEPT FORM OF THE EQUATION OF THE LINE.

C THE TWO-POINT FORM OF THE EQUATION OF THE LINE. WHAT IS ITS GENERAL …

7 FIND THE SLOPE AND y-INTERCEPT, IF THE EQUATION OF THE LINE IS:

A $\frac{1}{3}x - \frac{2}{3}y + 1 = y + x$ **B** $3(y - 2x) = y + \frac{1}{2}(1 - 2x)$.

8 A TRIANGLE HAS VERTICES AT A (–1, 1), B (1, 3) AND C (3, 1).

A FIND THE EQUATIONS OF THE LINES CONTAINING THE SIDES OF THE TRIANG…

B IS THE TRIANGLE A RIGHT-ANGLED TRIANGLE?

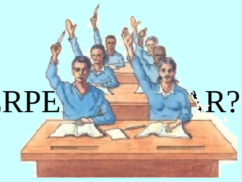
C WHAT ARE THE INTERCEPTS OF THE LINE PASSING THROUGH B AND C?

4.4 PARALLEL AND PERPENDICULAR LINES

SLOPES CAN BE USED TO SEE WHETHER TWO NON-VERTICAL LINES IN A PLANE ARE PARALLEL, PERPENDICULAR, OR NEITHER.

FOR INSTANCE, THE LINES $y = x + 1$ AND $y = x + 3$ ARE PARALLEL AND THE LINES $y = x$ AND $y = -x$ ARE PERPENDICULAR. HOW ARE THE SLOPES RELATED?

ACTIVITY 4.8



1 WHAT IS MEANT BY TWO LINES BEING PARALLEL? PERPENDICULAR?

2 IN FIGURE 4.19, ℓ_1 AND ℓ_2 ARE PARALLEL.

A CALCULATE THE SLOPE OF EACH LINE. **B** FIND THE EQUATION OF EACH LINE.

C DISCUSS HOW THEIR SLOPES ARE RELATED.

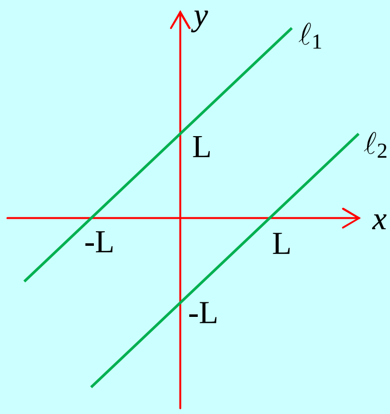


Figure 4.19

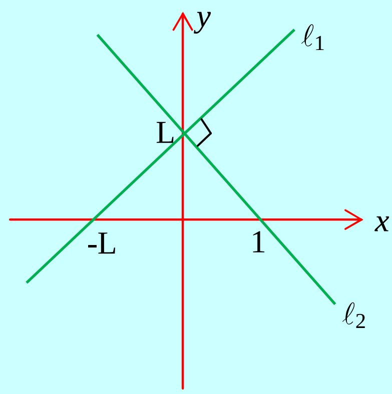


Figure 4.20

3 IN FIGURE 4.20 ABOVE, ℓ_1 AND ℓ_2 ARE PERPENDICULAR.

A CALCULATE THE SLOPE OF EACH LINE. **B** FIND THE EQUATION OF EACH LINE.

C DISCUSS HOW THEIR SLOPES ARE RELATED.

Theorem 4.1

If two non-vertical lines ℓ_1 and ℓ_2 are parallel to each other, then they have the same slope.

SUPPOSE YOU HAVE TWO NON-VERTICAL LINES ℓ_1 AND ℓ_2 WITH SLOPES m_1 AND m_2 AND INCLINATIONS AND, RESPECTIVELY AS SHOWN IN FIGURE 4.21.

IF ℓ_1 IS PARALLEL TO ℓ_2, THEN = (WHY?)

CONSEQUENTLY, m_1 = TAN = TAN = m_2

State and prove the converse of the above theorem.

What can be stated for two vertical lines? Are they parallel?

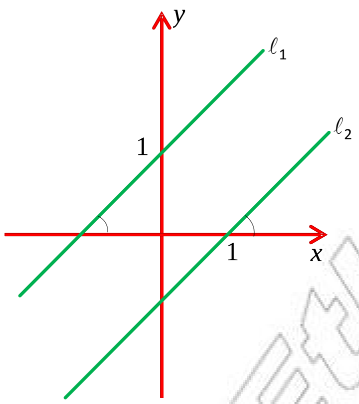


Figure 4.21

EXAMPLE 1 SHOW THAT THE LINE PASSING THROUGH A(−1, −1) AND B (2, −3) IS PARALLEL TO THE LINE PASSING THROUGH P(−3, −2) AND Q (3, −6).

SOLUTION: SLOPE OF $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-1)}{2-(-1)} = \frac{-3+1}{2+1} = -\frac{2}{3}$

SLOPE $\overleftrightarrow{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-(-2)}{3-(-3)} = \frac{-6+2}{3+3} = -\frac{2}{3}$

SINCE $\overleftrightarrow{AB}$ AND $\overleftrightarrow{PQ}$ HAVE THE SAME SLOPE, $\overleftrightarrow{AB}$ IS PARALLEL TO $\overleftrightarrow{PQ}$, I.E. $\overleftrightarrow{AB} // \overleftrightarrow{PQ}$

RECALL THAT TWO LINES ARE PERPENDICULAR, IF THEY FORM A RIGHT-ANGLE AT INTERSECTION.

Theorem 4.2

Two non-vertical lines having slopes m_1 and m_2 are perpendicular, if and only if $m_1 \cdot m_2 = -1$.

Proof: SUPPOSE ℓ_1 IS PERPENDICULAR TO ℓ_2.

Note: IF ONE OF THE LINES IS A VERTICAL LINE, THEN THE OTHER LINE MUST BE A HORIZONTAL LINE WHICH HAS SLOPE ZERO. SO, ASSUME THAT NEITHER LINE IS VERTICAL.

LET m_1 AND m_2 BE THE SLOPES OF ℓ_1 AND ℓ_2, RESPECTIVELY.

LET R(x_0, y_0) BE THE POINT OF INTERSECTION AND CHOOSE P(x_1, y_1) AND Q(x_2, y_2) ON ℓ_1 AND ℓ_2, RESPECTIVELY.

DRAW TRIANGLES QSR AND RTP AS SHOWN IN FIGURE 4.22.

ΔQSR AND ΔRTP ARE SIMILAR, (WHY?)

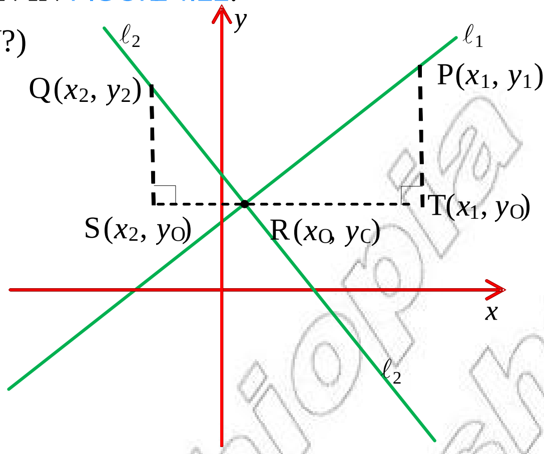


Figure 4.22

$$\frac{PT}{RT} = \frac{RS}{QS} \quad \text{(WHY?)}$$

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{x_0 - x_2}{y_2 - y_0} = -\left(\frac{x_2 - x_0}{y_2 - y_0}\right)$$

$$\frac{y_1 - y_o}{x_1 - x_o} = \frac{-1}{\dfrac{y_2 - y_o}{x_2 - x_o}}$$

$$m_1 = -\frac{1}{m_2} \text{ OR } m_1 m_2 = -1$$

AS AN EXERCISE, START WITH $\frac{QS}{RS} = \frac{RT}{PT}$ AND CONCLUDE THAT $m_2 = -\frac{1}{m_1}$

CONVERSELY, YOU COULD SHOW THAT IF TWO LINES HAVE SLOPES m_1 AND m_2 WITH $m_1 m_2 = -1$, THEN THE LINES ARE PERPENDICULAR. THIS CAN BE DONE BY REVERSING THE ABOV CONCLUDING THAT THE TWO TRIANGLES ARE SIMILAR. COMPLETE THE PROOF.

EXAMPLE 2 SUPPOSE ℓ_1 PASSES THROUGH P (–1, –3) AND Q (2, 6). FIND THE SLOPE m_2 OF ANY LINE ℓ_2 THAT IS:

A PARALLEL TO ℓ_1 **B** PERPENDICULAR TO ℓ_1

SOLUTION: THE SLOPE OF ℓ_1 IS

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-3)}{2-(-1)} = \frac{9}{3} = 3. \text{ SO,}$$

A THE SLOPE OF ℓ_2 PARALLEL TO ℓ_1 IS $m_2 = m_1 = 3$

B THE SLOPE OF ℓ_2 PERPENDICULAR TO ℓ_1 IS $m_2 = -\frac{1}{m_1} = -\frac{1}{3}$

EXAMPLE 3 FIND THE EQUATION OF THE LINE PASSING THROUGH (5, –1) AND PERPENDICULAR TO THE LINE $x - 3y = -7$.

SOLUTION: FROM $x - 3y = -7$, $y = \frac{1}{3}x + \frac{7}{3}$ SO, $m_1 = \frac{1}{3}$

LET THE SLOPE OF THE REQUIRED LINE BE m_2. THEN, $m_1 m_2 = -1$ GIVES $m_2 = \frac{-1}{m_1} = -3$

THEREFORE, THE REQUIRED EQUATION OF THE LINE IS $y + 1 = -3(x - 5)$ I.E. $y = -3x + 14$.

Exercise 4.6

1. In each of the following, determine whether the line through A and B is parallel to or perpendicular to the line through P and Q:
   - A. A (−1, 3) and B (2, −2); P (1, 4) and Q (−2, 9)
   - B. A (−3, 5) and B (2, −5); P (−1, 4) and Q (1, 5).
2. Find the slope of the line that is perpendicular to the line joining P (2, −3) and Q (−3, −2).
3. Use slope to show that the quadrilateral ABCD with vertices A (−5, −2), B (−3, 1), C (3, 0) and D (1, −3) is a parallelogram.
4. Let ℓ be the line with equation $2x - 3y = 6$. Find the slope-intercept form of the equation of the line that passes through the point P (2, −1) and is:
   - A. parallel to ℓ
   - B. perpendicular to ℓ
5. Find the equation of a line passing through P and parallel to the line ℓ for:
   - A. $\ell: 2x - 5y - 4 = 0$; P (−1, 2)
   - B. $\ell: 3x + 6 = 0$; P (4, −6).
6. Determine which of the following pairs of lines whose equations are given are perpendicular or parallel or neither:
   - A. $3x - y + 5 = 0$ and $x + 3y - 1 = 0$
   - B. $3x - 4y + 1 = 0$ and $4x - 3y + 1 = 0$
   - C. $4x - 10y + 8 = 0$ and $10x + 6y - 3 = 0$
   - D. $2x + 2y = 4$ and $x + y = 10$.
7. Find the equation of the line passing through the point (2, 5) and:
   - A. parallel to the line passing through the points A (3, −1) and B (−1, 3)
   - B. parallel to the line $\ell: x + y = 2$
   - C. perpendicular to the line joining the points A (−1, 2) and B (4, −2)
   - D. perpendicular to the line $\ell: y = x + 1$.
8. Determine k so that the line with equation $4x + ky = 12$ will be:
   - A. parallel to the line with equation $x = 3y$
   - B. perpendicular to the line with equation $x - 3y = 5$.
9. Show that the plane figure with vertices:
   - A. A (6, 1), B (5, 6), C (−4, 3) and D (−3, −2) is a parallelogram
   - B. A (2, 4), B(1, 5), C (−2, 2) and D (−1, 1) is a rectangle.
10. The vertices of a triangle are A (−2, 5), B (3, 8) and C (6, −4). Show that the line joining the mid-points of sides $\overline{AB}$ and $\overline{BC}$ is parallel to and one-half the length of side $\overline{AC}$.

Key Terms

analytic geometry	***general equation of a line***	***point-slope form***
angle of inclination	***horizontal line***	***slope (gradient)***
coordinate geometry	***inclination of a line***	***slope-intercept form***
coordinates	***mid-point***	***steepness***
equation of a line	***non-vertical line***	***two-point form***

Summary

1 IF A POINT P HAS COORDINATES (a, b), THEN THE NUMBER a IS CALLED THE ***x-coordinate*** OR ***abscissa*** OF P AND b IS CALLED THE ***y-coordinate*** OR ***ordinate*** OF P.

2 THE ***distance d*** BETWEEN POINTS P (x_1, y_1) AND Q (x_2, y_2) IS GIVEN BY THE FORMULA

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3 THE POINT R (x_0, y_0) DIVIDING THE LINE SEGMENT PQ ***internally, in the ratio m:n*** IS GIVEN BY

$$\text{R}\,(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right),$$

WHERE P (x_1, y_1) AND Q (x_2, y_2) ARE THE END-POINTS.

4 THE ***mid-point*** OF A LINE SEGMENT WHOSE END-POINTS ARE P (x_1, y_1) AND Q (x_2, y_2) IS GIVEN BY

$$\text{M}\,(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

5 IF P (x_1, y_1) AND Q (x_2, y_2) ARE POINTS ON A LINE WITH $x_1 \neq x_2$, THEN THE ***slope*** (***gradient***) OF THE LINE IS GIVEN BY

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

6 IF IS THE ANGLE BETWEEN THE x-AXIS AND THE LINE PASSING THROUGH THE POINTS P (x_1, y_1) AND Q (x_2, y_2), $x_1 \neq x_2$, THEN THE ***slope*** OF THE LINE IS GIVEN BY

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \text{TAN}$$

7 THE GRAPH OF THE EQUATION $x = c$ IS THE ***vertical line*** THROUGH P $(c, 0)$ AND HAS NO SLOPE.

8 THE ***equation of the line*** WITH SLOPE m AND PASSING THROUGH THE POINT P (x_1, y_1) IS GIVEN BY

$$y - y_1 = m\,(x - x_1)$$

9 The equation of the line with slope m and y-intercept b is given by

$$y = mx + b$$

10 The equation of the line passing through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),\ x_1 \neq x_2$$

11 The graph of every first degree (linear) equation $Ax + By + C = 0$, $A, B \neq 0$ is a ***straight line*** and every straight line is a graph of a first degree equation.

12 Two non-vertical lines are ***parallel*** if and only if they have the same slope.

13 Let ℓ_1 be a line with slope m_1 and ℓ_2 be a line with slope m_2. Then ℓ_1 and ℓ_2 are ***perpendicular*** lines if and only if $m_1 m_2 = -1$.

Review Exercises on Unit 4

1 Show that the points A(1, −1), B(−1, 1) and $C(\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle.

2 Find the coordinates of the three points that divide the line segment joining P(−4, 7) and Q(10, −9) into four parts of equal length.

3 Find the equation of the line which passes through the points P(−4, −2) and Q(3, 6).

4 Find the equation of the line:

- **A** with slope −3 that passes through P(8, 3).
- **B** with slope $-\frac{1}{2}$ that passes through Q(−2, 5).

5 In each of the following, show that the three points are vertices of a right angled triangle:

- **A** A(0, 0), B(1, 1), C(2, 0)
- **B** P(3, 1), Q(−3, 4), R(−3, 1).

6 Find the slope and y-intercept of the line with the following equations:

- **A** $2x - 3y = 4$
- **B** $2y - 5x - 2 = 0$
- **C** $5y + 6x - 4 = 0$
- **D** $3y = 7x + 1$.

7 Find the equation of the straight line passing through P(−2, 10) and

- **A** parallel to the line with equation $-3y = 12x$
- **B** perpendicular to the line with equation $5y + 6x = 10$.

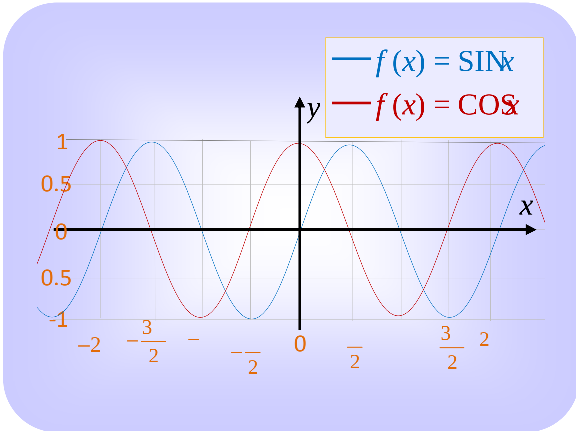
8 Let ℓ be the line through A(−4, 5) and B(3, t) that is perpendicular to the line through P(1, 3) and Q(−4, 2). Find the value of t

9 Let ℓ be the line through A(−4, 3) and B(t, −2) that is parallel to the line through P(−2, 4) and Q(4, −1). Find the value of t

10 Prove that the condition for lines $Ax + By + C = 0$ and $ax + by + c = 0$ to be perpendicular may be written in the form

$Aa + Bb = 0$, where $B, b \neq 0$.

Unit 5



TRIGONOMETRIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know principles and methods for sketching graphs of basic trigonometric functions.*
- *understand important facts about reciprocals of basic trigonometric functions.*
- *identify trigonometric identities.*
- *solve real life problems involving trigonometric functions.*

Main Contents

5.1 Basic trigonometric functions

5.2 The reciprocals of the basic trigonometric functions

5.3 Simple trigonometric identities

5.4 Real life application problems

Key Terms

Summary

Review Exercises

INTRODUCTION

IN MATHEMATICS, THE trigonometric functions (ALSO CALLED CIRCULAR FUNCTIONS) ARE FUNCTIONS OF ANGLES. THEY WERE ORIGINALLY USED TO RELATE THE ANGLES OF A LENGTHS OF THE SIDES OF A TRIANGLE. LOOSELY TRANSLATED, trigonometry MEANS triangle measure. TRIGONOMETRIC FUNCTIONS ARE HIGHLY USEFUL IN THE STUDY OF TRIANGLE AND ALSO IN MANY DIFFERENT PHENOMENA IN REAL LIFE.

THE MOST FAMILIAR TRIGONOMETRIC FUNCTIONS ARE sine, cosine AND tangent. IN THIS UNIT, YOU WILL BE STUDYING THE PROPERTIES OF THESE FUNCTIONS IN DETAIL, INCLUDING AND SOME PRACTICAL APPLICATIONS. ALSO, YOU WILL EXTEND YOUR STUDY WITH AN TO THREE MORE TRIGONOMETRIC FUNCTIONS.

5.1 BASIC TRIGONOMETRIC FUNCTIONS

HISTORICAL NOTE:

Astronomy led to the development of trigonometry. The Greek astronomer **Hipparchus** (140 BC) is credited for being the originator of trigonometry. To aid his calculations regarding astronomy, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Hipparchus (190-120 BC)

Ptolomy, another great Greek astronomer of the time, extended this table in his major published work *Almagest* which was used by astronomers for the next 1000 years. In fact much of Hipparchus' work is known through the writings of Ptolomy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 6th century AD, drew up a table of the lengths of half-chords of a circle with radius one unit. Aryabhata actually drew up the first table of sine values.

In the late 16th century, Rhaeticus produced a comprehensive and remarkably accurate table of all the six trigonometric functions. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.

OPENING PROBLEM

FROM AN OBSERVER O, THE ANGLES OF ELEVATION OF THE BOTTOM AND THE TOP OF A FLAGPOLE ARE 36° AND 38° RESPECTIVELY. FIND THE HEIGHT OF THE FLAGPOLE.

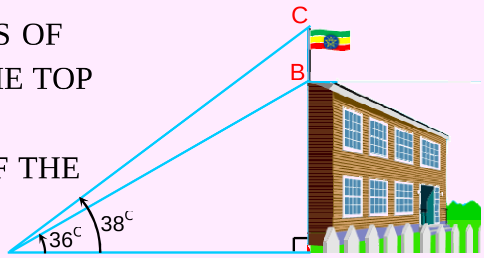


Figure 5.1

5.1.1 The Sine, Cosine and Tangent Functions

Basic terminologies

IF A GIVEN RAY OA (WRITTEN $\overrightarrow{OA}$) ROTATES AROUND A POINT O FROM ITS INITIAL POSITION
NEW POSITION, IT FORMS AN ANGLE AS SHOWN BELOW.

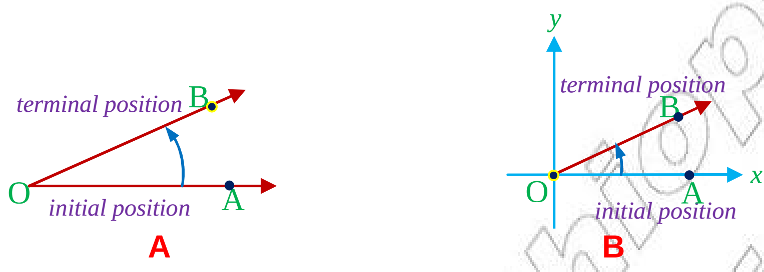


Figure 5.2

$\overrightarrow{OA}$ (INITIAL POSITION) IS CALLED THE **initial side** OF .

$\overrightarrow{OB}$ (TERMINAL POSITION) IS CALLED THE **terminal side** OF .

THE ANGLE FORMED BY A RAY ROTATING ANTICLOCKWISE IS TAKEN TO BE A POSITIVE

AN ANGLE FORMED BY A RAY ROTATING CLOCKWISE IS TAKEN TO BE A NEGATIVE ANG

EXAMPLE 1

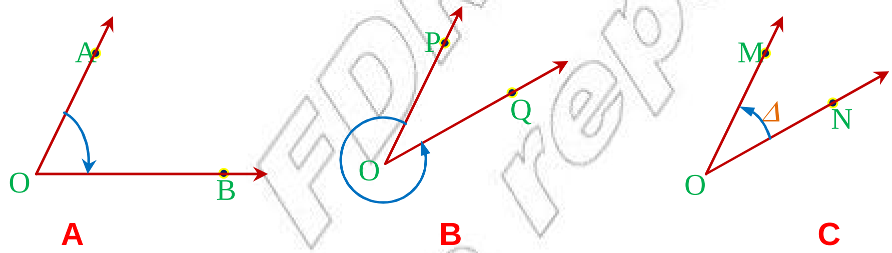


Figure 5.3

- ✓ ANGLE IN FIGURE 5.3A IS A NEGATIVE ANGLE WITH INITIAL SIDE $\overrightarrow{OA}$ AND TERMINAL SIDE $\overrightarrow{OB}$
- ✓ ANGLE IN FIGURE 5.3B IS A POSITIVE ANGLE WITH INITIAL SIDE $\overrightarrow{OP}$ AND TERMINAL SIDE $\overrightarrow{OQ}$
- ✓ ANGLE IN FIGURE 5.3C IS A POSITIVE ANGLE WITH INITIAL SIDE $\overrightarrow{ON}$ AND TERMINAL SIDE $\overrightarrow{OM}$

Angles in standard position

AN ANGLE IN THE COORDINATE PLANE IS SAID TO BE IN **standard position**, IF

1 ITS VERTEX IS AT THE ORIGIN, AND

2 ITS INITIAL SIDE LIES ON THE POSITIVE x-AXIS.

EXAMPLE 2 THE FOLLOWING ANGLES ARE ALL IN STANDARD POSITION:

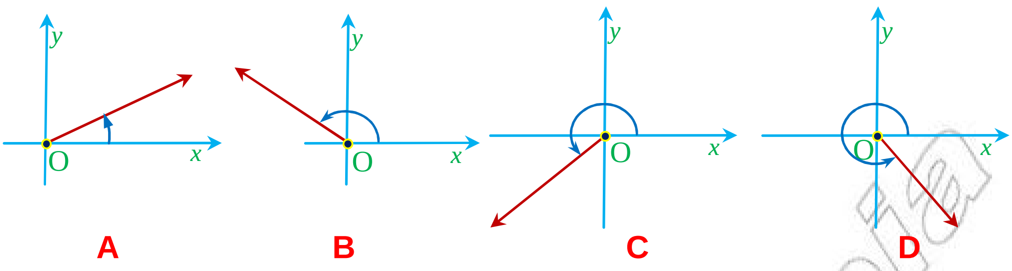


Figure 5.4

First, second, third and fourth quadrant angles

- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE FIRST QUADRANT, THEN IT IS CALLED A first quadrant angle.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE SECOND QUADRANT, THEN IT IS CALLED A second quadrant angle.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE THIRD QUADRANT, THEN IT IS CALLED A third quadrant angle.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE FOURTH QUADRANT, THEN IT IS CALLED A fourth quadrant angle.

EXAMPLE 3 THE FOLLOWING ARE ANGLES IN DIFFERENT QUADRANTS:

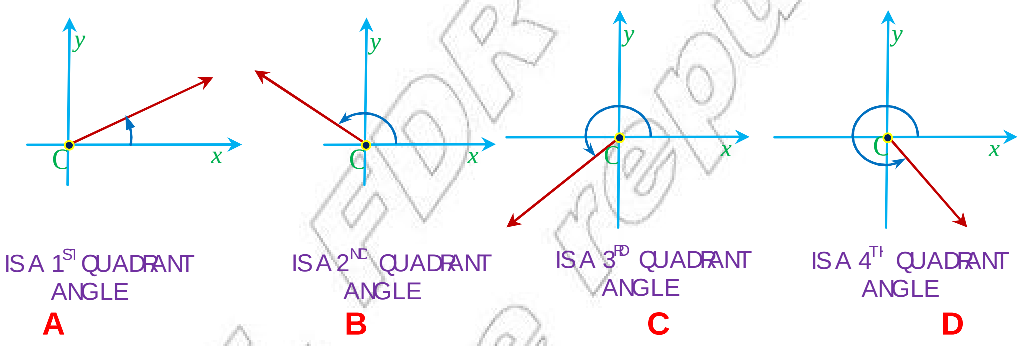
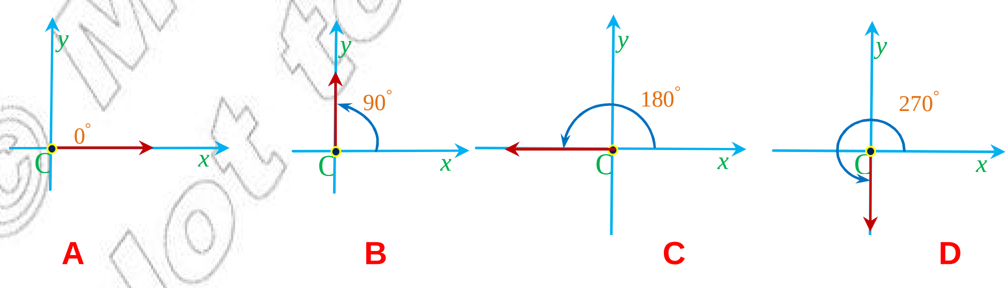


Figure 5.5

Quadrantal angles

IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES ON THE x-AXIS OR THE y-AXIS, THEN THE ANGLE IS CALLED A quadrantal angle.

EXAMPLE 4 THE FOLLOWING ARE ALL QUADRANTAL ANGLES.



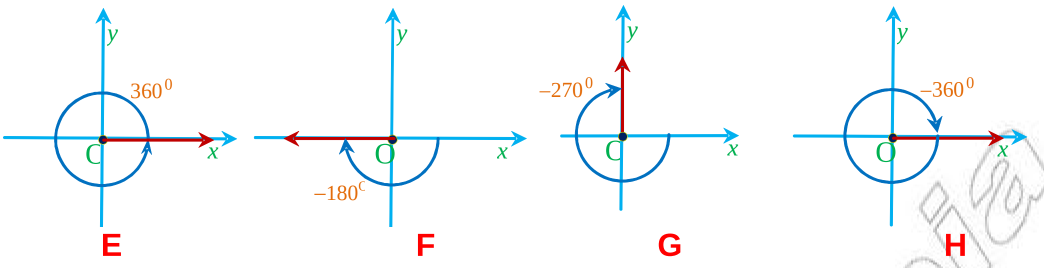


Figure 5.6

ANGLES WITH MEASURES -360^O, -270^O, -180^O, -90^O, 0^O, 90^O, 180^O, 270^O, 360^O ARE EXAMPLES OF QUADRANTAL ANGLES BECAUSE THEIR TERMINAL SIDES LIE ALONG THE x-AXIS OR THE y-AXIS.

EXAMPLE 5 THE FOLLOWING ARE MEASURES OF DIFFERENT ANGLES. PUT THE ANGLES IN STANDARD POSITION AND INDICATE TO WHICH QUADRANT THEY BELONG:

A 200^O **B** 1125^O **C** -900^O

SOLUTION:

A $200^O = 180^O + 20^O$

$\therefore$ AN ANGLE WITH MEASURE 200^O IS A THIRD QUADRANT ANGLE.

B $1125^O = 3(360)^O + 45^O$

1125^O IS A MEASURE OF A FIRST QUADRANT ANGLE.

C $-900^O = 2(-360)^O + (-180^O)$

-900^O IS A MEASURE OF A QUADRANTAL ANGLE.

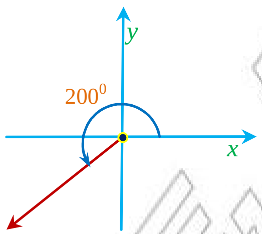


Figure 5.7

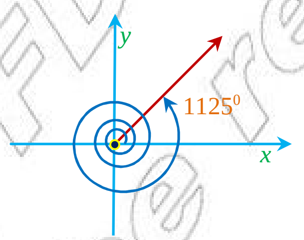


Figure 5.8

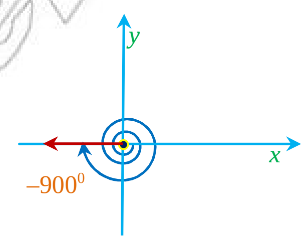


Figure 5.9

Exercise 5.1

THE FOLLOWING ARE MEASURES OF DIFFERENT ANGLES. PUT THE ANGLES IN STANDARD INDICATE TO WHICH QUADRANT THEY BELONG:

A 240^O **B** 350^O **C** 620^O **D** 666^O

E -350^O **F** -480^O **G** 550^O **H** -1080^O

Radian measure of angles

SO FAR WE HAVE MEASURED ANGLES IN DEGREES. HOWEVER, ANGLES CAN ALSO BE MEASURED IN RADIANS. SCIENTISTS, ENGINEERS, AND MATHEMATICIANS USUALLY WORK WITH ANGL

Group Work 5.1

1 DRAW A CIRCLE OF RADIUS 5 CM ON A SHEET OF PA

2 USING A THREAD MEASURE THE CIRCUMFERENCE O... ND RECORD YOUR RESULT IN CENTIMETRES.

3 DIVIDE THE RESULT OBTAINED IN 2 BY LENGTH OF DIAMETER OF THE CIRCLE) AND GIVE YOUR ANSWER IN CENTIMETRES.

4 COMPARE THE ANSWER YOU OBTAINED IN 3 WITH THE VALUE OF

5 USING A THREAD, MEASURE AN ARC LENGTH OF 5 CM ON THE CIRCUMFERENCE OF THE CIRCLE AND NAME THE END POINTS A AND B AS SHOWN IN FIGURE 5.10.

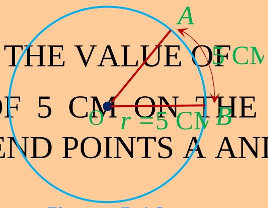


Figure 5.10

6 USING YOUR PROTRACTOR MEASURE ANGLE *AOB*

7 IF YOU REPRESENT THE MEASURE OF THE CENTRAL ANGLE *AOB*, WHICH IS SUBTENDED BY AN ARC EQUAL IN LENGTH TO THE RADIUS AS 1 RADIAN, WHAT WILL BE THE APPROXIM 1 RADIAN IN DEGREES?

8 CAN YOU APPROXIMATE 180^0 AND 360^0 IN RADIANS?

9 DISCUSS YOUR FINDINGS AND FIND A FORMULA THAT CONVERTS DEGREE MEASUR MEASURE.

THE ANGLE SUBTENDED AT THE CENTRE OF A CIRCLE BY AN ARC EQUAL IN LENGTH TO T

1 *radian*. THAT IS $= \frac{r}{r} = 1$ *radian*. (*See* FIGURE 5.11A)

Figure 5.11

IN GENERAL, IF THE LENGTH OF THE ARC IS *s* UNITS AND THE RADIUS IS *r* UNITS, THEN $= \frac{s}{r}$ RADIANS

(*See* FIGURE 5.11B) THIS INDICATES THAT THE SIZE OF THE ANGLE IS THE RATIO OF THE ARC TO THE LENGTH OF THE RADIUS.

EXAMPLE 6 IF s = 3 CM AND r = 2 CM, CALCULATE IN RADIANS.

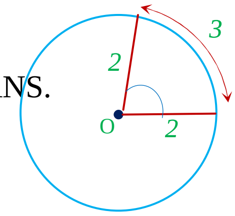


Figure 5.12

SOLUTION: $= \frac{s}{r} = \frac{3}{2} = 1.5$ RADIA

EXAMPLE 7 CONVERT 360^0 TO RADIANS.

SOLUTION: A CIRCLE WITH RADIUS R UNITS HAS CIRCUMFERENCE 2 r

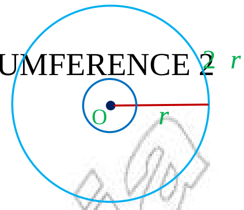


Figure 5.13

IN THIS CASE $\frac{s}{r}$ BECOMES = $\frac{2\ \ r}{r} \Rightarrow$ = 2

I.E., 360^0 = 2 RADIANS.

EXAMPLE 8 CAN YOU CONVERT 180^0 TO *radian* MEASURE?

SOLUTION: SINCE $360° = 2$ RADIANS, $180° =$ RAD … *because* $180° = \frac{360^0}{2}$

IT FOLLOWS THAT 1 RAD = $\frac{180°}{}$ = $57.3°$

Rule 1

TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{}{180^o}$

I.E., *radians* = *degrees* $\times \frac{}{180^o}$.

EXAMPLE 9

A CONVERT 30^O TO RADIANS. **B** CONVERT 240^0 TO RADIANS.

SOLUTION:

A $30^O = 30° \times \frac{}{180°} = \frac{}{6}$ RADIAN. **B** $240^O = 240 \times \frac{}{180} = \frac{4}{3}$ RADIANS.

Rule 2

TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^o}{}$

I.E., *degrees* = *radians* $\times \frac{180^o}{}$.

EXAMPLE 10

A $\frac{}{2}$ RAD $= \frac{}{2} \times \frac{180^O}{} = 90^0$ **B** -4 RAD $= -4 \times \frac{180^o}{} = -720^0$

Exercise 5.2

1 CONVERT EACH OF THE FOLLOWING *degrees* TO *radians*

A 60 **B** 45 **C** –150 **D** 90 **E** –270 **F** 135

2 CONVERT EACH OF THE FOLLOWING *radians* TO *degrees*

A $\frac{}{12}$ **B** $-\frac{}{6}$ **C** $\frac{2}{3}$ **D** $\frac{5}{6}$ **E** $-\frac{10}{3}$ **F** 3

Definition of the sine, cosine and tangent functions

THE Sine, Cosine and Tangent Functions ARE THE THREE basic trigonometric functions.

TRIGONOMETRIC FUNCTIONS WERE ORIGINALLY USED TO RELATE THE ANGLES OF A TRIANGL LENGTHS OF THE SIDES OF A TRIANGLE. IT IS FROM THIS PRACTICE OF MEASURING T TRIANGLE WITH THE HELP OF ITS ANGLES (OR VICE VERSA) THAT THE NAME TRIGONOME

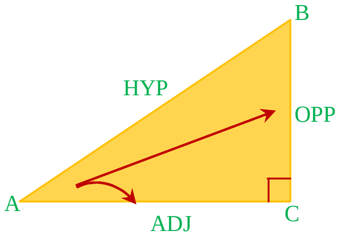


Figure 5.14

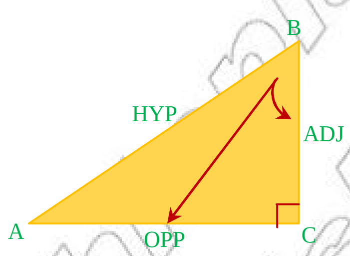


Figure 5.15

LET US CONSIDER THE RIGHT ANGLED TRIANGLES IN FIGURE 5.14 AND FIGURE 5.15.

YOU ALREADY KNOW THAT, FOR A GIVEN RIGHT ANGLED TRIANGLE, THE hypotenuse (HY SIDE WHICH IS OPPOSITE THE RIGHT ANGLE AND IS THE LONGEST SIDE OF THE TRIANGL

FOR THE ANGLE MARKED BY (See FIGURE 5.14)

- ✓ $\overline{BC}$ IS THE SIDE opposite (OPP) ANGLE Æ
- ✓ $\overline{AC}$ IS THE SIDE adjacent (ADJ) ANGLE Æ.

SIMILARLY, FOR THE ANGLE MARKED BY (See FIGURE 5.15)

- ✓ $\overline{AC}$ IS THE SIDE opposite (OPP) ANGLE .
- ✓ $\overline{BC}$ IS THE SIDE adjacent (ADJ) ANGLE .

Definition 5.1

If is an angle in standard position and $P(a,b)$ is a point on the terminal side of , other than the origin $\mathbf{O}(0, 0)$, and r is the distance of point P from the origin $\mathbf{O}$, then

$$\sin = \frac{OPP}{HYP} = \frac{b}{r}$$

$$\cos = \frac{ADJ}{HYP} = \frac{a}{r}$$

$$\tan = \frac{OPP}{ADJ} = \frac{b}{a}$$

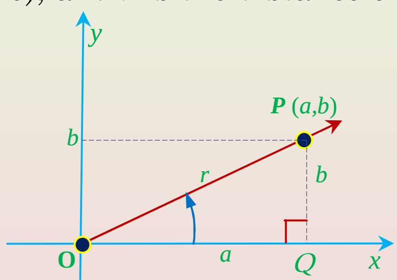


Figure 5.16

REMEMBER THAT OPQ IS A RIGHT ANGLE TRIANGLE.

(BY THE PYTHAGORAS THEOREM, $r = \sqrt{a^2 + b^2}$)

(SIN, COS AND TAN ARE ABBREVIATIONS OF SINE, COSINE AND TANGENT, RESPECTIVELY.)

TRIGONOMETRIC FUNCTIONS CAN BE CONSIDERED IN THE SAME WAY AS ANY GENERAL LINEAR, QUADRATIC, EXPONENTIAL OR LOGARITHMIC.

THE INPUT VALUE FOR A TRIGONOMETRIC FUNCTION IS AN angle THAT COULD BE MEASURED IN DEGREES OR RADIANS. THE OUTPUT VALUE FOR A TRIGONOMETRIC FUNCTION IS A pure number WITH NO UNIT.

EXAMPLE 11 IF θ IS AN ANGLE IN STANDARD POSITION AND P (3, 4) IS A POINT ON THE TERMINAL SIDE OF θ, THEN EVALUATE THE SINE, COSINE AND TANGENT OF θ.

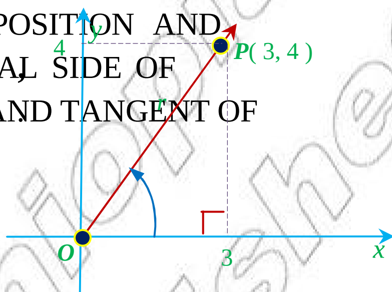


Figure 5.17

SOLUTION: THE DISTANCE $r = \sqrt{3^2+4^2} = 5$ UNITS

SO $\sin\theta = \dfrac{OPP}{HYP} = \dfrac{4}{5}$ $\cos\theta = \dfrac{ADJ}{HYP} = \dfrac{3}{5}$ AND

$\tan\theta = \dfrac{OPP}{ADJ} = \dfrac{4}{3}$.

Exercise 5.3

EVALUATE THE SINE, COSINE AND TANGENT FUNCTIONS OF θ IF θ IS IN STANDARD POSITION AND ITS TERMINAL SIDE CONTAINS THE GIVEN POINT P(x, y):

A	P (3, – 4)	**B**	P (– 6, – 8)	**C**	P (1, –1)
D	P $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$	**E**	P $\left(4\sqrt{5}, -2\sqrt{5}\right)$	**F**	P (1, 0)

The unit circle

THE CIRCLE WITH CENTRE AT (0, 0) AND RADIUS 1 UNIT IS CALLED THE unit circle.

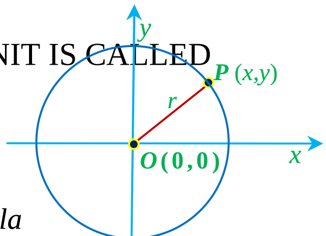


Figure 5.18

CONSIDER A POINT P(x, y) ON THE CIRCLE. (*See* FIGURE 5.18)

SINCE OP = r, THEN $\sqrt{(x-0)^2+(y-0)^2} = r$ … *by distance formula*

$\therefore x^2 + y^2 = r^2$ … *squaring both sides*

WE SAY THAT $x^2 + y^2 = r^2$ IS THE EQUATION OF A CIRCLE WITH CENTRE (0, 0) AND RADIUS r. ACCORDINGLY, THE EQUATION OF THE *unit circle* IS $x^2 + y^2 = 1$. (AS $r = 1$)

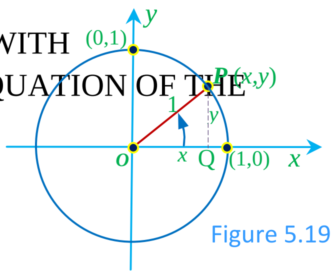


Figure 5.19

LET THE TERMINAL SIDE OF θ INTERSECT THE *unit circle* AT POINT P(x, y). SINCE $r = x^2 + y^2 = 1$, THE sine, cosine AND tangent FUNCTIONS OF θ ARE GIVEN AS FOLLOWS:

$$\text{SIN} = \frac{OPP}{HYP} = \frac{y}{r} = \frac{y}{1} = y \quad \ldots \textit{ the y-coordinate of P}$$

$$\text{COS} = \frac{ADJ}{HYP} = \frac{x}{r} = \frac{x}{1} = x \quad \ldots \textit{ the x-coordinate of P}$$

$$\text{TAN} = \frac{OPP}{ADJ} = \frac{y}{x}$$

EXAMPLE 12 USING THE UNIT CIRCLE, FIND THE VALUES OF THE sine, cosine AND tangent OF θ; IF $\theta = 90^O, 180^O, 270^O$.

SOLUTION: AS SHOWN IN FIGURE 5.20, THE TERMINAL SIDE OF THE ANGLE 90^O INTERSECTS THE UNIT CIRCLE AT (0, 1). SO $(x, y) = (0, 1)$.

HENCE, $\text{SIN } 90^O = y = 1$, $\text{COS } 90^O = x = 0$ AND $\text{TAN } 90^O$ IS UNDEFINED SINCE $\frac{y}{x} = \frac{1}{0}$

THE TERMINAL SIDE OF THE ANGLE 180^O INTERSECTS THE UNIT CIRCLE AT (–1, 0). (*See* FIGURE 5.21) SO, $(x, y) = (-1, 0)$.

HENCE, $\text{SIN } 180^O = y = 0$, $\text{COS } 180^O = x = -1$ AND $\text{TAN } 180^O = \frac{y}{x} = \frac{0}{-1} = 0$.

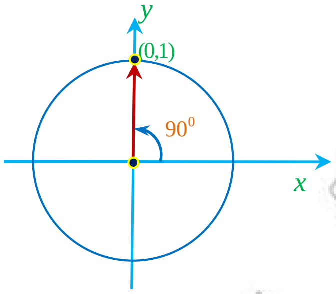


Figure 5.20

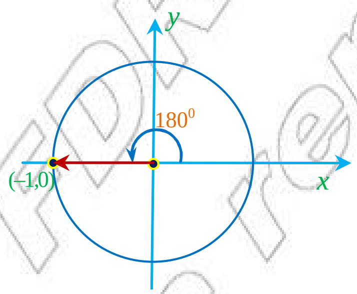


Figure 5.21

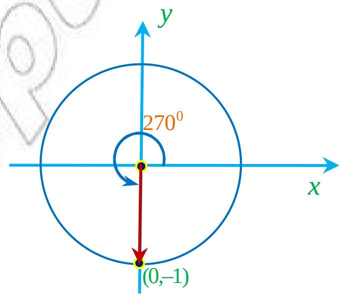


Figure 5.22

THE TERMINAL SIDE OF THE ANGLE 270^O INTERSECTS THE UNIT CIRCLE AT (0, –1). (*See* FIGURE 5.22) SO $(x, y) = (0, -1)$. HENCE, $\text{SIN } 270^O = y = -1$, $\text{COS } 270^O = x = 0$ AND $\text{TAN } 270^O$ IS UNDEFINED SINCE $\frac{y}{x} = \frac{-1}{0}$.

Exercise 5.4

1 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANGENT FU THE FOLLOWING QUADRANTAL ANGLES:

A 0^O **B** 360^O **C** 450^O

D 540^O **E** 630^O

Trigonometric values of 30°, 45° and 60°

THE FOLLOWING GROUP WORK WILL HELP YOU TO FIND THE TRIGONOMETRIC VALUES OF THE S ANGLE 45^{o}.

Group Work 5.2

CONSIDER THE ISOSCELES RIGHT ANGLE TRIANGLE IN FIG

- **a** CALCULATE THE LENGTH OF THE HYPOTENUSE *AB*.
- **b** FROM THE PROPERTIES OF AN ISOSCELES RIGHT ANGLE TRIANGLE WHAT IS THE MEASURE OF ANGLE *A*?
- **c** ARE THE ANGLES *A* AND *B* CONGRUENT?
- **d** WHICH SIDE IS OPPOSITE TO ANGLE *A*? WHICH SIDE IS ADJACENT TO ANGLE *A*?
- **e** FIND SIN *A*, COS *A* AND TAN *A*.

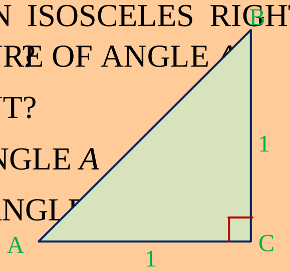


Figure 5.23

FROM GROUP WORK 5.2 YOU HAVE FOUND THE VALUES OF SIN 45^{o}, COS 45^{o} AND TAN 45^{o}.

ANOTHER WAY OF FINDING THE TRIGONOMETRIC VALUES OF 45^{o} IS TO PLACE THE ANGLE IN STANDARD POSITION AS SHOWN IN FIGURE 5.24.

WHEN WE PLACE THE ANGLE 45^{o} IN STANDARD POSITION, ITS TERMINAL SIDE INTERSECTS THE CIRCLE AT $P(x, y)$.

TO CALCULATE THE COORDINATES OF *P*, DRAW *PD* PARALLEL TO THE *y*-AXIS.

ΔOPD IS AN ISOSCELES RIGHT ANGLE TRIANGLE.

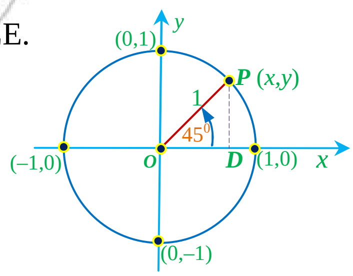


Figure 5.24

BY PYTHAGORAS RULE, $(OD)^2 + (PD)^2 = (OP)^2$

SINCE $OD = PD$, $(PD)^2 + (PD)^2 = (OP)^2$.

THAT IS $y^2 + y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2}$

$$\Rightarrow y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

SINCE THE TRIANGLE IS ISOSCELES, BOTH THE *x*- AND *y*-COORDINATES ARE THE SAME.

THEREFORE THE TERMINAL SIDE OF THE 45^{o} ANGLE INTERSECTS THE UNIT CIRCLE AT $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

HENCE, SIN 45^{o} = $y = \frac{\sqrt{2}}{2}$; COS 45^{o} = $x = \frac{\sqrt{2}}{2}$ AND TAN 45^{o} = $\frac{y}{x} = \frac{\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = 1$

Trigonometric values for 30° and 60°

CONSIDER THE EQUILATERAL TRIANGLE ABC IN FIGURE 5.25, WITH SIDE LENGTH 2 UNITS. THE ALTITUDE $\overline{BD}$ BISECTS ∠B AS WELL AS SIDE $\overline{AC}$. HENCE ∠$ABD = 30^{O}$ AND $AD = 1$ (HALF OF THE LENGTH OF AC).

BY PYTHAGORAS THEOREM, THE LENGTH OF THE ALTITUDE IS h WHERE

$$h^2 + 1^2 = 2^2 \quad \Rightarrow h^2 = 4 - 1 = 3 \quad \Rightarrow h = \sqrt{3}$$

NOW IN THE RIGHT-ANGLED TRIANGLE ABD,

$$\sin 30^{O} = \frac{1}{2} = 0.5 \qquad \sin 60^{O} = \frac{\sqrt{3}}{2}$$

$$\cos 30^{O} = \frac{\sqrt{3}}{2} \qquad \cos 60^{O} = \frac{1}{2} = 0.5$$

$$\tan 30^{O} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60^{O} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

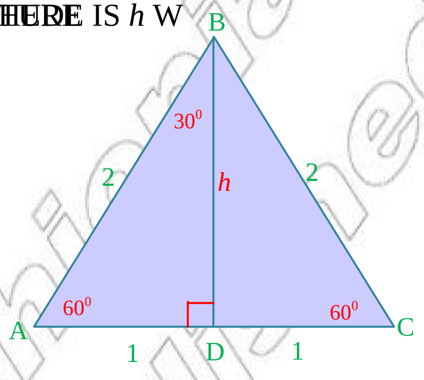


Figure 5.25

Trigonometric values of negative angles

Remember that AN ANGLE IS positive IF MEASURED ANTICLOCKWISE AND negative IF CLOCKWISE.

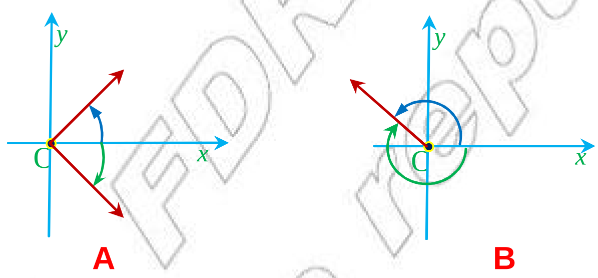


Figure 5.26

θ IS A POSITIVE ANGLE WHEREAS β IS A NEGATIVE ANGLE.

EXAMPLE 13 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANGENT FUNCTIONS OF θ WHEN θ = -180^{O}.

THE TERMINAL SIDE OF -180^{O} INTERSECTS THE UNIT CIRCLE AT (–1, 0). SO $(x, y) = (-1, 0)$.

HENCE, $\sin(-180^{O}) = y = 0$,

$\cos(-180^{O}) = x = -1$

AND $\tan(-180^{O}) = \frac{y}{x} = \frac{0}{-1} = 0.$

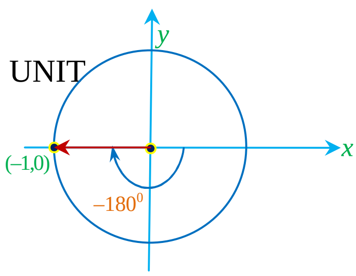


Figure 5.27

EXAMPLE 14 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANG FUNCTIONS OF θ WHEN θ = -45^{O}.

SOLUTION: PLACE THE -45° ANGLE IN STANDARD POSITION. ITS TERMINAL SIDE INTERSECTS THE UNIT CIRCLE AT Q(x, y).

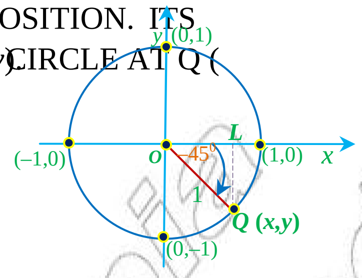


Figure 5.28

TO DETERMINE THE COORDINATES OF Q, DRAW $\overline{QL}$ PARALLEL TO THE y-AXIS.

ΔOQL IS AN ISOSCELES RIGHT TRIANGLE.

BY PYTHAGORAS THEOREM, $(OL)^2 + (QL)^2 = (OQ)^2$

SINCE $OL = QL$, $(QL)^2 + (QL)^2 = (OQ)^2$.

THAT IS $y^2 + y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$

$\therefore y = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. … ***Remember that*** *y is negative in the fourth quadrant*

SINCE THE TRIANGLE IS ISOSCELES, $QL = OL = \frac{\sqrt{2}}{2}$

THEREFORE, THE x COORDINATE OF Q IS $\frac{\sqrt{2}}{2}$ … *Note that x is positive in the fourth quadrant*

SO, THE TERMINAL SIDE OF THE ANGLE -45° INTERSECTS THE UNIT CIRCLE AT P$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

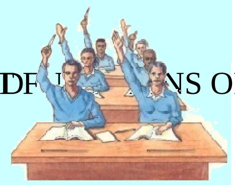
I.E., $(x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

HENCE, SIN $(-45^{\circ}) = y = \frac{\sqrt{2}}{2}$; COS $(-45^{\circ}) = x = \frac{\sqrt{2}}{2}$ AND TAN $(-45^{\circ}) = \frac{y}{x} = \frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = -1$.

OBSERVE THAT FROM THE TRIGONOMETRIC VALUES OF 45° AND -45°,

SIN $(-45^{\circ}) = -$SIN 45°, COS $(-45^{\circ}) =$ COS 45° AND TAN$(-45^{\circ}) = -$TAN45°.

ACTIVITY 5.1



1 FIND THE VALUES OF THE SINE, COSINE AND TANGENT FUNCTIONS OF AND COMPLETE THE FOLLOWING TWO TABLES:
(USE A DASH "–" IF IT IS UNDEFINED).

	0°	30°	45°	60°	90°	180°	270°	360°
sin	0				1		–1	
cos						–1		
tan					–			

	-30^o	-45^o	-60^o	-90^o	-180^o	-270^o	-360^o
SIN	$-\frac{1}{2}$		$-\frac{\sqrt{3}}{2}$				
COS	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$	0			
TAN	$-\frac{\sqrt{3}}{3}$		$-\sqrt{3}$	–			

2 WHICH OF THE FOLLOWING PAIRS OF VALUES ARE EQUAL?

- **A** SIN(–30°) AND SIN(30°)
- **B** COS(–30°) AND COS(30°)
- **C** TAN(–30°) AND TAN(30°)
- **D** SIN(– 45°) AND SIN(45°)
- **E** COS(– 45°) AND COS(45°)
- **F** TAN(– 45°) AND TAN(45°)
- **G** SIN(– 60°) AND SIN(60°)
- **H** COS(– 60°) AND COS(60°)
- **I** TAN(– 60°) AND TAN(60°)

3 HOW DO YOU COMPARE THE VALUES OF:

- **A** SIN (–θ) AND SIN θ
- **B** COS (–θ) AND COS θ
- **C** TAN (–θ) AND – TAN θ

FROM ACTIVITY 5.10 YOU CONCLUDE THE FOLLOWING:

IF θ IS ANY ANGLE, THEN SIN(–θ) = –SIN θ, COS(–θ) = COS θ and TAN(–θ) = –TAN θ.

LET US REFER TO FIGURE 5.29 TO JUSTIFY THE ABOVE.

$$\sin\theta = \frac{y}{r},\ \sin(-\theta) = \frac{-y}{r} = -\left(\frac{y}{r}\right) \therefore \sin(-\theta) = -\sin\theta$$

$$\cos\theta = \frac{x}{r},\ \cos(-\theta) = \frac{x}{r} \qquad \therefore \cos(-\theta) = \cos\theta$$

$$\tan\theta = \frac{y}{x},\ \tan(-\theta) = \frac{-y}{x} = -\left(\frac{y}{x}\right) \therefore \tan(-\theta) = -\tan\theta$$

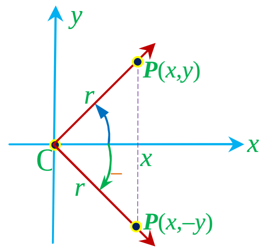


Figure 5.29

5.1.2 Values of Trigonometric Functions for Related Angles

The signs of sine, cosine and tangent functions

IN THIS SUB-SECTION YOU WILL CONSIDER WHETHER THE SIGN OF EACH OF THE TRIGONOMETRIC FUNCTIONS OF AN ANGLE IS POSITIVE OR NEGATIVE.

THE SIGN (WHETHER SIN θ, COS θ AND TAN θ ARE POSITIVE OR NEGATIVE) DEPENDS ON THE QUADRANT TO WHICH θ BELONGS.

EXAMPLE 1 CONSIDER AN ANGLE IN THE FIRST AND SECOND QUADRANTS.

IF IS A FIRST QUADRANT ANGLE, THEN THE SIGN OF

SIN $= \frac{opp}{hyp} = \frac{y}{r}$ IS POSITIVE

COS $= \frac{adj}{hyp} = \frac{x}{r}$ IS POSITIVE

TAN $= \frac{opp}{adj} = \frac{y}{x}$ IS POSITIVE

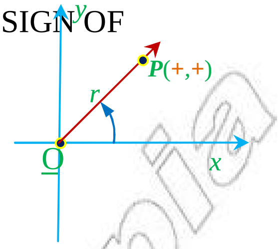


Figure 5.30

IF IS A SECOND QUADRANT ANGLE THEN, THE SIGN OF

SIN $= \frac{opp}{hyp} = \frac{y}{r}$ IS POSITIVE

COS $= \frac{adj}{hyp} = \frac{x}{r}$ IS NEGATIVE SINCE x IS NEGATIVE

TAN $= \frac{opp}{adj} = \frac{y}{x}$ IS NEGATIVE

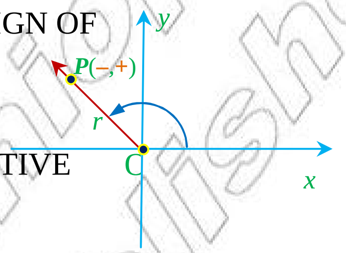


Figure 5.31

ACTIVITY 5.2

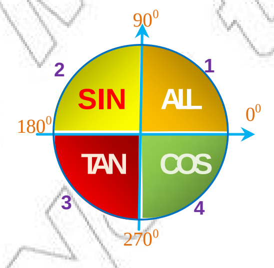
1 DETERMINE WHETHER THE SIGNS OF SIN, COS AND TAN ARE POSITIVE OR NEGATIVE:

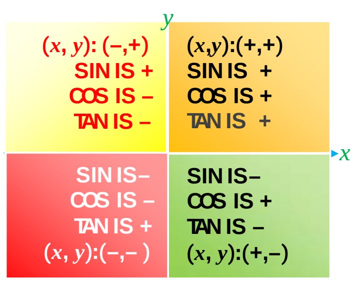
A IF IS A THIRD QUADRANT ANGLE **B** IF IS A FOURTH QUADRANT ANGLE

2 DECIDE WHETHER THE THREE TRIGONOMETRIC FUNCTIONS ARE POSITIVE OR NEGA COMPLETE THE FOLLOWING TABLE:

	has terminal side in quadrant			
	I	II	III	IV
sin	+			–
cos		–		
tan			+	

IN GENERAL, THE SIGNS OF THE SINE, COSINE AND TANGENT FUNCTIONS IN ALL OF THE CAN BE SUMMARIZED AS BELOW:





- IN THE FIRST QUADRANT all the three TRIGONOMETRIC FUNCTIONS ARE POSITIVE.
- IN THE SECOND QUADRANT ONLY sine IS POSITIVE.
- IN THE THIRD QUADRANT ONLY tangent IS POSITIVE.
- IN THE FOURTH QUADRANT ONLY cosine IS POSITIVE.

Do you want an easy way to remember this? KEEP IN MIND THE FOLLOWING STATEMENT:

All Students Take Chemistry

TAKING THE FIRST LETTER OF EACH WORD WE HAVE

All	All are positive
Students	Sine is positive
Take	Tangent is positive
Chemistry	Cosine is positive

EXAMPLE 2 DETERMINE THE SIGN OF:

A SIN 195^{O} **B** TAN 336^{O} **C** COS 895^{O}

SOLUTION:

A OBSERVE THAT $180^{O} < 195^{O} < 270^{O}$. SO ANGLE 195^{O} IS A THIRD QUADRANT ANGLE. IN THE THIRD QUADRANT THE SINE FUNCTION IS NEGATIVE.

∴ SIN 195^{O} IS NEGATIVE

B SINCE $270^{O} < 336^{O} < 360^{O}$, THE ANGLE WHOSE MEASURE IS 336^{O} IS A FOURTH QUADRANT ANGLE. IN THE FOURTH QUADRANT THE TANGENT FUNCTION IS N

HENCE TAN 336^{O} IS NEGATIVE.

C SINCE $2(360)^{O} < 895^{O} < 2(360)^{O} + 180^{O}$, THE ANGLE WHOSE MEASURE IS 895^{O} IS A SECOND QUADRANT ANGLE. IN THE SECOND QUADRANT THE COSINE FUNCTION IS NEGATIVE.

HENCE, COS 895^{O} IS NEGATIVE.

Group Work 5.3

1 DISCUSS AND ANSWER EACH OF THE FOLLOWING:

A IF TAN > 0 AND COS < 0, THEN IS IN QUADRANT______.

B IF SIN > 0 AND COS < 0, THEN IS IN QUADRANT______.

C IF COS > 0 AND TAN < 0, THEN IS IN QUADRANT______.

D IF SIN < 0 AND TAN > 0, THEN IS IN QUADRANT______.

2 DETERMINE THE SIGN OF:

A COS 267^{O} **B** TAN (-280^{O}) **C** SIN (-815^{O})

3 DETERMINE THE SIGNS OF SIN AND TAN, IF IS AN ANGLE IN STANDARD POSITION AND P (2, −5) IS A POINT ON ITS TERMINAL SIDE.

Complementary angles

ANY TWO ANGLES ARE SAID TO BE **COMPLEMENTARY**, IF THE SUM OF THEIR MEASURES IS EQUAL TO 90^{0}

EXAMPLE 3 ANGLE WITH MEASURES OF 30^{O} AND 60^{O}, 20^{O} AND 70^{O}, 40^{O} AND 50^{O}, 45^{O} AND 45^{O}, 10^{O} AND 80^{O} ARE EXAMPLES OF COMPLEMENTARY ANGLES.

ACTIVITY 5.3

1 REFERRING TO FIGURE 5.32,

A FIND SIN 30^{O}, COS 30^{O}, TAN 30^{O}, SIN 60^{O}, COS 60^{O}, TAN 60^{O}

B **I** COMPARE THE RESULTS OF SIN 30^{O} AND COS 60^{O}

II COMPARE THE RESULTS OF SIN 60^{O} AND COS 30^{O}

III COMPARE THE RESULTS OF TAN 30^{O} AND TAN 60^{O}

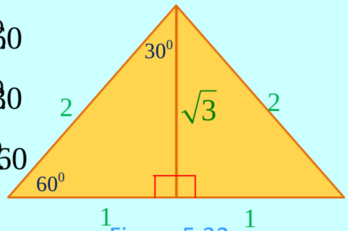


Figure 5.32

2 REFER TO FIGURE 5.33 ON THE RIGHT AND FIND

A SIN , COS , TAN, SIN , COS AND TAN .

B **I** COMPARE THE RESULTS OF SIN AND COS .

II COMPARE THE RESULTS OF SIN AND COS .

III COMPARE THE RESULTS OF TAN AND TAN .

C WHAT DO YOU CONCLUDE FROM YOUR FINDINGS?

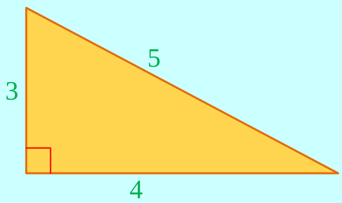


Figure 5.33

FROM ACTIVITY 5.3, THE FOLLOWING RELATIONSHIPS CAN BE CONCLUDED:

IF AND ARE COMPLEMENTARY ANGLES, THAT IS,

(+ = 90^{0}) (*See* FIGURE 5.34), THEN WE HAVE,

$$\text{SIN} = \frac{a}{c} \quad \text{COS} = \frac{a}{c} \quad \text{TAN} = \frac{b}{a}$$

$$\text{SIN} = \frac{b}{c} \quad \text{COS} = \frac{b}{c} \quad \text{TAN} = \frac{a}{b} = \frac{1}{\left(\frac{b}{a}\right)}$$

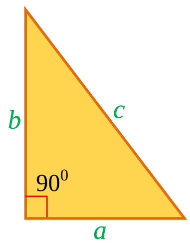


Figure 5.34

HENCE, FOR COMPLEMENTARY ANGLES AND ,

$$\text{SIN} = \text{COS} , \text{COS} = \text{SIN} \text{ AND } \text{TAN} = \frac{1}{\text{TAN}} .$$

Exercise 5.5

ANSWER EACH OF THE FOLLOWING QUESTIONS:

A IF $\sin 31^\circ = 0.5150$, THEN WHAT IS $\cos 59^\circ$?

B IF $\sin\theta = \frac{3}{5}$, THEN WHAT IS $\cos(90^\circ - \theta)$?

C IF $\cos\theta = \frac{4}{5}$, THEN WHAT IS $\sin(90^\circ - \theta)$?

D IF $\sin\theta = k$, THEN WHAT IS $\cos(90^\circ - \theta)$?

E IF $\cos\theta = r$, THEN WHAT IS $\sin(90^\circ - \theta)$?

F IF $\tan\beta = \frac{m}{n}$, THEN WHAT IS $\frac{1}{\tan(90^\circ - \beta)}$?

Reference angle (θ_R)

IF θ IS AN ANGLE IN STANDARD POSITION WHOSE TERMINAL SIDE DOES NOT LIE ON A COORDINATE AXIS, THEN A **reference angle** θ_R FOR θ IS THE **acute angle** FORMED BY THE TERMINAL SIDE OF θ AND THE x-AXIS AS SHOWN IN THE FOLLOWING FIGURES:

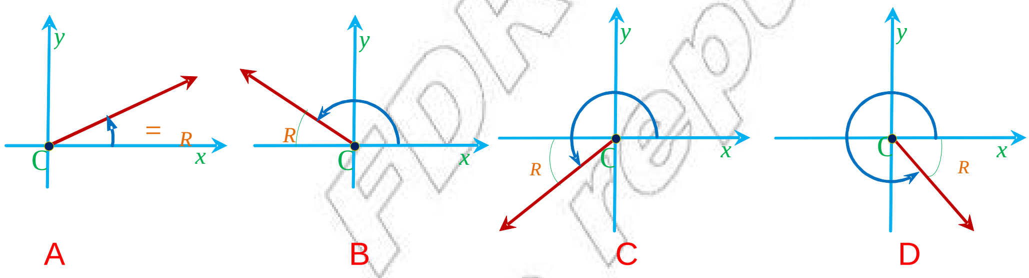


Figure 5.35

EXAMPLE 4 FIND THE REFERENCE ANGLE θ_R FOR θ IF:

A $\theta = 110°$ **B** $\theta = 212°$ **C** $\theta = 280°$

SOLUTION:

A SINCE $\theta = 110°$ IS A SECOND QUADRANT ANGLE,

$\theta_R = 180 - 110° = 70°$

B SINCE $\theta = 212°$ IS A THIRD QUADRANT ANGLE,

$\theta_R = 212° - 180° = 32°$

C SINCE $\theta = 280°$ IS A FOURTH QUADRANT ANGLE,

$\theta_R = 360° - 280° = 80°$

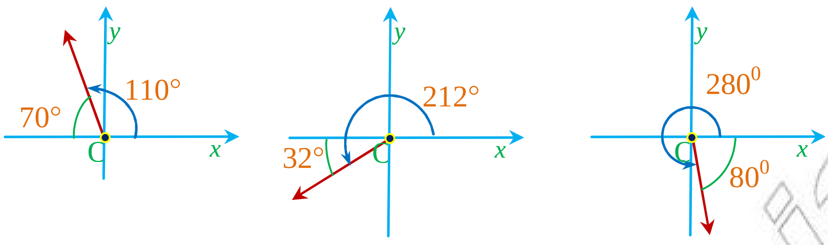


Figure 5.36 Figure 5.37 Figure 5.38

Exercise 5.6

FIND THE REFERENCE ANGLE θ_R FOR θ IF:

A θ = 150°	**B** θ = 170°	**C** θ = 240°	**D** θ = 320°
E θ = 99°	**F** θ = 225°	**G** θ = 315°	**H** θ = 840°

Values of the trigonometric functions of θ and its reference angle θ_R

LET US CONSIDER A SECOND QUADRANT ANGLE θ IN STANDARD POSITION AS SHOWN IN THE FIGURE 5.39, AND LET P($-x$, y) BE A POINT ON ITS TERMINAL SIDE. USING THE y-AXIS AS THE AXIS OF SYMMETRY, REFLECT P THROUGH THE y-AXIS. THIS WILL GIVE YOU ANOTHER POINT P'(x, y) WHICH IS THE IMAGE OF P ON THE TERMINAL SIDE OF θ_R.

THIS IMPLIES THAT $OP = OP'$, THAT IS, $OP = OP' = \sqrt{x^2 + y^2} = r$

HENCE, $\quad \sin\theta = \frac{y}{r}, \quad \sin\theta_R = \frac{y}{r} \Rightarrow \sin\theta = \sin\theta_R$

$$\cos\theta = \frac{-x}{r}, \quad \cos\theta_R = \frac{x}{r} \Rightarrow \cos\theta = -\cos\theta_R$$

$$\tan\theta = \frac{y}{-x} = -\frac{y}{x}, \quad \tan\theta_R = \frac{y}{x} \Rightarrow \tan\theta = -\tan\theta_R$$

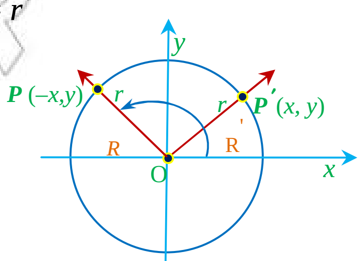


Figure 5.39

THE VALUES OF THE TRIGONOMETRIC FUNCTION OF A GIVEN ANGLE θ AND THE VALUES OF THE CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE REFERENCE ANGLE θ_R ARE THE SAME IN ABSOLUTE VALUE BUT THEY MAY DIFFER IN SIGN.

EXAMPLE 5 EXPRESS THE SINE, COSINE AND TANGENT FUNCTIONS OF 160° IN TERMS OF ITS REFERENCE ANGLE.

SOLUTION: ***Remember that*** AN ANGLE WITH MEASURE 160° IS A SECOND QUADRANT ANGLE .

IN QUADRANT II, ONLY SINE IS POSITIVE.

THE REFERENCE ANGLE θ_R = 180° – 160° = 20°

THEREFORE, SIN 160° = SIN 20°, COS 160° = – COS 20° AND TAN 160° = – TAN 20°

Supplementary angles

TWO ANGLES ARE SAID TO BE **supplementary**, IF THE SUM OF THEIR MEASURES IS EQUAL TO 180°.

EXAMPLE 6 PAIRS OF ANGLES WITH MEASURES OF 30° AND 150°, 120° AND 60°, 45° AND 135°, 75° AND 105°, 10° AND 170° ARE EXAMPLES OF SUPPLEMENTARY ANGLES.

EXAMPLE 7 FIND THE VALUES OF SIN 150°, COS 150° AND TAN 150°

SOLUTION: THE REFERENCE ANGLE $\theta_R = 180^{\circ} - 150^{\circ} = 30^{\circ}$

THEREFORE, $\sin 150^{\circ} = \sin 30^{\circ} = \frac{1}{2}$, $\cos 150^{\circ} = -\cos 30^{\circ} = -\frac{\sqrt{3}}{2}$

AND $\tan 150^{\circ} = -\tan 30^{\circ} = -\frac{\sqrt{3}}{3}$.

EXAMPLE 8 FIND THE VALUES OF SIN 240°, COS 240° AND TAN 240°

SOLUTION: THE REFERENCE ANGLE $\theta_R = 240^{0} - 180^{0} = 60^{0}$

$\sin 240^{\circ} = -\sin 60^{\circ} = -\frac{\sqrt{3}}{2}$, $\cos 240^{\circ} = -\cos 60^{\circ} = -\frac{1}{2}$ AND

$\tan 240^{\circ} = \tan 60^{\circ} = \sqrt{3}$.

… ***remember that*** *in quadrant III only tangent is positive.*

IN GENERAL,

IF θ IS A SECOND QUADRANT ANGLE, THEN ITS REFERENCE ANGLE WILL BE $(180^{\circ} - \theta)$. HENCE,

$\sin\theta = \sin(180^{\circ} - \theta)$ $\cos\theta = -\cos(180^{\circ} - \theta)$ $\tan\theta = -\tan(180^{\circ} - \theta)$

IF θ IS A THIRD QUADRANT ANGLE, ITS REFERENCE ANGLE WILL BE $\theta - 180^{0}$.

HENCE, $\sin\theta = -\sin(\theta - 180^{\circ})$ $\cos\theta = -\cos(\theta - 180^{\circ})$ AND $\tan\theta = \tan(\theta - 180^{\circ})$.

Exercise 5.7

1 EXPRESS THE SINE, COSINE AND TANGENT FUNCTIONS OF THE FOLLOWING ANGLE MEASURES IN TERMS OF THEIR REFERENCE ANGLE:

A 105° **B** 175° **C** 220°

D -260° **E** -300° **F** 380°

2 FIND THE VALUES OF:

A SIN 135°, COS 135° AND TAN 135° **B** COS 143°, IF COS 37° = 0.7986

C TAN 138° IF TAN 42° = 0.9004 **D** SIN 115°, IF SIN 65° = 0.9063

E TAN 159° IF TAN 21° = 0.3839 **F** COS 24°, IF COS 156° = – 0.9135

Co–terminal angles

Co–terminal angles are angles in standard position that have a common terminal side.

EXAMPLE 9

a The three angles with measures 30°, –330° and 390° are co-terminal angles. (*See* Figure 5.40)

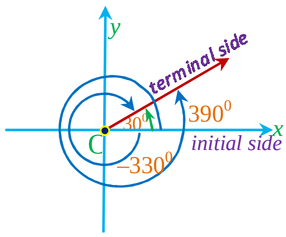


Figure 5.40

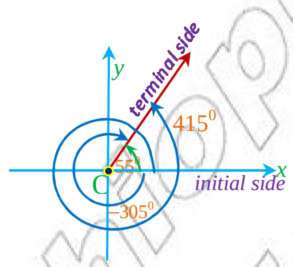
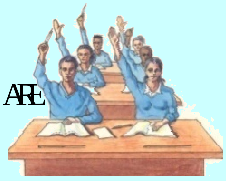


Figure 5.41

b The three angles with measures 55°, –305° and 415° are also co-terminal. (*See* Figure 5.41)

ACTIVITY 5.4



1 With the help of the following table find angles which are co-terminal with 60°.

Angles which are co–terminal with 60°	
$60^{o} + 1(360^{o}) = 420^{o}$	$60^{o} - 1(360^{o}) = -300^{o}$
$60^{o} + 2(360^{o}) = 780^{o}$	$60^{o} - 2(360^{o}) = -660^{o}$
________________	________________
________________	________________
________________	________________
$60^{o} + 6(360^{o}) = 2220^{o}$	$60^{o} - 6(360^{o}) = -2100^{o}$
.	.
.	.
.	.

2 Give a formula to find all angles which are co-terminal with 60°.

Given an angle , all angles which are co–terminal with are given by the formula

± n (360°), where n = 1 , 2, 3, . . .

EXAMPLE 10 Find a positive and a negative angle co-terminal with 75° .

SOLUTION: To find a positive and a negative angle co-terminal with a given angle, you can add or subtract 360°. Hence, 75° – 360° = –285°; 75° + 360° = 435°.

THEREFORE, –285° AND 435° ARE CO-TERMINAL WITH 75°.

THERE ARE AN INFINITE NUMBER OF OTHER ANGLES CO-TERMINAL WITH 75°. THEY BY 75° ± n (360°), n = 1, 2, 3, . . .

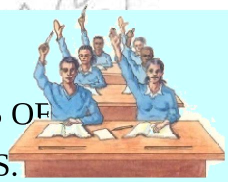
Exercise 5.8

FIND ANY TWO CO-TERMINAL ANGLES (ONE OF THEM POSITIVE AND THE OTHER NEGAT OF THE FOLLOWING ANGLE MEASURES:

A	70^{O}	**B**	110^{O}	**C**	220^{O}	**D**	270^{O}
E	-90^{O}	**F**	-37^{O}	**G**	-60^{O}	**H**	-70^{O}

Trigonometric values of co–terminal angles

ACTIVITY 5.5



CONSIDER FIGURE 5.42 AND FIND THE TRIGONOMETRIC VALUES OF AND .

P (x, y) IS A POINT ON THE TERMINAL SIDE OF BOTH ANGLES.

ANSWER EACH OF THE FOLLOWING QUESTIONS:

A ARE AND CO-TERMINAL ANGLES? WHY?

B WHICH ANGLE IS POSITIVE? WHICH ANGLE IS NEGATIVE?

C FIND THE VALUES OF SIN , COS , TAN IN TERMS OF x, y, r.

D FIND THE VALUES OF SIN , COS , TAN IN TERMS OF x, y, r.

E IS SIN = SIN ? IS COS = COS ? IS TAN = TAN ?

F WHAT CAN YOU CONCLUDE ABOUT THE TRIGONOMETRIC VALUES OF CO-TERMINAL ANGLES?

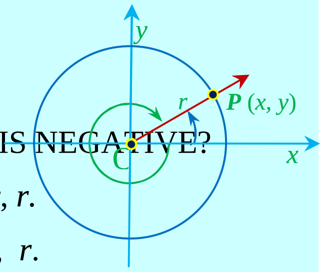


Figure 5.42

CO-TERMINAL ANGLES HAVE THE SAME TRIGONOMETRIC VALUES.

EXAMPLE 11 FIND THE TRIGONOMETRIC VALUES OF

A -330^{O} AND 30^{O} **B** 120^{O} AND -240^{O}

SOLUTION:

A OBSERVE THAT BOTH ANGLES ARE CO-TERMINAL WITH THEIR TERMINAL SIDE IN THE FIRST QUADRANT (SEE FIGURE 5.43).

$-330^{O} = 30^{O} - 1(360^{O})$. THIS GIVES US:

$\text{SIN } 30^{O} = \text{SIN } (-330^{O}) = \frac{1}{2}$

$\text{COS } 30^{O} = \text{COS } (-330^{O}) = \frac{\sqrt{3}}{2}$

$\text{TAN } 30^{O} = \text{TAN } (-330^{O}) = \frac{\sqrt{3}}{3}$

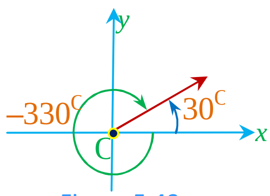


Figure 5.43

B BOTH 120° AND – 240° ANGLES ARE CO-TERMINAL.
THEIR TERMINAL SIDE LIES IN THE SECOND QUADRANT (*See* FIGURE 5.44)

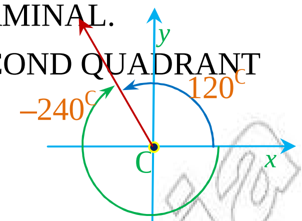


Figure 5.44

$-240^o = 120^o - 360^o$. THUS,

SIN 120° = SIN (– 240°) = SIN 60° = $\frac{\sqrt{3}}{2}$

… *a 60° angle is the reference angle for a 120° angle*

COS 120° = COS (– 240°) = – COS 60° = $-\frac{\sqrt{3}}{2}$

… *cosine is negative in quadrant II*

TAN 120° = TAN (– 240°) = – TAN 60° = $-\sqrt{3}$

… *tangent is also negative in quadrant II*

Angles larger than 360°

CONSIDER THE ANGLE 780°

780° = 360° + 360° + 60° = 2(360°) + 60°

… *a 60° angle is the co-terminal acute angle for a 780° angle*

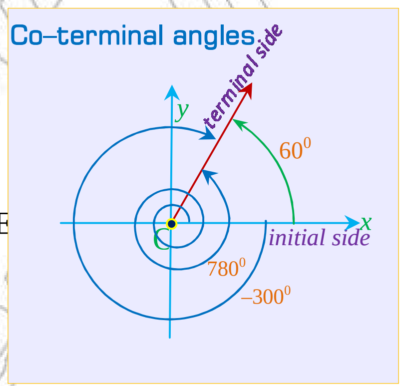


Figure 5.45

SINCE AN ANGLE AND ITS CO–TERMINAL HAVE THE SAME TRIGONOMETRIC VALUE,

SIN 780° = SIN 60° = $\frac{\sqrt{3}}{2}$, COS 780° = COS 60° = $\frac{1}{2}$

AND TAN 780° = TAN 60° = $\sqrt{3}$.

(***Remember that*** *since 780° is the measure of a first quadrant angle, all three of the functions are positive.*)

EXAMPLE 12 FIND THE TRIGONOMETRIC VALUES OF 945°

SOLUTION: 945° = 360° + 360° + 225° = 2(360°) + 225°

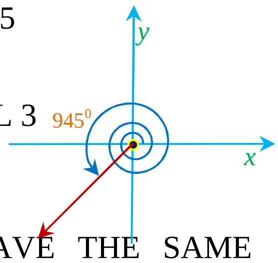


Figure 5.46

THIS MEANS 945° AND 225° ARE MEASURES OF CO-TERMINAL 3RD QUADRANT ANGLES.

THE REFERENCE ANGLE FOR 225° IS θ_R = 225° – 180° = 45°.

SINCE AN ANGLE AND ITS CO-TERMINAL HAVE THE SAME TRIGONOMETRIC VALUE, IT FOLLOWS THAT

SIN 945° = SIN 225° = – SIN 45° = $-\frac{\sqrt{2}}{2}$ … *sine is negative in quadrant III*

COS 945° = COS 225° = – COS 45° = $-\frac{\sqrt{2}}{2}$ … *cosine is negative in quadrant III*

TAN 945° = TAN 225° = TAN 45° = 1 … *tangent is positive in quadrant III*

Exercise 5.9

1 FIND THE VALUE OF EACH OF THE FOLLOWING:

- **A** SIN 390°, COS 390°, TAN 390°
- **B** SIN (-405°), COS (-405°), TAN (-405°)
- **C** SIN (-690°), COS (-690°), TAN (-690°)
- **D** SIN 1395°, COS 1395°, TAN 1395°

2 EXPRESS EACH OF THE FOLLOWING AS A TRIGONOMETRIC FUNCTION OF A POSITIVE ACUTE ANGLE:

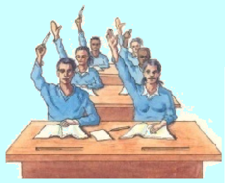
A	SIN 130°	**B**	SIN 200°	**C**	COS 165°	**D**	COS 310°
E	TAN 325°	**F**	SIN (-100°)	**G**	COS (-305°)	**H**	TAN 415°
I	SIN 1340°	**J**	TAN 1125°	**K**	SIN (-330°)	**L**	COS 1400°

5.1.3 Graphs of the Sine, Cosine and Tangent Functions

IN THIS SECTION, YOU WILL DRAW AND DISCUSS SOME OF THE GRAPHS OF THE THREE TRIGONOMETRIC FUNCTIONS: SINE, COSINE AND TANGENT.

Graph of the sine function

ACTIVITY 5.6



1 COMPLETE THE FOLLOWING TABLE OF VALUES FOR $y = \sin\theta$.

θ in deg	–360	–330	–270	–240	–180	–120	–90
$y = \sin\theta$							

θ in deg	0	90	120	180	240	270	330	360
$y = \sin\theta$								

2 MARK THE VALUES OF θ ON THE HORIZONTAL AXIS AND THE VALUES OF y ON THE VERTICAL AXIS AND PLOT THE POINTS YOU FIND.

3 CONNECT THESE POINTS USING A SMOOTH CURVE TO DRAW THE GRAPH OF $y = \sin\theta$.

4 WHAT ARE THE DOMAIN AND THE RANGE OF $y = \sin\theta$?

EXAMPLE 1 DRAW THE GRAPH OF $y = \sin\theta$, WHERE $-360^{\circ} \le \theta \le 360^{\circ}$

SOLUTION: TO DETERMINE THE GRAPH OF $y = \sin\theta$, WE CONSTRUCT A TABLE OF VALUES FOR $y = \sin\theta$, WHERE $-360^{\circ} \le \theta \le 360^{\circ}$ (WHICH IS THE SAME AS $-2\pi \le \theta \le 2\pi$ IN *radians*.)

THE TABLES BELOW SHOW SOME OF THE VALUES OF $y = \sin\theta$ IN THE GIVEN INTERVAL.

in *deg*	−360	−330	−300	−270	−240	−210	−180	−150	−120	− 90	− 60	−30
in *rad*	-2	$\frac{-11}{6}$	$-\frac{5}{3}$	$-\frac{3}{2}$	$-\frac{4}{3}$	$-\frac{7}{6}$	$-\pi$	$-\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{\ }{2}$	$-\frac{\ }{3}$	$-\frac{\ }{6}$
y = sin	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	– 1	– 0.87	– 0.5

in deg	0	30	60	90	120	150	180	210	240	270	300	330	360
in rad	0	$\frac{\ }{6}$	$\frac{\ }{3}$	$\frac{\ }{2}$	$\frac{2}{3}$	$\frac{5}{6}$	π	$\frac{7}{6}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{11}{6}$	2
y = sin	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	–1	– 0.87	– 0.5	0

TO DRAW THE GRAPH WE MARK THE VALUES OF θ ON THE HORIZONTAL AXIS AND THE VALUES OF y ON THE VERTICAL AXIS. THEN WE PLOT THE POINTS AND CONNECT THEM USING A SMOOTH

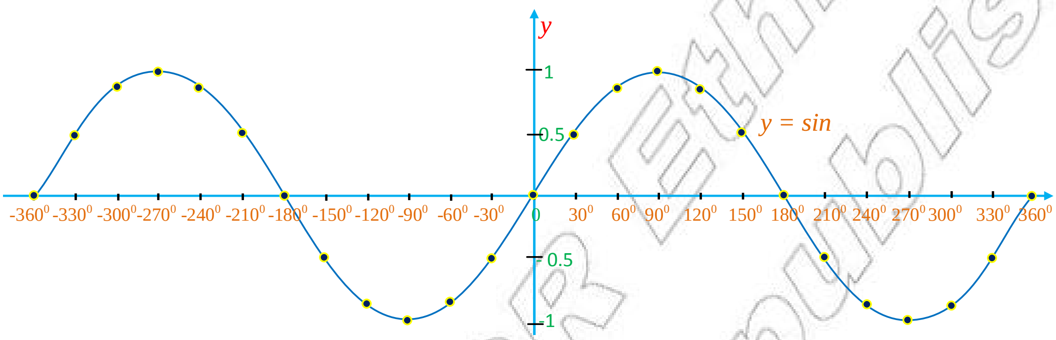


Figure 5.47

AFTER A COMPLETE REVOLUTION (EVERY 360° OR 2π RADIAN) THE VALUES OF THE SINE FUNCTION REPEAT THEMSELVES. THIS MEANS

SIN θ = SIN $\theta \pm 360^{O}$ = SIN $\theta \pm 2(360^{O})$ = SIN $\theta \pm 3(360^{O})$, ETC.

SIN 90^{O} = SIN $90^{O} \pm 360^{O}$ = SIN $90^{O} \pm 2(360^{O})$ = SIN $90^{O} \pm 3(360^{O})$, ETC.

SIN 180^{O} = SIN $180^{O} \pm 360^{O}$ = SIN $180^{O} \pm 2(360^{O})$ = SIN $180^{O} \pm 3(360^{O})$, ETC.

IN GENERAL, SIN θ = SIN ($\theta \pm n(360^{O})$) WHERE n IS AN INTEGER.

A FUNCTION THAT REPEATS ITS VALUES AT REGULAR INTERVALS IS CALLED A periodic function.

THE SINE FUNCTION REPEATS AFTER EVERY 360^{O} OR 2π RADIANS.

THEREFORE, 360^{O} OR 2π IS CALLED THE period OF THE SINE FUNCTION.

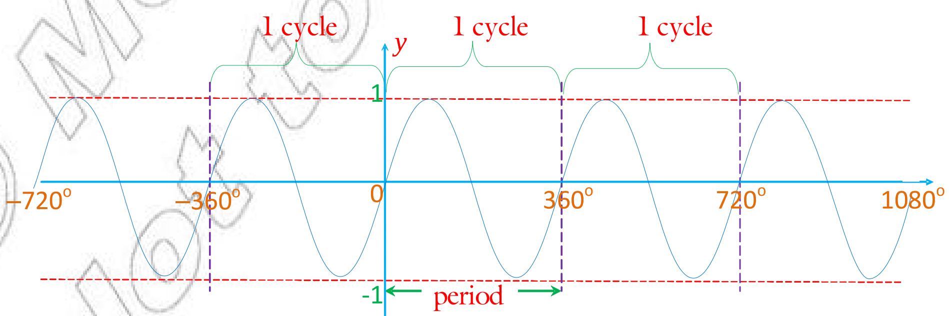


Figure 5.48 Graph of y = SIN θ for $-720^{o} \le \theta \le 1080^{o}$

Domain and range

FOR ANY ANGLE θ TAKEN ON THE UNIT CIRCLE, THERE IS SOME POINT P(x, y) ON ITS TERMINAL SIDE. SINCE SIN $\theta = \frac{y}{1} = y$, THE FUNCTION y = SIN θ IS DEFINED FOR ANY ANGLE θ TAKEN ON THE UNIT CIRCLE.

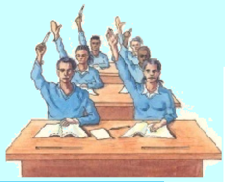
THEREFORE, THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS.

ALSO, NOTE FROM THE GRAPH THAT THE VALUE OF Y IS NEVER LESS THAN –1 OR GREAT

Note: THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS.
THE RANGE OF THE SINE FUNCTION IS $\{y \mid -1 \le y \le 1\}$

Graph of the cosine function

ACTIVITY 5.7



1 COMPLETE THE FOLLOWING TABLES OF VALUES FOR y = COS θ.

θ in deg	– 360	– 300	– 270	– 240	– 180	– 120	– 90	– 60
y = cos θ								

θ in deg	0	60	90	120	180	240	270	300	360
y = cos θ									

2 SKETCH THE GRAPH OF y = COS θ.

3 WHAT ARE THE DOMAIN AND THE RANGE OF y = COS θ?

4 WHAT IS THE PERIOD OF THE COSINE FUNCTION?

FROM ACTIVITY 5.7 YOU CAN SEE THAT y = COS θ IS NEVER LESS THAN –1 OR GREATER THAN +1.

JUST LIKE THE SINE FUNCTION, THE COSINE FUNCTION IS PERIODIC. IT REPEATS EVERY 360° OR 2π RADIANS.

THEREFORE, 360° OR 2π IS CALLED THE period OF THE COSINE FUNCTION.

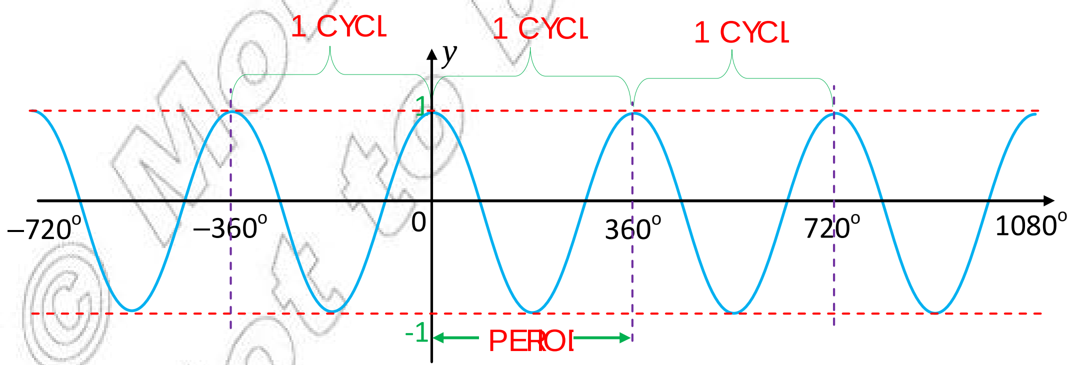


Figure 5.49 Graph of y = COS θ for $-720° \le \theta \le 1080°$

Note: THE DOMAIN OF THE COSINE FUNCTION IS THE SET OF REAL NUMBERS.

THE RANGE OF THE COSINE FUNCTION IS $\{y \mid -1 \le y \le 1\}$.

FIGURE 5.50 REPRESENTS THE SINE AND COSINE FUNCTIONS DRAWN ON THE SAME COORDINATE SYSTEM.

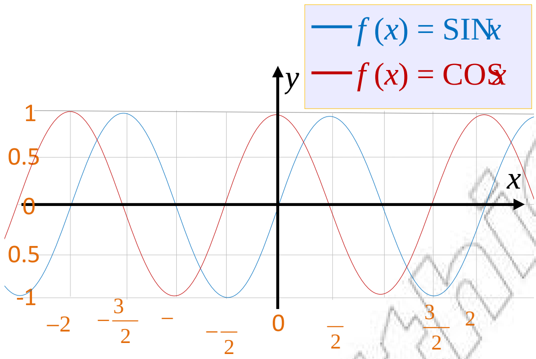


Figure 5.50

FROM THIS DIAGRAM YOU CAN SEE THAT BOTH THE SINE AND COSINE CURVES HAVE THE SAME SHAPE.

THE CURVES "FOLLOW" EACH OTHER, ALWAYS $\frac{\pi}{2}$ RADIANS (90°) APART.

Graph of the tangent function

ACTIVITY 5.8

1 COMPLETE THE FOLLOWING TABLES OF VALUES FOR $y = \tan x$.

x in deg	–360	–315	–270	–225	–180	–135	–90	–45
$y = \tan x$								

x in deg	0	45	90	135	180	225	270	315	360
$y = \tan x$									

2 USE THE TABLE YOU CONSTRUCTED ABOVE TO SKETCH THE GRAPH OF $y = \tan x$.

3 FOR WHICH VALUES OF x IS $y = \tan x$ UNDEFINED?

4 WHAT ARE THE DOMAIN AND THE RANGE OF $y = \tan x$?

5 WHAT IS THE PERIOD OF THE TANGENT FUNCTION?

THE ACTIVITY 5.8 YOU HAVE DONE ABOVE GIVES YOU A HINT ON WHAT THE GRAPH OF $y = \tan x$ LOOKS LIKE. NEXT, YOU WILL SEE THE GRAPH IN DETAIL.

EXAMPLE 2 DRAW THE GRAPH OF $y = \tan$, WHERE $-360° \le \quad \le 360°$.

SOLUTION: THE TABLES BELOW SHOW SOME OF THE VALUES OF $y = \tan$, WHERE $-2 \le \quad \le 2$

θ in deg	–360	–315	–270	–225	–180	–135	–90	–45	0
θ in rad	–2	$-\frac{7}{4}$	$-\frac{3}{2}$	$-\frac{5}{4}$	–	$-\frac{3}{4}$	$-\frac{}{2}$	$-\frac{}{4}$	0
y = tan θ	0	1	–	–1	0	1	–	–1	0

θ in deg	45	90	135	180	225	270	315	360
θ in rad	$\frac{}{4}$	$\frac{}{2}$	$\frac{3}{4}$		$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
y = tan θ	1	–	–1	0	1	–	–1	0

Remember that IF IS IN A STANDARD POSITION AND $P(x, y)$ IS A POINT WHERE THE TERMINAL SIDE OF INTERSECTS THE UNIT CIRCLE, THEN TAN $= \frac{y}{x}$. HOWEVER, $\frac{y}{x}$ IS NOT DEFINED IF $x = 0$.

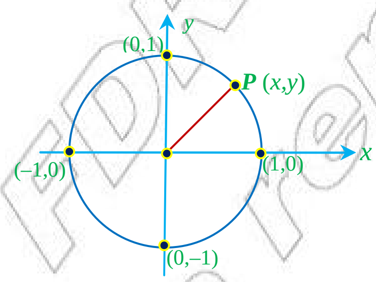


Figure 5.51

SO TAN IS NOT DEFINED IF

$= 90^O$, $= 90^O \pm 180^O$, $= 90^O \pm 2(180^O)$, $= 90^O \pm 3(180^O)$, ETC.

IN GENERAL, TAN IS UNDEFINED IF $= 90^O \pm n\,(180^O)$ OR IF $= \frac{}{2} + n$, WHERE n IS AN INTEGER.

THE GRAPH OF $y = \tan$ DOES NOT CROSS THE VERTICAL LINES AT $= \frac{}{2} + n$, n IS INTEGER.

MOREOVER, IF WE CLOSELY INVESTIGATE THE BEHAVIOUR OF TAN AS INCREASES FROM $-\frac{}{2}$ TO $\frac{}{2}$, WE CAN SEE THAT TAN INCREASES FROM NEGATIVE INFINITY TO POSITIVE INFINITY (FROM $-\infty$ TO ∞). A SKETCH OF THE GRAPH OF $y = \tan$ FOR $-\frac{}{2} < \quad < \frac{}{2}$, IS SHOWN IN FIGURE 5.52.

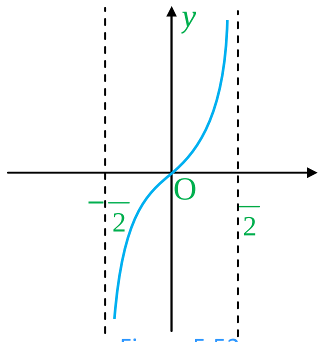


Figure 5.52

FROM THE GRAPH WE SEE THAT THE TANGENT FUNCTION REPEATS ITSELF EVERY 180° OR π RADIANS.

THEREFORE, 180° OR π IS THE PERIOD FOR THE TANGENT FUNCTION.

SINCE TAN IS PERIODIC WITH PERIOD π, WE CAN EXTEND THE ABOVE GRAPH FOR AS MANY REPETITIONS (CYCLES) AS WE WANT.

FOR EXAMPLE, THE GRAPH OF $y = \tan\theta$ FOR $-2\pi \leq \theta \leq 2\pi$ IS SHOWN BELOW.

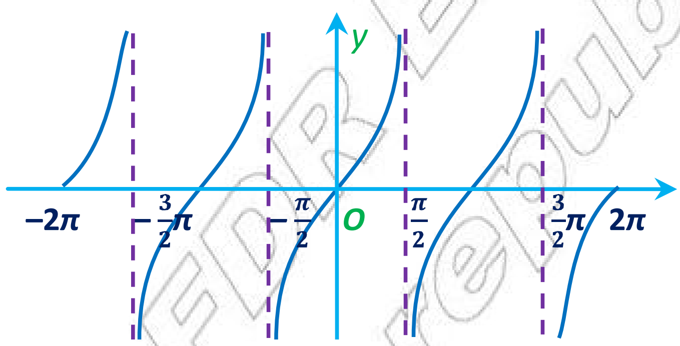


Figure 5.53

WHAT ARE THE DOMAIN AND THE RANGE OF $y = \tan\theta$?

FOR WHICH VALUES OF θ IS $y = \tan\theta$ NOT DEFINED?

USING A UNIT CIRCLE WE CAN SEE THAT $\tan\theta = \frac{y}{x}$ IS UNDEFINED WHENEVER THE x-COORDINATE ON THE UNIT CIRCLE IS 0.

THIS HAPPENS WHEN $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}$, ETC. THEREFORE THE DOMAIN OF THE TANGENT FUNCTION MUST EXCLUDE THESE ODD MULTIPLES OF $\frac{\pi}{2}$.

HENCE, THE DOMAIN OF THE TANGENT FUNCTION IS $\{\theta \mid \theta \neq \frac{n\pi}{2}$, WHERE n IS AN ODD INTEGER$\}$.

THE RANGE OF $y = \tan\theta$ IS THE SET OF REAL NUMBERS.

Group Work 5.4

1 USE THE GRAPH OF THE COSINE FUNCTION TO FIND VALUES OF θ FOR WHICH COS θ = 0.

2 FROM THE GRAPH OF Y = SIN θ, FIND THE VALUES OF θ FOR WHICH SIN θ = –1.

3 GRAPH THE SINE CURVE FOR THE INTERVAL $-540^\circ \le \theta \le 0^\circ$

Exercise 5.10

1 REFER TO THE GRAPH OF $y = \sin\theta$ OR THE TABLE OF VALUES FOR $y = \sin\theta$ TO DETERMINE HOW THE SINE FUNCTION BEHAVES AS θ INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:

A AS θ INCREASES FROM 0° TO 90°, SIN θ INCREASES FROM __0__ TO __1__.

B AS θ INCREASES FROM 90° TO 180°, SIN θ DECREASES FROM ______ TO _____.

C AS θ INCREASES FROM 180° TO 270°, SIN θ DECREASES FROM _____ TO _____.

D AS θ INCREASES FROM 270° TO 360°, SIN θ INCREASES FROM ______ TO ____.

2 REFER TO THE GRAPH OF $y = \cos\theta$ OR THE TABLE OF VALUES FOR $y = \cos\theta$ TO DETERMINE HOW THE COSINE FUNCTION BEHAVES AS θ INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:

A AS θ INCREASES FROM 0° TO 90°, COS θ DECREASES FROM __1__ TO __0__.

B AS θ INCREASES FROM 90° TO 180°, COS θ DECREASES FROM ______ TO _____.

C AS θ INCREASES FROM 180° TO 270°, COS θ INCREASES FROM _____ TO _____.

D AS θ INCREASES FROM 270° TO 360°, COS θ INCREASES FROM ______ TO ____.

3 DETERMINE HOW THE TANGENT FUNCTION BEHAVES AS θ INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:

A AS θ INCREASES FROM 0° TO 90°, TAN θ INCREASES FROM __0__ TO __POSITIVE INFINITY (+∞)__.

B AS θ INCREASES FROM 90° TO 180°, TAN θ INCREASES FROM ___ TO ____.

C AS θ INCREASES FROM 180° TO 270°, TAN θ INCREASES FROM __ TO _____.

D AS θ INCREASES FROM 270° TO 360°, TAN θ ______ FROM –∞ TO 0.

5.2 THE RECIPROCAL FUNCTIONS OF THE BASIC TRIGONOMETRIC FUNCTIONS

IN THIS SECTION, YOU WILL LEARN ABOUT THREE MORE TRIGONOMETRIC FUNCTIONS, WHICH ARE CALLED THE RECIPROCALS OF THE SINE, COSINE AND TANGENT FUNCTIONS, NAMED RES cosecant, secant AND cotangent FUNCTIONS.

5.2.1 The Cosecant, Secant and Cotangent Functions

Definition 5.2

If is an angle in standard position and **P**(*x, y*) is a point on the terminal side of , different from the origin **O**(0, 0), and ***r*** is the distance of point **P** from the origin O, then

$$\csc = \frac{HYP}{OPP} = \frac{r}{y}$$

$$\sec = \frac{HYP}{ADJ} = \frac{r}{x}$$

$$\cot = \frac{ADJ}{OPP} = \frac{x}{y}$$

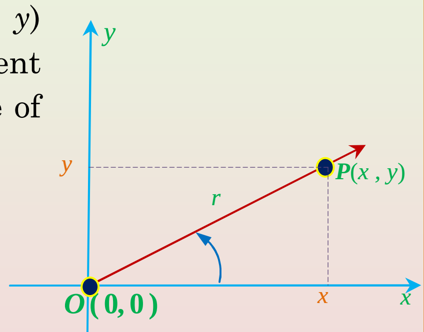


Figure 5.54

CSC, SEC AND COT ARE ABBREVIATIONS FOR COSECANT, SECANT AND COTANGENT RESPECTIVELY.

EXAMPLE 1 IF IS AN ANGLE IN STANDARD POSITION AND P(3, 4) IS A POINT ON THE TERMINAL SIDE OF , THEN EVALUATE THE COSECANT, SECANT AND COTANGENT FUNCTIONS.

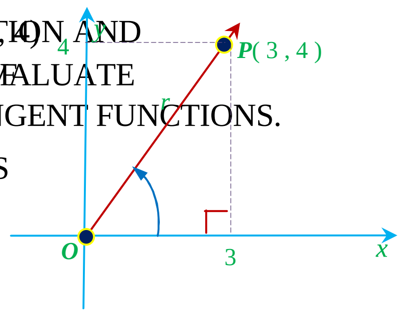


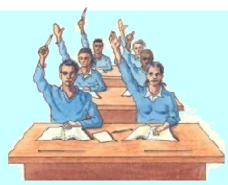
Figure 5.55

SOLUTION: THE DISTANCE $r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ UNITS

SO, $CSC = \frac{HYP}{OPP} = \frac{5}{4}$,

$SEC = \frac{HYP}{ADJ} = \frac{5}{3}$ AND $COT = \frac{ADJ}{OPP} = \frac{3}{4}$

ACTIVITY 5.9



REFERRING TO FIGURE 5.55 FIND:

1 SIN, COS AND TAN

2 COMPARE SIN WITH CSC, COS WITH SEC, TAN WITH COT

3 HOW DO THEY RELATE? ARE THEY EQUAL? ARE THEY OPPOSITE? ARE THEY RECIPROCALS?

FROM THE RESULTS OF ACTIVITY 5.9, YOU CAN CONCLUDE THE FOLLOWING:

$\text{CSC} = \frac{r}{y}$ WHEREAS $\text{SIN} = \frac{y}{r}$

$\text{SEC} = \frac{r}{x}$ WHEREAS $\text{COS} = \frac{x}{r}$

$\text{COT} = \frac{x}{y}$ WHEREAS $\text{TAN} = \frac{y}{x}$

Have you noticed that one is the reciprocal of the other?

THAT IS,

$$\text{CSC} = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\text{SIN}}, \quad \text{SEC} = \frac{r}{x} = \frac{1}{\left(\frac{x}{r}\right)} = \frac{1}{\text{COS}} \quad \text{AND}$$

$$\text{COT} = \frac{x}{y} = \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{\text{TAN}}$$

THEREFORE,

$$\text{CSC}\,\theta = \frac{1}{\text{SIN}}, \quad \text{SEC}\,\theta = \frac{1}{\text{COS}} \quad \text{AND COT}\,\theta = \frac{1}{\text{TAN}}.$$

HENCE, CSC AND SIN ARE RECIPROCALS

SEC AND COS ARE RECIPROCALS

TAN AND COT ARE RECIPROCALS

EXAMPLE 2 IF $= 30^{O}$, THEN FIND CSC, SEC, COT.

SOLUTION:

$$\text{CSC} = \frac{1}{\text{SIN}} = \frac{1}{\left(\frac{1}{2}\right)} = 2 \quad \ldots \textbf{\textit{remember that}} \sin 30^{0} = \frac{1}{2} = 0.5$$

$$\text{SEC} = \frac{1}{\text{COS}} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad \ldots \textbf{\textit{remember that}} \cos 30^{0} = \frac{\sqrt{3}}{2}$$

$$\text{COT} = \frac{1}{\text{TAN}} = \frac{1}{\left(\frac{\sqrt{3}}{3}\right)} = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \ldots \textbf{\textit{remember that}} \tan 30^{0} = \frac{\sqrt{3}}{3}$$

EXAMPLE 3 IF SIN IS 0.5, THEN CSC IS $\frac{1}{0.5} = 2$

IF COS IS – 0.1035, THEN SEC IS $\frac{1}{-0.1035} = -9.6618$

IF TAN IS $-\frac{1}{4}$, THEN COT IS $\frac{1}{\left(-\frac{1}{4}\right)} = -4$

EXAMPLE 4 USING A UNIT CIRCLE, FIND THE VALUES OF THE COSECANT, SECANT AND COTANGENT FUNCTIONS IF = 90^{O}, 180^{O}, 270^{O}.

SOLUTION: AS YOU CAN SEE IN THE ADJACENT FIGURE, THE TERMINAL SIDE OF THE ANGLE 90^{O} INTERSECTS THE UNIT CIRCLE AT (0 ,1)

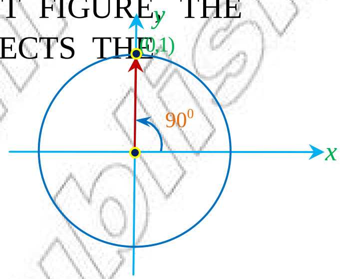


Figure 5.56

HENCE, $\csc 90^{O} = \frac{r}{y} = \frac{1}{1} = 1$

$\sec 90^{O} = \frac{r}{x} = \frac{1}{0}$ *is undefined*

$\cot 90^{O} = \frac{x}{y} = \frac{0}{1} = 0$

THE TERMINAL SIDE OF THE ANGLE 180^{O} INTERSECTS THE *unit circle* AT (–1, 0).

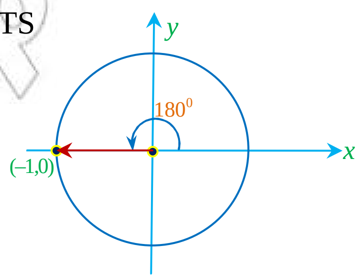


Figure 5.57

HENCE, $\csc 180^{O} = \frac{r}{y} = \frac{1}{0}$ *is undefined*

$\sec 180^{O} = \frac{r}{x} = \frac{1}{-1} = -1$

$\cot 180^{O} = \frac{x}{y} = \frac{-1}{0}$ *is undefined*

SIMILARLY THE TERMINAL SIDE OF THE ANGLE 270^{O} INTERSECTS THE *unit circle* AT (0 ,–1).

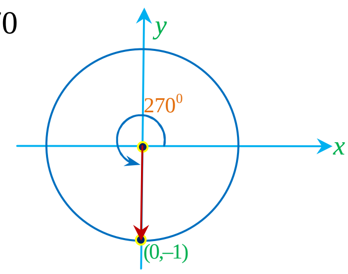


Figure 5.58

HENCE, $\csc 270^{O} = \frac{r}{y} = \frac{1}{-1} = -1$

$\sec 270^{O} = \frac{r}{x} = \frac{1}{0}$ *is undefined*

$\cot 270^{O} = \frac{x}{y} = \frac{0}{-1} = 0$

EXAMPLE 5 USING A UNIT CIRCLE, FIND THE VALUES OF THE COSECANT, SECANT AND COTANGENT FUNCTIONS IF = 360^{O}.

SOLUTION: THE TERMINAL SIDE OF ANGLE 360° INTERSECTS THE UNIT CIRCLE AT (1, 0).

HENCE, $\csc 360^{\circ} = \frac{r}{y} = \frac{1}{0}$ *is undefined*

$\sec 360^{\circ} = \frac{r}{x} = \frac{1}{1} = 1$

$\cot 360^{\circ} = \frac{x}{y} = \frac{1}{0}$ *is undefined*

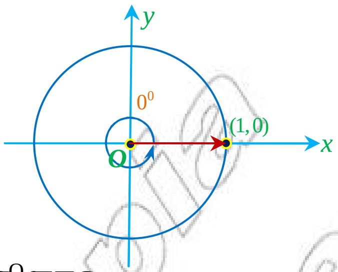


Figure 5.59

Remember that THESE RESULTS ARE ALSO TRUE FOR 0°, 720°, 1080°, ETC.

WHEN DO YOU THINK THE FUNCTIONS CSC, SEC AND COT ARE UNDEFINED?

FOR EXAMPLE, $\csc\theta = \frac{r}{y}$ IS UNDEFINED WHEN $y = 0$. THE VALUE OF y ON THE UNIT CIRCLE WILL BE 0 WHEN $\theta = 0^{\circ}, \pm 180^{\circ}, \pm 2(180^{\circ}), \pm 3(180^{\circ}), \pm 4(180^{\circ})$, ETC.

IN GENERAL, $\csc\theta$ IS UNDEFINED FOR $\theta = \pm n\,(180^{\circ})$, WHERE n IS AN INTEGER.

Group Work 5.5

1 DECIDE IF THE FOLLOWING TRIGONOMETRIC FUNC
POSITIVE OR NEGATIVE AND COMPLETE THE FOLLOW

	has terminal side in quadrant			
	I	II	III	IV
csc	+			
sec			–	
cot				–

2 COMPLETE THE FOLLOWING TABLE OF VALUES:

in deg	–360	–300	–270	–240	–180	–120	–90	–60	0
y = csc									
y = sec									

in deg	60	90	120	180	240	270	300	360
y = csc								
y = sec								

3 SKETCH THE GRAPHS OF $y = \csc\theta$ AND $y = \sec\theta$ ON A SEPARATE COORDINATE SYSTEM.

4 CONSTRUCT A TABLE OF VALUES FOR $y = \cot\theta$ AND SKETCH THE GRAPH.

Hint: USE THE TABLE OF VALUES FOR $y = \tan\theta$

5 DISCUSS AND IDENTIFY THE VALUES OF θ WHERE CSC, SEC AND COT WILL BE UNDEFINED.

Exercise 5.11

1 SUPPOSE THE FOLLOWING POINTS LIE ON THE TERMINAL SIDE OF AN ANGLE θ. DETERMINE THE COSECANT, SECANT AND COTANGENT FUNCTIONS OF θ:

A P (12, 5) **B** P (– 8, 15) **C** P (– 6, 8) **D** P (5, 3)

E P (2, 0) **F** P $(\frac{4}{5}, \frac{-3}{5})$ **G** P $(\sqrt{2}, \sqrt{5})$ **H** P $(\sqrt{6}, \sqrt{3})$

2 COMPLETE EACH OF THE FOLLOWING:

A IF SIN θ IS – 0.35, THEN CSC θ IS ____. **B** IF SEC θ IS 2.6, THEN COS θ IS ____.

C IF CSC θ IS 30.5, THEN SIN θ IS ____. **D** IF TAN θ IS 1, THEN COT θ IS ____.

E IF TAN θ IS $\frac{\sqrt{3}}{3}$, THEN COT θ IS ____. **F** IF TAN θ IS 0, THEN COT θ IS ____.

3 FIND THE VALUES OF CSC θ, SEC θ AND COT θ, IF θ IN DEGREES IS:

A 30 **B** 45 **C** 60 **D** 120

E 150 **F** 210 **G** 240 **H** 300

I – 390 **J** – 405 **K** – 420 **L** 780.

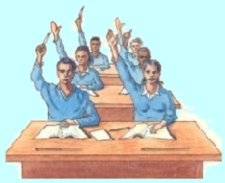
4 IF COT θ = $\frac{3}{8}$ AND θ IS IN THE FIRST QUADRANT, FIND THE OTHER FIVE TRIGONOMETRIC FUNCTIONS OF θ.

Co–functions

WHAT KINDS OF FUNCTIONS ARE CALLED CO-FUNCTIONS?

IN ORDER TO UNDERSTAND THE CONCEPT OF A CO-FUNCTION, TRY THE FOLLOWING ACTIVITY.

ACTIVITY 5.10



ABC IS A RIGHT ANGLE TRIANGLE. α AND β ARE ACUTE ANGLES. SINCE THEIR SUM IS 90°, THEY ARE complementary angles. FIND THE VALUES OF THE SIX TRIGONOMETRIC FUNCTIONS FOR BOTH α AND β, AND COMPARE THE RESULTS.

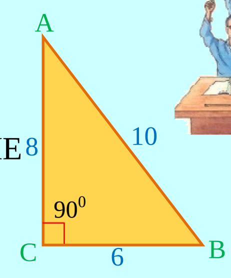


Figure 5.60

IDENTIFY THE FUNCTIONS THAT HAVE THE SAME VALUE.

FROM ACTIVITY 5.10, YOU MAY CONCLUDE THE FOLLOWING:

OBSERVE THAT ABC IS A RIGHT ANGLE TRIANGLE WITH m(∠C) = 90°, α + β = 90°. THIS MEANS THE ACUTE ANGLES α AND β ARE complementary.

HENCE WE HAVE THE FOLLOWING RELATIONSHIP:

SIN $= \frac{a}{c} =$ COS   CSC $= \frac{c}{a} =$ SEC

COS $= \frac{b}{c} =$ SIN   SEC $= \frac{c}{b} =$ CSC

TAN $= \frac{a}{b} =$ COT   COT $= \frac{b}{a} =$ TAN

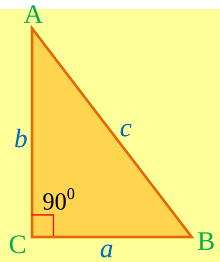


Figure 5.61

NOTE THAT, FOR THE TWO COMPLEMENTARY ANGLES α AND β

- ✓ THE SINE OF ANY ANGLE IS EQUAL TO THE COSINE OF ITS COMPLEMENTARY ANGLE.
- ✓ THE TANGENT OF ANY ANGLE IS EQUAL TO THE COTANGENT OF ITS COMPLEMENTARY ANGLE.
- ✓ THE SECANT OF ANY ANGLE IS EQUAL TO THE COSECANT OF ITS COMPLEMENTARY ANGLE.

THUS, THE PAIR OF FUNCTIONS **sine** AND **cosine** ARE CALLED **co-functions**.

SIMILARLY, **tangent** AND **cotangent**, **secant** AND **cosecant** ARE ALSO CO-FUNCTIONS.

ANY TRIGONOMETRIC FUNCTION OF AN ACUTE ANGLE IS EQUAL TO THE CO-FUNCTION OF ITS COMPLEMENTARY ANGLE. THAT IS, IF $0^\circ \le \theta \le 90^\circ$, THEN

SIN θ = COS $(90^0 - \theta)$ CSC θ = SEC $(90^0 - \theta)$

COS θ = SIN $(90^0 - \theta)$ SEC θ = CSC $(90^0 - \theta)$

TAN θ = COT $(90^0 - \theta)$ COT θ = TAN $(90^0 - \theta)$

EXAMPLE 6

A SIN 30^0 = COS 60^0 **B** SEC 40^0 = CSC 50^0 **C** TAN $\frac{\pi}{3}$ = COT $\frac{\pi}{6}$

Exercise 5.12

1 FIND THE SIZE OF ACUTE ANGLE θ IN DEGREES IF:

A SIN 20^0 = COS θ **B** SEC θ = CSC 80^0 **C** TAN 55^0 = COT θ

D COS $\frac{\pi}{9}$ = SIN θ **E** SEC θ = CSC $\frac{5\pi}{12}$ **F** COT 1^0 = TAN θ

2 ANSWER EACH OF THE FOLLOWING:

A IF COS 35^0 = 0.8387, THEN SIN 55^0 = ______

B IF SIN 77^0 = 0.9744, THEN COS 13^0 = ______

C IF TAN 45^0 = 1, THEN COT 45^0 = ______

D IF SEC $15^0 = x$, THEN CSC 75^0 = ______

E IF CSC $\alpha = \frac{a}{b}$ AND SEC $\beta = \frac{a}{b}$, THEN $\alpha + \beta$ = ________

F IF COT $55^0 = y$ AND TAN $\theta = y$, THEN θ = __________

5.3 SIMPLE TRIGONOMETRIC IDENTITIES

Pythagorean identities

USING THE DEFINITIONS OF THE SIX TRIGONOMETRIC FUNCTIONS DISCUSSED SO FAR, IT IS POSSIBLE TO FIND SPECIAL RELATIONSHIPS THAT EXIST BETWEEN THEM.

LET θ BE AN ANGLE IN STANDARD POSITION AND P(x, y) BE A POINT ON THE TERMINAL SIDE OF θ (See FIGURE 5.62)

FROM PYTHAGORAS' THEOREM, WE KNOW THAT

$$x^2 + y^2 = r^2$$

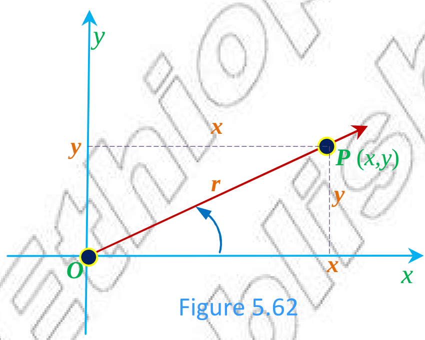


Figure 5.62

IF WE DIVIDE BOTH SIDES BY r^2, WE HAVE

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\therefore (\cos\theta)^2 + (\sin\theta)^2 = 1$$

IF WE DIVIDE BOTH SIDES OF $x^2 + y^2 = r^2$ BY x^2, THEN WE HAVE

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$$1 + (\tan\theta)^2 = (\sec\theta)^2$$

IF WE DIVIDE BOTH SIDES OF $x^2 + y^2 = r^2$ BY y^2, THEN WE HAVE

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

$$(\cot\theta)^2 + 1 = (\csc\theta)^2$$

HENCE WE HAVE THE FOLLOWING RELATIONS:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

THE ABOVE RELATIONS ARE KNOWN AS PYTHAGOREAN IDENTITIES.

Note:

$(\sin\theta)^2 = \sin^2\theta$ AND $(\cos\theta)^2 = \cos^2\theta$, ETC.

EXAMPLE 1 IF SIN $= \frac{1}{2}$ AND IS IN THE FIRST QUADRANT, FIND THE VALUES OF THE OTHER FIVE TRIGONOMETRIC FUNCTIONS OF .

SOLUTION: FROM $SIN^2 + COS^2 = 1$, WE HAVE

$COS^2 = 1 - SIN^2$

$$SO,\ COS = \sqrt{1 - SIN^2} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$SEC = \frac{1}{CO} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}};\ CSC = \frac{1}{SIN} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

FROM $1 + TAN^2 = SEC^2$, WE HAVE, $TAN^2 = SEC^2 - 1$

$$SO\ TAN = \sqrt{SEC^2 - 1} = \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 - 1} = \sqrt{\frac{4}{3} - 1} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

FROM $COT^2 + 1 = CSC^2$, WE HAVE $COT^2 = CSC^2 - 1$, THIS IMPLIES THAT

$$COT = \sqrt{CSC^2 - 1} = \sqrt{2^2 - 1} = \sqrt{4 - 1} = \sqrt{3}.$$

Exercise 5.13

1 USING THE PYTHAGOREAN IDENTITIES FIND THE VALUES OF THE OTHER FIVE TRIGONOMETRIC FUNCTIONS IF:

A SIN $= \frac{15}{17}$ AND IS IN QUADRANT I.

B COS $= \frac{-4}{5}$ AND IS IN QUADRANT II

C COT $= \frac{7}{24}$ AND IS IN QUADRANT III.

D COS $= \frac{24}{25}$ AND IS IN QUADRANT IV.

2 REFERRING TO THE RIGHT ANGLE TRIANGLE *ABC* (*See* FIGURE 5.63), FIND:

A SIN **B** COS **C** SIN (90^0 –)

D COS (90^0 –) **E** CSC (90^0 –) **F** COT (90^0 –)

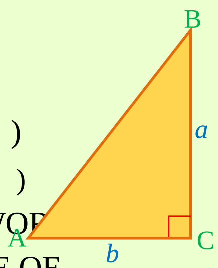


Figure 5.63

3 FILL IN THE BLANK SPACE WITH THE APPROPRIATE WORD.

A THE SINE OF AN ANGLE IS EQUAL TO THE COSINE OF ______

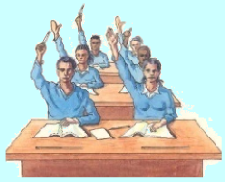
B THE COSECANT OF AN ANGLE IS EQUAL TO THE SECANT OF ______

C THE TANGENT OF AN ANGLE IS EQUAL TO THE ______ OF THE COMPLEMENTARY ANGLE.

Quotient identities

THE FOLLOWING ARE ADDITIONAL RELATIONSHIPS DERIVED FROM THE SIX TRIGONOMETRIC FUNCTIONS:

ACTIVITY 5.11



LET θ BE AN ANGLE IN STANDARD POSITION AND P(*x*, *y*) BE A POINT ON THE TERMINAL SIDE OF θ (*See* FIGURE 5.64).

THEN ANSWER THE FOLLOWING:

A WHAT ARE THE VALUES OF SIN θ, COS θ, TAN θ AND COT θ?

B HOW DO THE VALUES $\frac{\sin\theta}{\cos\theta}$ AND TAN θ COMPARE?

C HOW DO THE VALUES $\frac{\cos\theta}{\sin\theta}$ AND COT θ COMPARE?

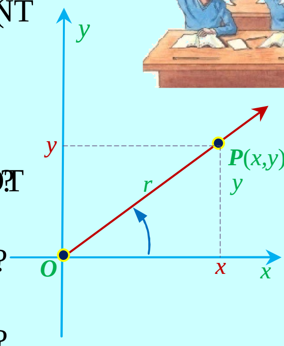


Figure 5.64

REFERRING TO FIGURE 5.64, WE CAN DERIVE THE FOLLOWING RELATIONSHIPS BETWEEN THE SIX TRIGONOMETRIC FUNCTIONS:

SIN θ $= \frac{y}{r}$ AND COS θ $= \frac{x}{r}$. FROM THIS WE HAVE, $\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x}$ = TAN θ

SIMILARLY, $\frac{\cos\theta}{\sin\theta} = \frac{\left(\frac{x}{r}\right)}{\left(\frac{y}{r}\right)} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y}$ = COT θ

HENCE THE RELATIONS:

TAN θ $= \frac{\sin\theta}{\cos\theta}$ AND COT θ $= \frac{\cos\theta}{\sin\theta}$ WHICH ARE KNOWN AS **quotient** IDENTITIES.

EXAMPLE 2 IF SIN θ $= \frac{4}{5}$ AND COS θ $= \frac{3}{5}$, THEN FIND TAN θ AND COT θ.

SOLUTION: FROM QUOTIENT IDENTITY, TAN θ $= \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = \frac{4}{3}$

COT θ $= \frac{\cos\theta}{\sin\theta} = \frac{\left(\frac{3}{5}\right)}{\left(\frac{4}{5}\right)} = \frac{3}{4}$

Note: AN IDENTITY IS AN EQUATION THAT IS TRUE FOR ALL VALUES OF THE VARIABLE FOR WHICH BOTH SIDES OF THE EQUATION ARE DEFINED.

ALL IDENTITIES ARE EQUATIONS BUT ALL EQUATIONS ARE NOT NECESSARILY IDENTITIES. THIS IS BECAUSE, UNLIKE IDENTITIES , EQUATIONS MAY NOT BE TRUE FOR SOME VALUES IN TH

FOR EXAMPLE CONSIDER THE EQUATION SIN = COS

FOR MOST VALUES OF, THIS EQUATION IS NOT TRUE (*for instance*, SIN 30° ≠ COS 30°)

HENCE THE EXPRESSION SIN = COS REPRESENTS A SIMPLE TRIGONOMETRIC EQUATION, BUT NOT AN IDENTITY.

Group Work 5.6

USE THE PYTHAGOREAN AND QUOTIENT IDENTITIES TO SOLVE EACH OF THE FOLLOWING:

1. COS $\alpha = \frac{-4}{5}$ AND α IS IN QUADRANT II. FIND TAN α AND COT α.
2. SIN $\alpha = \frac{8}{17}$ AND α IS IN QUADRANT I. FIND TAN α AND COT α.
3. SIN 330° $= -\frac{1}{2}$. FIND TAN 330° AND COT 330°.
4. COS 150° $= -\frac{\sqrt{3}}{2}$. FIND TAN 150° AND COT 150°.
5. SEC 60° = 2 . FIND TAN 60° AND COT 60°.
6. SUPPOSE α IS AN ACUTE ANGLE SUCH THAT SIN $\alpha = x$ AND SIN (90° − α) = y; FIND TAN (90° − α) AND COT (90° − α).

Using tables of the trigonometric functions

SO FAR YOU HAVE SEEN HOW TO DETERMINE THE VALUES OF TRIGONOMETRIC FUNCTIONS OF SOME SPECIAL ANGLES. THE SAME METHODS CAN IN THEORY BE APPLIED TO ANY ANGLE RESULTS FOUND IN THIS WAY ARE APPROXIMATIONS. THEREFORE WE USE PUBLISH VALUES, WHERE VALUES ARE GIVEN TO FOUR DECIMAL PLACES OF ACCURACY.

SINCE THE TRIGONOMETRIC FUNCTIONS OF A POSITIVE ACUTE ANGLE AND THE CORRESPONDING CO-FUNCTIONS OF THE COMPLEMENTARY ANGLE (90° −) ARE EQUAL, TRIGONOMETRIC TABLES ARE OFT CONSTRUCTED ONLY FOR VALUES BETWEEN 0° AND 45°.

TO FIND THE TRIGONOMETRIC FUNCTIONS OF ANGLES BETWEEN 45° AND 90°, A TABLE CONSTRUCTED FOR VALUES OF BETWEEN 0° AND 45° IS USED BY READING FROM BOTTOM UP. CORRESPONDING TO EACH ANGLE BETWEEN 0° AND 45° LISTED IN THE LEFT HAND COLUMN, THE COMPLEMENTARY ANGLE (90° −) IS LISTED IN THE RIGHT HAND COLUMN. CORRESPONDING TO EACH TRIGONOMETRIC FUNCTION LISTED AT THE TOP, THE CO-FUNCTION IS LISTED AT THE BOTTOM. THUS, FOR ANGLES F THE TRIGONOMETRIC FUNCTIONS ARE READ USING THE BOTTOM ROW AND THE RIGHT HA

(*A part of the trigonometric table is given below for your reference*).

	sin	cos	tan	cot	
0°	0.0000	1.0000	0.0000	–	**90°**
1°	0.0175	0.9998	0.0175	57.29	**89°**
2°	--------	--------	--------	--------	**88°**
. . .					**. . .**
5°	0.0872	--------	0.0875	--------	**85°**
. . .					**. . .**
45°	--------	--------	--------	--------	**45°**
	cos	**sin**	**cot**	**tan**	

FOR INSTANCE, SIN 5° AND COS 85° ARE BOTH FOUND AT THE SAME PLACE IN THE TABLE AND EACH IS APPROXIMATELY EQUAL TO 0.0872. SIMILARLY, TAN 5° = COT 85° ≈ 0.0875, ETC.

EXAMPLE 3 USE THE TABLE GIVEN AT THE END OF THE BOOK TO FIND THE APPROXIMATE VALUES OF:

A COS 20° **B** COT 50°

SOLUTION:

A SINCE 20° < 45°, WE BEGIN BY LOCATING 20° IN THE VERTICAL COLUMN ON THE LEFT SIDE OF THE DEGREE TABLE. THEN WE READ THE ENTRY 0.9397 UNDER THE COLUMN LABELLED COS AT THE TOP.

∴ COS 20° = 0.9397 .

B SINCE 50° > 45°, WE USE THE VERTICAL COLUMN ON THE RIGHT SIDE (READING UPWARD) TO LOCATE 50° AND READ ABOVE THE BOTTOM CAPTION "COT" TO GET 0.8391;

∴ COT 50° = 0.8391.

EXAMPLE 4 FIND θ SO THAT:

A SEC θ = 1.624 **B** SIN θ = 0.5831

SOLUTION: FINDING AN ANGLE WHEN THE VALUE OF ONE OF ITS FUNCTIONS IS GIVEN IS THE REVERSE PROCESS OF THAT ILLUSTRATED IN THE ABOVE EXAMPLE.

A GIVEN SEC θ = 1.624, LOOKING UNDER THE SECANT COLUMN OR ABOVE THE SECANT COLUMN, WE FIND THE ENTRY 1.624 ABOVE THE SECANT COLUMN AND THE CORRESPONDING VALUE OF θ IS 52°. THEREFORE, θ = 52°.

B REFERRING TO THE "SINE" COLUMNS OF THE TABLE, 0.5831 DOES NOT APPEAR THERE. THE TWO VALUES IN THE TABLE CLOSEST TO 0.5831 (ONE SMA ONE LARGER) ARE 0.5736 AND 0.5878. THESE VALUES CORRESPOND TO 35° AND 36°, RESPECTIVELY. AS SHOWN BELOW, THE DIFFERENCE BETWEEN AND THE VALUE OF SIN 36° IS SMALLER THAN THE DIFFERENCE BETWEEN SIN 35° AND SIN. THEREFORE USE THE VALUE 36° FOR BECAUSE SIN IS CLOSER TO SIN 36° THAN IT IS TO SIN 35°.

SIN = 0.5831 SIN 36° = 0.5878

SIN 35° = 0.5736 SIN = 0.5831

DIFFERENCE = 0.0095 DIFFERENCE = 0.0047

$\therefore$ = 36° (NEAREST DEGREE).

THE FOLLOWING EXAMPLES ILLUSTRATE HOW TO DETERMINE THE VALUES OF T FUNCTIONS FOR ANGLES MEASURED IN DEGREES (OR RADIANS) WHOSE MEASURES AR

0° AND 90° (OR 0 AND $\frac{}{2}$).

EXAMPLE 5 USE THE NUMERICAL TABLE, REFERENCE ANGLES, TRIGONOMETRIC FUNCTIONS OF NEGATIVE ANGLES AND PERIODICITY OF THE FUNCTIONS TO DETERMINE T EACH OF THE FOLLOWING:

A SIN 236° **B** COS 693°

SOLUTION:

A TO FIND SIN 236°, FIRST WE CONSIDER THE QUADRANT THAT THE ANGLE 236° BELONGS TO. THIS IS DONE BY PLACING THE ANGLE IN STANDARD POSITION AS SH FIGURE 5.65. WE SEE THAT THE ANGLE 236° LIES IN QUADRANT III SO THAT THE SINE VALUE IS NEGATIVE. THE REFERENCE ANGLE CORRESPONDING TO 236° IS

$_R$ = 236° – 180° = 56°. THUS, SIN 236° = – SIN 56°.

SINCE 56° > 45°, WE LOCATE 56° IN THE VERTICAL COLUMN ON THE RIGHT SIDE OF THE TRIGONOMETRIC TABLE. LOOKING IN THE VERTICAL COLUMN ABOVE TH BOTTOM CAPTION "SIN", WE SEE THAT SIN 56° = 0.8290.

SO SIN 236° = – SIN 56° = – 0.8290.

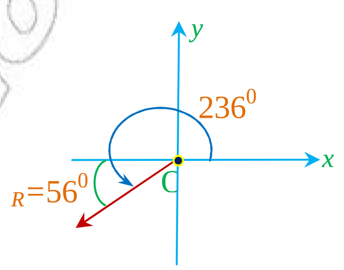


Figure 5.65

b To find the value of cos 693°, first observe that 693° is greater than 360°. The period of cosine function is 360°. Dividing 693° by 360° we obtain

$$693° = 1 \times 360° + 333°$$

This means that the angle 693° is co terminal with the 333° angle. i.e., cos 693° = cos 333°.

Since the terminal side of 333° is in quadrant IV, the reference angle is θ_R = 360° – 333° = 27° (*See* Figure 5.66)

Cosine is positive in quadrant IV, so cos 333° = cos 27° = 0.8910.

Hence, cos 693° = 0.8910.

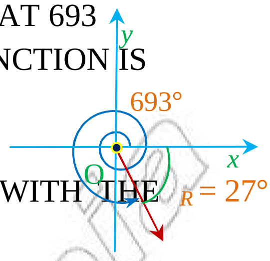


Figure 5.66

Exercise 5.14

1 Using trigonometric table, find:

a sin 59° **b** cos 53° **c** tan 36° **d** sec 162°

e sin 593° **f** tan 593° **g** cos (–143°)

2 In each of the following problems, find A to the nearest degree:

a sin A = 0.5299 **b** cos A = 0.6947 **c** tan A = 1.540

d csc A = 1.000 **e** sec A = 2.000 **f** cot A = 1.808

5.4 REAL LIFE APPLICATION PROBLEMS

Even though trigonometry was originally used to relate the angles of a triangle to the lengths of the sides of a triangle, trigonometric functions are important not only in the study of triangles but also in modeling many periodic phenomena in rea… section you will see some of the real life applications of trigonometry.

Solving right-angled triangles

Many applications of trigonometry involve "solving a triangle". A triangle has basically seven components; namely three sides, three angles and an area. Thus, solv… means to find the lengths of the three sides, the measures of all the three… measure of its area.

Revision of the properties of right angle triangles

We already know that, for a given right angled triangle, the hypotenuse (HYP) is the side which is opposite the right angle and is the longest side of the triangle. For the angle marked θ in Figure 5.67:

✓ $\overline{BC}$ is the side opposite (OPP) angle θ

✓ $\overline{AC}$ is the side adjacent (ADJ) angle θ.

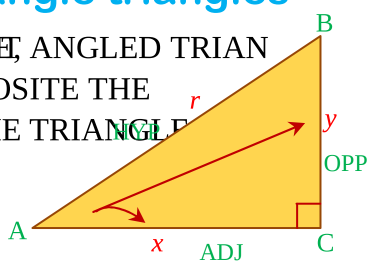


Figure 5.67

HENCE,

1	$x^2 + y^2 = r^2$	**2**	$\text{SIN} = \frac{y}{r}$ $\quad \text{CSC} = \frac{r}{y} = \frac{1}{\text{SIN}}$ $\text{COS} = \frac{x}{r}$ $\quad \text{SEC} = \frac{r}{x} = \frac{1}{\text{COS}}$ $\text{TAN} = \frac{y}{x}$ $\quad \text{COT} = \frac{x}{y} = \frac{1}{\text{TAN}}$
3	$\text{SIN}^2 + \text{COS}^2 = 1$ $1 + \text{TAN}^2 = \text{SEC}^2$ $1 + \text{COT}^2 = \text{CSC}^2$	**4**	$\text{TAN} = \frac{\text{SIN}}{\text{COS}}$ $\text{COT} = \frac{\text{COS}}{\text{SIN}}$

EXAMPLE 1 SOLVE THE RIGHT-ANGLED TRIANGLE WITH AN ACUTE ANGLE OF 25° AND HYPOTENUSE OF LENGTH 10 CM.

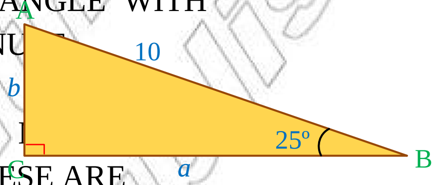


Figure 5.68

SOLUTION: IT IS REQUIRED TO FIND THE UNKNOWN ELEMENTS OF THE TRIANGLE. THESE ARE

A $m(\angle A)$ **B** LENGTH OF SIDE BC
C LENGTH OF SIDE AC **D** THE AREA OF THE TRIANGLE

DRAW THE TRIANGLE AND LABEL ALL KNOWN PARTS (*See* Figure 5.68)

A $m(\angle A) = 90^\circ - 25^\circ = 65^\circ$

B TO FIND a, OBSERVE THAT THE SIDE $\overline{BC}$ IS OPPOSITE TO THE 65° ANGLE, AND THE LENGTH OF THE HYPOTENUSE IS 10 CM. SO $\text{SIN } 65^\circ = \frac{a}{10}$.

MULTIPLYING BOTH SIDES OF THE EQUATION BY 10, WE OBTAIN

$$a = 10 \times \text{SIN } 65^\circ$$

USING THE TRIGONOMETRIC TABLE, WE HAVE

$$a = 10 \times \text{SIN } 65^\circ \approx 10 \times 0.9063 = 9.063 \text{ CM}$$

C TO FIND b, WE CAN USE THE PYTHAGOREAN THEOREM OR THE SINE FUNCTION.

$$\text{SIN } 25^\circ = \frac{b}{10}$$

MULTIPLYING BOTH SIDES BY 10 WE OBTAIN $b = 10 \times \text{SIN } 25^\circ$

USING TRIGONOMETRIC TABLE WE HAVE $b = 10 \times \text{SIN } 25^\circ \approx 10 \times (0.4226) \approx 4.226$ CM.

D AREA OF $\triangle ABC = \frac{1}{2}ab \approx \frac{1}{2} \times 9.063 \times 4.226 \approx 19.150 \text{ CM}^2$

EXAMPLE 2 SOLVE THE RIGHT ANGLE TRIANGLE WHOSE HYPOTENUSE IS 20 UNITS AND ONE OF THE LEGS IS 17 UNITS.

SOLUTION: THE MISSING ELEMENTS OF THE TRIANGLE ARE

A $m(\angle A)$ **C** LENGTH OF SIDE AC

B $m(\angle B)$ **D** THE AREA OF THE TRIANGLE

DRAW THE TRIANGLE (See FIGURE 5.69).

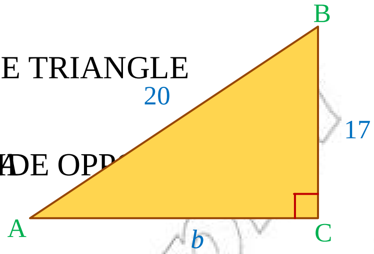


Figure 5.69

A SINCE THE HYPOTENUSE AND THE SIDE OPP
ARE GIVEN,

$$\sin A = \frac{17}{20} = 0.8500$$

THUS, FROM THE TRIGONOMETRIC TABLE, WE SEE THAT $m(\angle A) \approx 58^{O}$

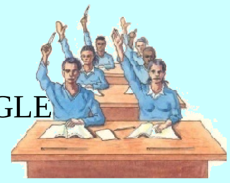
B $m(\angle B) = 90^{O} - m(\angle A) = 90^{O} - 58^{O} = 32^{O}$

C TO FIND b, USE $\cos A = \frac{b}{20}$ WHICH GIVES

$$b = 20\cos A \approx 20\cos 58^{O} \approx 20(0.5299) \approx 10.598$$

D AREA OF $\triangle ABC = \frac{1}{2} \times b \times 17 = \frac{1}{2} \times 10.598 \times 17 = 90.083$ UNITS^2

ACTIVITY 5.12



1. SOLVE THE RIGHT ANGLED TRIANGLE ABC WITH THE RIGHT ANGLE AT B, AB = 2 CM AND BC = 3 CM.
2. SOLVE THE RIGHT ANGLED TRIANGLE ABC WITH THE RIGHT ANGLE AT B, $m(\angle A) = 24°$ AND AB = 20 CM.

Angle of elevation and angle of depression

THE line of sight OF AN OBJECT IS THE LINE JOINING THE EYE OF AN OBSERVER AND THE OBJECT. IF THE OBJECT IS ABOVE THE HORIZONTAL PLANE THROUGH THE EYE OF THE OBSER
BETWEEN THE LINE OF SIGHT AND THIS HORIZONTAL PLANE IS CALLED THE angle of elevation (*See* FIGURE 5.70). IF THE OBJECT IS below THIS HORIZONTAL PLANE, THE ANGLE IS THEN CALL
THE angle of depression.

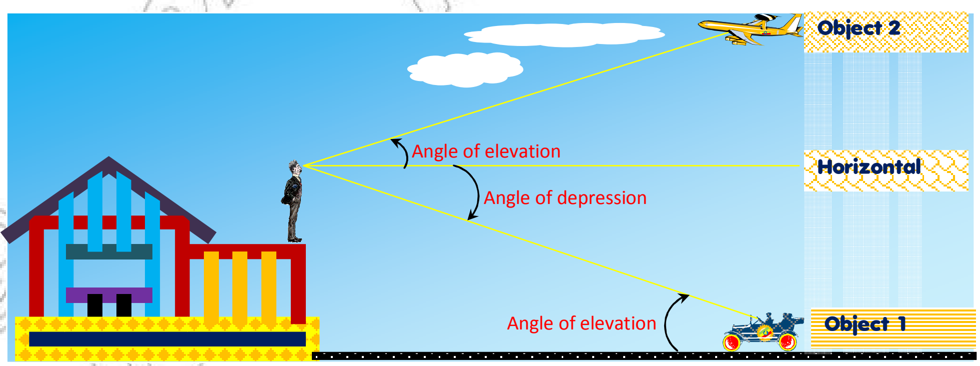


Figure 5.70

EXAMPLE 3 FIND THE HEIGHT OF A TREE WHICH CASTS A SHADOW OF 12.4 M WHEN THE ANGLE OF ELEVATION OF THE SUN IS 52°.

SOLUTION: LET h BE THE HEIGHT OF THE TREE IN METRES. FOR THE 52° ANGLE, THE OPPOSITE SIDE IS h AND THE ADJACENT SIDE 12.4 M.

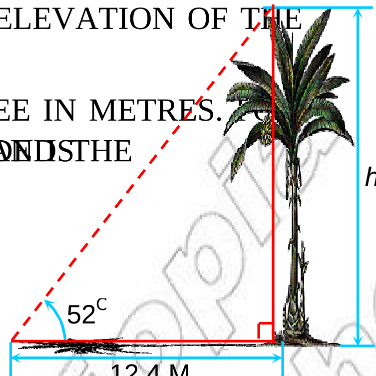


Figure 5.71

THEREFORE, $\tan 52^{\circ} = \frac{h}{12.4}$

$\therefore h = 12.4 \times \tan 52^{\circ} = 15.9$ M.

THEREFORE, THE TREE IS 15.9 M HIGH.

EXAMPLE 4 FROM THE TOP OF A BUILDING, THE ANGLE OF DEPRESSION OF A POINT ON THE GROUND 7 M AWAY FROM THE BASE OF THE BUILDING IS 60°. FIND THE HEIGHT OF THE BUILDING.

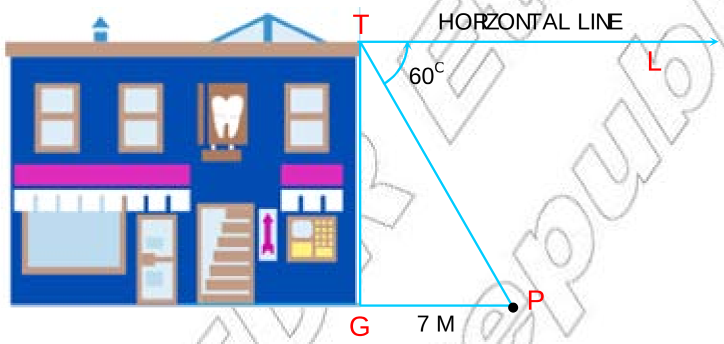


Figure 5.72

SOLUTION: IN FIGURE 5.72, T IS A POINT ON THE TOP OF THE BUILDING, P IS THE POINT ON THE GROUND, AND $\overline{TL}$ IS A HORIZONTAL RAY THROUGH THE PLANE OF TOP.

$m(\angle GPT) = m(\angle LTP) = 60^{\circ}$ (WHY?)

$$\frac{GT}{PG} = \tan m(\angle GPT) \Rightarrow \tan 60^{\circ} = \frac{GT}{7} \Rightarrow GT = 7 \tan 60^{\circ} \approx 7 \times 1.732 \approx 12 \text{ M}.$$

THEREFORE, THE HEIGHT OF THE BUILDING IS ABOUT 12 METRES.

EXAMPLE 5 A PERSON STANDING ON THE EDGE OF ONE BANK OF A CANAL OBSERVES A LAMP POST ON THE EDGE OF THE OTHER BANK OF THE CANAL. THE PERSON'S EY 152 CM ABOVE THE GROUND. THE ANGLE OF ELEVATION FROM EYE LEVE TOP OF THE LAMP POST IS 12°, AND THE ANGLE OF DEPRESSION FROM EYE LEVEL T THE BOTTOM OF THE LAMP POST IS 7°. HOW HIGH IS THE LAMP POST? HOW WIDE IS THE CANAL? (*See* FIGURE 5.73A.)

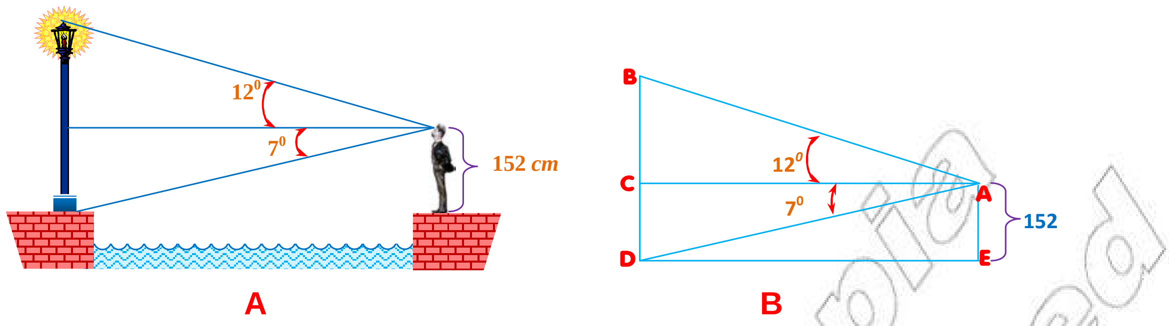


Figure 5.73

SOLUTION: CONSIDERING THE ESSENTIAL INFORMATION, WE CAN DRAW THE DIAGRAM AS IN FIGURE 5.73B.

WE WANT TO FIND THE HEIGHT OF THE LAMP POST BD AND THE WIDTH OF THE CANAL AC. THE EYE LEVEL HEIGHT AE OF THE OBSERVER IS 152CM. SINCE $\overline{AC}$ AND $\overline{ED}$ ARE PARALLEL, $\overline{CD}$ ALSO HAS LENGTH 152 CM. IN THE RIGHT ANGLED TRIANGLE ACD, WE KNOW THAT THE SIDE CD IS OPPOSITE TO THE ANGLE OF 7°.

SO, $\tan 7° = \frac{opp}{adj} = \frac{152}{AC}$ GIVING $AC = \frac{152}{\tan 7°}$

THEREFORE, $AC = \frac{152}{\tan 7°} = \frac{152}{0.1228} = 1237.79$ CM

SO THE CANAL IS APPROXIMATELY 12.4 METRES WIDE.

NOW, USING THE RIGHT TRIANGLE ACB, WE SEE THAT

$$\tan 12° = \frac{opp}{adj} = \frac{BC}{AC} = \frac{BC}{1237.79}$$

THEREFORE, $BC = 1237.79 \times \tan 12° = 1237.79 \times 0.2126 = 263.15$ CM.

SO THE HEIGHT OF THE LAMP POST BD IS

$BC + CD = 263.15 + 152 = 415.15$ CM ≈ 4.15 M

Exercise 5.15

1 IN PROBLEMS A TO F, ΔABC IS A RIGHT ANGLE TRIANGLE WITH $m(\angle C) = 90°$. LET a, b, c BE ITS SIDES WITH c THE LENGTH OF ITS HYPOTENUSE, a THE SIDE LENGTH OPPOSITE ANGLE A AND b ITS SIDE LENGTH OPPOSITE B. USING THE INFORMATION BELOW, FIND THE MISSING ELEMENTS OF EACH RIGHT ANGLE TRIANGLE, GIVING ANSWERS CORRECT TO THE NUMBER.

- **A** $m(\angle B) = 50°$ AND $c = 20$ UNITS
- **B** $m(\angle A) = 54°$ AND $a = 12$ UNITS
- **C** $m(\angle A) = 36°$ AND $b = 8$ UNITS
- **D** $m(\angle B) = 55°$ AND $a = 10$ UNITS
- **E** $m(\angle A) = 38°$ AND $c = 20$ UNITS
- **F** $m(\angle A) = 17°$ AND $a = 14$ UNITS.

2 **A** A LADDER 6 METRES LONG LEANS AGAINST A BUILDING. IF THE ANGLE FORMED BY THE LADDER AND THE GROUND IS 65°, HOW FAR FROM THE BUILDING IS THE FOOT OF THE LADDER?

B A MONUMENT IS 50 METRES HIGH. WHAT IS THE LENGTH OF THE SHADOW CAST BY THE MONUMENT IF THE ANGLE OF ELEVATION OF THE SUN IS 60°?

C WHEN THE SUN IS 35° ABOVE THE HORIZON, HOW LONG IS THE SHADOW CAST BY A BUILDING 15 METRES HIGH?

D FROM AN OBSERVER O, THE ANGLES OF ELEVATION OF THE BOTTOM AND THE TOP OF A FLAGPOLE ARE 40° AND 45° RESPECTIVELY. FIND THE HEIGHT OF THE FLAGPOLE.

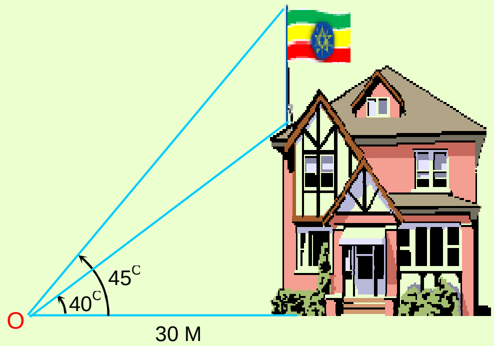


Figure 5.74

E FROM THE TOP OF A CLIFF 200 METRES ABOVE SEA LEVEL, THE ANGLES OF DEPRESSION TO TWO FISHING BOATS ARE 40° AND 45° RESPECTIVELY. HOW FAR APART ARE THE BOATS?

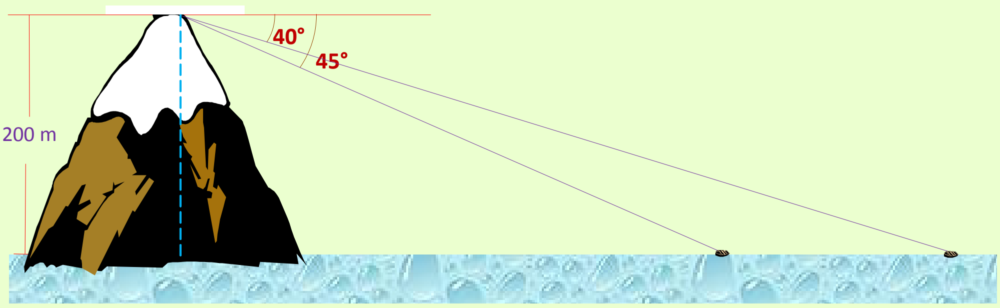


Figure 5.75

F A SURVEYOR STANDING AT *A* SIGHTS TWO OBJECTS *B* AND *C* ON THE OPPOSITE SIDE OF A CANAL. THE OBJECTS ARE 120 M APART. IF THE ANGLE OF SIGHT BETWEEN THEM IS 37°, HOW WIDE IS THE CANAL?

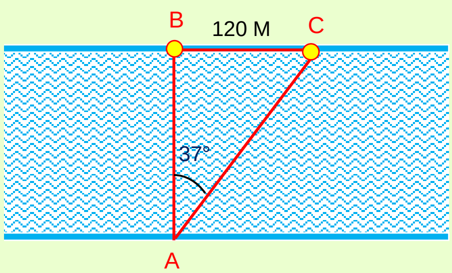


Figure 5.76

Key Terms

angle in standard position	negative angle	radian
angle of depression	period	reference angle
angle of elevation	periodic function	special angle
co-function	positive angle	supplementary angles
complementary angles	pythagorean identity	trigonometric function
co-terminal angles	quadrantal angle	trigonometry
degree	quotient identity	unit circle

Summary

1 AN ANGLE IS DETERMINED BY THE ROTATION OF A RAY ABOUT ITS VERTEX FROM AN INITIAL POSITION TO A TERMINAL POSITION.

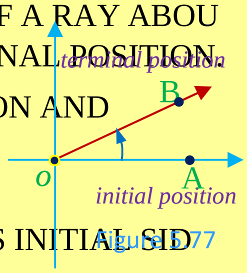


Figure 5.77

2 AN ANGLE IS positive FOR ANTICLOCKWISE ROTATION AND negative FOR CLOCKWISE ROTATION.

3 AN ANGLE IN THE COORDINATE PLANE IS IN standard Position, IF ITS VERTEX IS AT THE ORIGIN AND ITS INITIAL SIDE LIES ALONG THE POSITIVE x-AXIS.

4 RADIAN MEASURE OF ANGLES:

2 RADIANS = 360^{0} RADIANS = 180^{0}

5 TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{\ }{180^{o}}$.

6 TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^{o}}{\ }$.

7 IF IS AN ANGLE IN STANDARD POSITION AND $P(x, y)$ IS A POINT ON THE TERMINAL SIDE OF , OTHER THAN THE ORIGIN $O(0,0)$, AND r IS THE DISTANCE OF P FROM THE ORIGIN O, THEN

SIN $= \frac{y}{r}$ CSC $= \frac{r}{y} = \frac{1}{\text{SIN}}$

COS $= \frac{x}{r}$ SEC $= \frac{r}{x} = \frac{1}{\text{COS}}$

TAN $= \frac{y}{x}$ COT $= \frac{x}{y} = \frac{1}{\text{TAN}}$

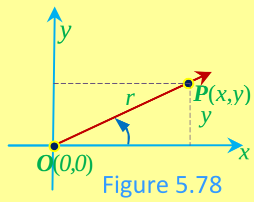


Figure 5.78

$r = \sqrt{x^2 + y^2}$ (PYTHAGORAS RULE)

8 Signs of sine, cosine and tangent functions:

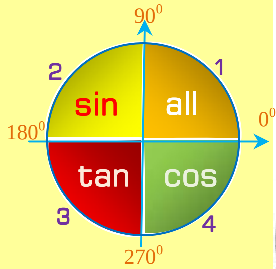


Figure 5.79

- ✓ IN THE FIRST QUADRANT all the three TRIGONOMETRIC FUNCTIONS ARE POSITIVE.
- ✓ IN THE SECOND QUADRANT sine IS ONLY POSITIVE.
- ✓ IN THE THIRD QUADRANT tangent IS ONLY POSITIVE.
- ✓ IN THE FOURTH QUADRANT cosine IS ONLY POSITIVE.

All Students Take Chemistry

9 Functions of negative angles:

IF IS AN ANGLE IN STANDARD POSITION, THEN

SIN(–) = – SIN COS(–) = COS TAN(–) = – TAN

10 Complementary angles:

TWO ANGLES ARE SAID TO BE COMPLEMENTARY, IF THEIR SUM IS EQUAL TO 90^0

IF α AND β ARE ANY TWO COMPLEMENTARY ANGLES, THEN

$$\sin\alpha = \cos\beta \qquad \cos\alpha = \sin\beta \qquad \tan\alpha = \frac{1}{\tan\beta}$$

11 Reference angle R:

IF IS AN ANGLE IN STANDARD POSITION WHOSE TERMINAL SIDE DOES NOT LIE ON COORDINATE AXIS, THEN THE reference angle $_R$ FOR IS THE positive acute angle FORMED BY THE TERMINAL SIDE OF AND THE AXIS.

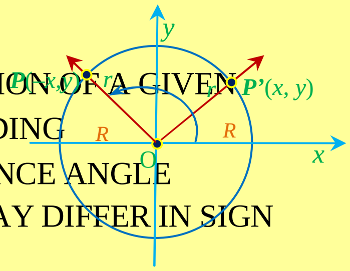


Figure 5.80

12 THE VALUES OF THE TRIGONOMETRIC FUNCTIONS OF A GIVEN ANGLE AND THE VALUES OF THE CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE REFERENCE ANGLE $_R$ ARE THE SAME IN ABSOLUTE VALUE BUT THEY MAY DIFFER IN SIGN

13 Supplementary angles:

TWO ANGLES ARE SAID TO BE SUPPLEMENTARY, IF THEIR SUM IS EQUAL TO 180^0. IF IS A SECOND QUADRANT ANGLE, THEN ITS SUPPLEMENT WILL BE (180^0 –).

SIN = SIN (180^0 –),

COS = – COS (180^0 –),

TAN = – TAN (180^0 –)

14 CO-TERMINAL ANGLES ARE ANGLES IN STANDARD POSITION (ANGLES WITH THE INITIAL SIDE ON THE POSITIVE x-AXIS) THAT HAVE A COMMON TERMINAL SIDE.

15 CO-TERMINAL ANGLES HAVE THE SAME TRIGONOMETRIC VALUES.

16 THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS.

17 THE RANGE OF THE SINE FUNCTION IS $\{y \mid -1 \leq y \leq 1\}$.

18 THE GRAPH OF THE SINE FUNCTION REPEATS ITSELF EVERY 360^{O} OR 2 RADIANS.

19 THE DOMAIN OF THE COSINE FUNCTION IS THE SET OF ALL REAL NUMBERS.

20 THE RANGE OF THE COSINE FUNCTION IS $\{y \mid -1 \leq y \leq 1\}$.

21 THE GRAPH OF THE COSINE FUNCTION REPEATS ITSELF EVERY 360^{O} OR 2 RADIANS.

22 THE DOMAIN OF THE TANGENT FUNCTION IS $\left\{ x \mid x \neq n\frac{\ }{2}, \text{ WHERE } n \text{ IS AN ODD INTEGER} \right\}$

23 THE RANGE OF y = TAN IS THE SET OF ALL REAL NUMBERS.

24 THE TANGENT FUNCTION HAS PERIOD 180^{O} OR RADIANS.

25 THE GRAPH OF y = TAN IS INCREASING FOR $-\frac{\ }{2} < \ < \frac{\ }{2}$.

26 ANY TRIGONOMETRIC FUNCTION OF AN ACUTE ANGLE IS EQUAL TO THE CO-FUNCTION OF ITS COMPLEMENTARY ANGLE.

THAT IS, IF $0^{O} \leq \ \leq 90^{O}$, THEN

SIN = COS (90^{O} –)	CSC = SEC (90^{O} –)
COS = SIN (90^{O} –)	SEC = CSC (90^{O} –)
TAN = COT (90^{O} –)	COT = TAN (90^{O} –)

27 **Reciprocal relations:**

$$\text{CSC} = \frac{1}{\text{SIN}}, \quad \text{SEC} = \frac{1}{\text{COS}}, \quad \text{COT} = \frac{1}{\text{TAN}}$$

28 **Pythagorean identities:**

$$\text{SIN}^2 + \text{COS}^2 = 1 \qquad 1 + \text{TAN}^2 = \text{SEC}^2 \qquad \text{COT}^2 + 1 = \text{CSC}^2$$

29 **Quotient identities:**

$$\text{TAN} = \frac{\text{SIN}}{\text{COS}} \qquad \text{COT} = \frac{\text{COS}}{\text{SIN}}$$

Review Exercises on Unit 5

1 INDICATE TO WHICH QUADRANT EACH OF THE FOLLOWING ANGLES BELONG:

A 225^{O} **B** 333^{O} **C** -300^{O} **D** 610^{O}
E -700^{O} **F** 900^{O} **G** -765^{O} **H** -1238^{O}
I 1440^{O} **J** 2010^{O}.

2 FIND TWO CO-TERMINAL ANGLES (ONE POSITIVE AND THE OTHER NEGATIVE) FOR EACH OF THE FOLLOWING ANGLES:

A 80^{O} **B** 140^{O} **C** 290^{O} **D** 375^{O} **E** 2900^{O}
F -765^{O} **G** -900^{O} **H** -1238^{O} **I** -1440^{O} **J** -2010^{O}.

3 CONVERT EACH OF THE FOLLOWING TO RADIANS:

A 40^{O} **B** 75^{O} **C** 240^{O} **D** 330^{O} **E** -95^{O}
F -180^{O} **G** -220^{O} **H** -420^{O} **I** -3060^{O}.

4 CONVERT EACH OF THE FOLLOWING ANGLES IN RADIANS TO DEGREES:

A $\frac{2}{6}$ **B** $\frac{-2}{3}$ **C** $\frac{7}{18}$ **D** $\frac{43}{6}$
E $-\frac{4}{9}$ **F** 5 **G** $\frac{-3}{12}$ **H** $\frac{-}{24}$.

5 USE A UNIT CIRCLE TO FIND THE VALUES OF SINE, COSINE AND TANGENT WHEN IS:

A 810^{O} **B** -450^{O} **C** 900^{O} **D** -630^{O}
E 990^{O} **F** -990^{O} **G** 1080^{O} **H** -1170^{O}

6 FIND THE VALUES OF SINE, COSINE AND TANGENT WHEN IN RADIANS IS:

A $\frac{5}{6}$ **B** $\frac{7}{6}$ **C** $\frac{4}{3}$ **D** $\frac{3}{2}$
E $\frac{5}{3}$ **F** $\frac{-5}{3}$ **G** $\frac{-7}{4}$ **H** $\frac{-11}{6}$.

7 STATE WHETHER EACH OF THE FOLLOWING FUNCTIONAL VALUES ARE POSITIVE OR NEGATIVE:

A SIN 310^{O} **B** COS 220^{O} **C** COS (-220^{O}) **D** TAN 765^{O}
E SIN (-90^{O}) **F** SEC (-70^{O}) **G** TAN 327^{O} **H** COT $\frac{5}{3}$
I CSC 1387^{O} **J** SIN $\left(\frac{-11}{6}\right)$

8 GIVE A REFERENCE ANGLE FOR EACH OF THE FOLLOWING;

A 140^{O} **B** 260^{O} **C** 355^{O} **D** 414^{O}
E -190^{O} **F** -336^{O} **G** 1238^{O} **H** -1080^{O}.

9 REFERRING TO THE VALUES GIVEN IN THE TABLE FOR $0° \le \theta \le 360°$, ROUGHLY SKETCH THE GRAPHS OF THE SINE, COSINE AND TANGENT FUNCTIONS.

Degrees	Radians	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	UNDEFINED	1	UNDEFINED
30°	$\frac{}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{}{2}$	1	0	UNDEFINED	0	UNDEFINED	1
120°	$\frac{2}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{-\sqrt{3}}{3}$	–2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{3}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	–1	–1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180°	π	0	–1	0	UNDEFINED	–1	UNDEFINED
210°	$\frac{7}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	–2
225°	$\frac{5}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	–2	$\frac{-2\sqrt{3}}{3}$
270°	$\frac{3}{2}$	–1	0	UNDEFINED	0	UNDEFINED	–1
300°	$\frac{5}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{3}$	2	$-\frac{2\sqrt{3}}{3}$
315°	$\frac{7}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	–1	–1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	–2
360°	2π	0	1	0	UNDEFINED	1	UNDEFINED

10 FIND THE VALUE OF EACH OF THE FOLLOWING:

A SIN (-120^0) **B** COS 600^0 **C** TAN (-300^0)

D CSC 990^0 **E** SEC 450^0 **F** COT (-420^0).

11 EVALUATE THE SIX TRIGONOMETRIC FUNCTIONS OF , IF IS IN STANDARD POSITION AND ITS TERMINAL SIDE CONTAINS THE GIVEN POINT P (x, y):

A P (5, 12) **B** P (– 7, 24) **C** P (5, – 6) **D** P (– 8, – 17)

E P (15, 8) **F** P (1, – 8) **G** P (– 3, –4) **H** P (0, 1)

12 LET BE AN ANGLE IN STANDARD POSITION. IDENTIFY THE QUADRANT IN WHICH BELONGS GIVEN THE FOLLOWING CONDITIONS:

A IF SIN < 0 AND COS < 0 **B** IF SIN > 0 AND TAN > 0

C IF SIN > 0 AND SEC < 0 **D** IF SEC > 0 AND COT < 0

E IF COS < 0 AND COT > 0 **F** IF SEC < 0 AND CSC > 0.

13 FIND THE ACUTE ANGLE , IF:

A SIN $60^0 = \dfrac{1}{\text{CSC}}$ **B** SIN = COS **C** SIN70^0 = COS

D $1 = \dfrac{\text{SIN}}{\text{COS } 80^0}$ **E** $\dfrac{\text{SIN}}{\text{CO}} = \text{COT } 35^0$ **F** $\dfrac{\text{SIN } 70^0}{\text{COS } 70^0} = \dfrac{\text{COS}}{\text{SIN}}$

14 IF IS OBTUSE AND COS $= \dfrac{-4}{5}$, FIND:

A SIN **B** TAN **C** CSC **D** COT.

15 IF $-90^0 <$ < 0 AND TAN $= -\dfrac{2}{3}$, FIND COS.

16 IN PROBLEMS A TO D BELOW, ΔABC IS A RIGHT ANGLE TRIANGLE WITH $m(\angle C) = 90°$. LET a, b, c BE ITS SIDES WITH c THE HYPOTENUSE, a THE SIDE OPPOSITE ANGLE A AND b THE SIDE OPPOSITE ANGLE B. USING THE INFORMATION BELOW, FIND THE MISSING ELEM EACH RIGHT TRIANGLE, ROUNDING ANSWERS CORRECT TO THE NEAREST WHOLE N

A $m(\angle B) = 60°$ AND a = 18 UNITS. **B** $m(\angle A) = 45°$ AND c = 16 UNITS.

C $m(\angle A) = 22°$ AND b = 10 UNITS. **D** $m(\angle B) = 52°$ AND c = 47 UNITS.

17 **A** FIND THE HEIGHT OF A TREE, IF THE ANGLE OF ELEVATION OF ITS TOP CHANGES FROM $25°$ TO $50°$ AS THE OBSERVER ADVANCES 15 METRES TOWARDS ITS BASE.

B THE ANGLE OF DEPRESSION OF THE TOP AND THE FOOT OF A POLE AS SEEN FROM THE TOP OF A BUILDING 145 METRES AWAY ARE $26°$ AND $34°$ RESPECTIVELY. FIND THE HEIGHTS OF THE POLE AND THE BUILDING.

C TO THE NEAREST DEGREE, FIND THE ANGLE OF ELEVATION OF THE SUN WHEN A 9 METRE VERTICAL FLAGPOLE CASTS A SHADOW 3 METRES LONG.

Unit 6

Leonardo da Vinci obtained the "Mona Lisa" smile by tilting the lips so that the ends lie on a circle which touches the outer corners of the eyes. The outline of the top of the head is the arc of another circle exactly twice as large as the first.

PLANE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *know more theorems special to triangles.*
- *know basic theorems specific to quadrilaterals.*
- *know theorems about circles and angles inside, on and outside a circle.*
- *solve geometrical problems involving quadrilaterals, circles and regular polygons.*

Main Contents

INTRODUCTION

WHY DO YOU STUDY GEOMETRY?

- GEOMETRY TEACHES YOU HOW TO THINK CLEARLY. OF ALL THE SUBJECTS TAUGHT AT HIGH SCHOOL LEVEL, GEOMETRY IS ONE OF THE LESSONS THAT GIVES THE BEST TRAINING AND ACCURATE METHODS OF THINKING.
- THE STUDY OF GEOMETRY HAS A PRACTICAL VALUE. IF SOMEONE WANTS TO BE AN ARTIST, A DESIGNER, A CARPENTER, A TINSMITH, A LAWYER OR A DENTIST, THE FACTS AND IN GEOMETRY ARE OF GREAT VALUE.

  Abraham Lincoln BORROWED A GEOMETRY TEXT AND LEARNED THE PROOFS OF MOS PLANE GEOMETRY THEOREMS SO THAT HE COULD MAKE BETTER ARGUMENTS IN C

  Leonardo da Vinci OBTAINED THE “MONA LISA” SMILE BY TILTING THE LIPS SO THA ENDS LIE ON A CIRCLE WHICH TOUCHES THE OUTER CORNERS OF THE EYES. THE O TOP OF THE HEAD IS THE ARC OF ANOTHER CIRCLE EXACTLY TWICE AS LARGE AS T SAME ARTIST’S “LAST SUPPER”, THE VISIBLE PART OF CHRIST CONFORMS TO THE S EQUILATERAL TRIANGLE.

PLANE GEOMETRY (SOMETIMES CALLED EUCLIDEAN GEOMETRY) IS A BRANCH OF DEALING WITH THE PROPERTIES OF FLAT SURFACES AND PLANE FIGURES, SUCH QUADRILATERALS OR CIRCLES.

6.1 THEOREMS ON TRIANGLES

IN PREVIOUS GRADES, YOU HAVE LEARNT THAT A TRIANGLE IS A POLYGON WITH THRE THE SIMPLEST TYPE OF POLYGON.

THREE OR MORE POINTS THAT LIE ON ONE LINE ARE CALLED ***collinear points***. THREE OR MORE LINES THAT PASS THROUGH ONE POINT ARE CALLED ***concurrent lines***

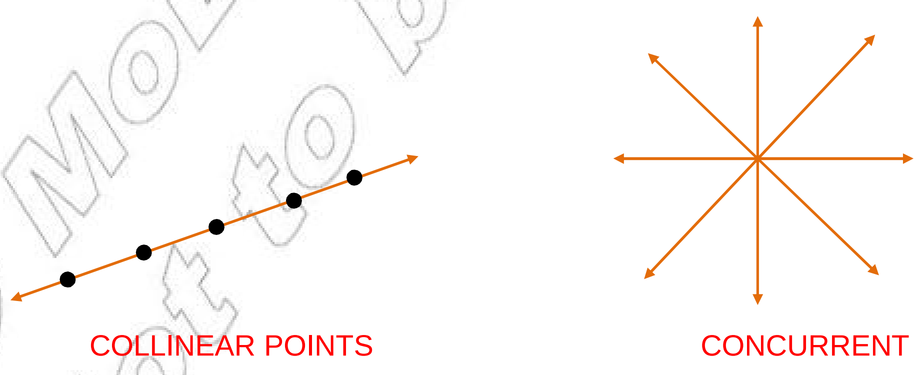
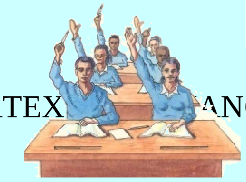


Figure 6.1

ACTIVITY 6.1



1 WHAT DO YOU CALL A LINE SEGMENT JOINING A VERTEX ... ANGLE TO THE MID-POINT OF THE OPPOSITE SIDE?

2 HOW MANY MEDIANS DOES A TRIANGLE HAVE?

3 DRAW TRIANGLE ABC WITH $\angle C = 90^{O}$, AC = 8 CM AND CB = 6 CM. DRAW THE MEDIAN FROM A TO $\overline{BC}$. HOW LONG IS THIS MEDIAN? CHECK YOUR RESULT USING PYTHAGORAS THEOREM.

4 DRAW A TRIANGLE. CONSTRUCT ALL THE THREE MEDIANS. ARE THEY CONCURREN... THINK THAT THIS IS TRUE FOR ALL TRIANGLES? TEST THIS BY DRAWING MORE TRI...

5 IS IT POSSIBLE FOR THE MEDIANS OF A TRIANGLE TO MEET OUTSIDE THE TRIANGLE...

THEOREMS ABOUT COLLINEAR POINTS AND CONCURRENT LINES ARE CALLED ***incidence theorems***. SOME SUCH THEOREMS ARE STATED BELOW.

RECALL THAT A LINE THAT DIVIDES AN ANGLE INTO TWO CONGRUENT ANGLES IS C... BISECTOR OF THE ANGLE.

A LINE THAT DIVIDES A LINE SEGMENT INTO TWO CONGRUENT LINE SEGMENTS IS CALL... ***bisector*** OF THE LINE SEGMENT. WHEN A BISECTOR OF A LINE SEGMENT FORMS RIGHT ANGLE WITH THE LINE SEGMENT, THEN IT IS CALLED THE ***perpendicular bisector*** OF THE LINE SEGMENT.

Median of a triangle

A ***median*** OF A TRIANGLE IS A LINE SEGMENT DRAWN FROM ANY VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

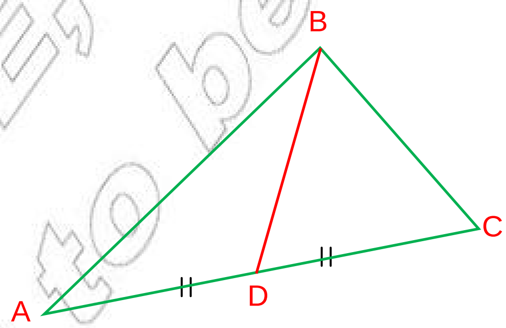


Figure 6.2

$\overline{BD}$ IS A MEDIAN OF TRIANGLE ABC.

ACTIVITY 6.2

COPY △ABC IN FIGURE 6.3.

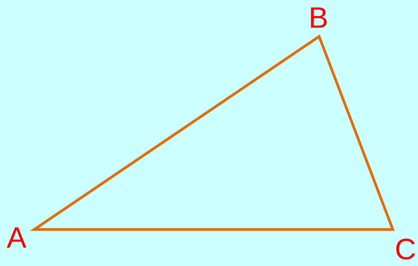


Figure 6.3

1. CONSTRUCT ALL THE MEDIANS OF △ABC CAREFULLY.
2. I. MARK THE MID-POINT OF $\overline{BC}$ AS E.
   II. MARK THE MID-POINT OF $\overline{AC}$ AS F.
   III. MARK THE MID-POINT OF $\overline{AB}$ AS D.
3. DID THE MEDIANS INTERSECT AT A POINT?
   IF YOUR ANSWER IS YES, MARK THE POINT O.
4. MEASURE EACH OF THE FOLLOWING SEGMENTS AND DETERMINE THE INDICATED R
   I. A. $\overline{AO}$ B. $\overline{OE}$ $AO : OE$
   II. A. $\overline{CO}$ B. $\overline{OD}$ $CO : OD$
   III. A. $\overline{BO}$ B. $\overline{OF}$ $BO : OF$
5. HOW DO YOU RELATE THE RATIOS OBTAINED IN QUESTION 4 ABOVE?

THE ABOVE ACTIVITY HELPS YOU TO OBSERVE THE FOLLOWING THEOREM.

Theorem 6.1

The medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side.

Proof:-

SUPPOSE $\overline{AE}$ AND $\overline{DC}$ ARE MEDIANS OF $\triangle ABC$ THAT ARE INTERSECTING AT POINT O. (*See* FIGURE 6.4).

	Statement		Reason
1	IN ΔABC, $\overline{AE}$ AND $\overline{DC}$ ARE MEDIANS INTERSECT POINT O	1	GIVEN
2	DRAW $\overline{DE}$	2	CONSTRUCTION
3	DRAW $\overline{EG}$ PARALLEL TO $\overline{DC}$ WITH G ON THE EXTENSION $\overline{AC}$	3	CONSTRUCTION
4	DRAW $\overline{EF}$ PARALLEL TO $\overline{AB}$ WITH F ON $\overline{AC}$	4	CONSTRUCTION
5	DRAW $\overline{FH}$ PARALLEL TO $\overline{DC}$ WITH H ON $\overline{AB}$	5	CONSTRUCTION
6	DRAW LINE ℓ PARALLEL TO $\overline{DC}$ PASING THROUGH A.	6	CONSTRUCTION
7	$AFED$ AND $CGED$ ARE PARALLELOGRAMS WITH SIDE $\overline{DE}$	7	STEPS 3 AND 4
8	THEREFORE, $AF = DE = CG$	8	STEP 7
9	$DE = \frac{1}{2}AC = AF$	9	$\Delta ABC \sim \Delta DBE$ FROM STEP 1
10	$AF = FC = CG$	10	STEPS 8 AND 9
11	$\overline{AG}$ IS TRISECTED BY PARALLEL LINES ℓ, $\overleftrightarrow{HF}$, $\overleftrightarrow{DC}$ AND $\overleftrightarrow{EG}$	11	STEPS 3, 5 AND 10
12	$\overline{AE}$ IS TRISECTED BY ℓ, $\overline{HF}$, $\overline{DC}$ AND $\overline{EG}$	12	STEP 11 AND PROPERTY OF PARALLEL LINES

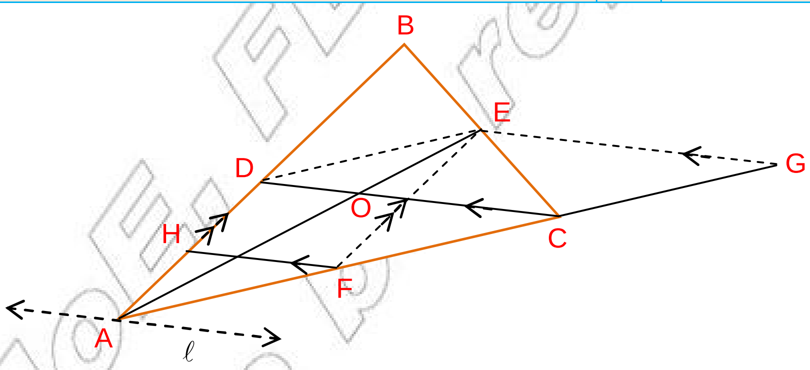


Figure 6.4

THEREFORE, $OE = \frac{1}{3}AE$, $AO = \frac{2}{3}AE$.

YOU HAVE PROVED THAT THE MEDIANS $\overline{DC}$ AND $\overline{AE}$ MEET AT POINT O SUCH THAT $AO = \frac{2}{3}AE$.

YOUR NEXT TASK IS TO PROVE THAT THE MEDIANS $\overline{AE}$ AND $\overline{BF}$ INTERSECT AT THE SAME POINT O. WITH THE SAME ARGUMENT USED ABOVE, LET O' BE THE POINT OF INTERSECTION OF $\overline{AE}$ AND $\overline{BF}$ WHOSE DISTANCE FROM A IS $\frac{2}{3}AE$ THAT IS $AO' = \frac{2}{3}AE$

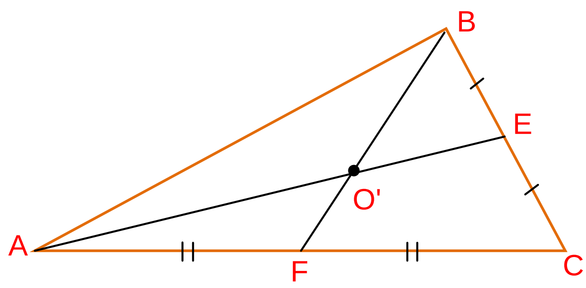


Figure 6.5

IT FOLLOWS THAT $AO = AO'$ AND HENCE $O = O'$ AS O AND O' ARE ON $\overline{AE}$.

THEREFORE, ALL THE THREE MEDIANS OF $\triangle ABC$ ARE CONCURRENT AT A SINGLE POINT O LOCATED $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOS

EXAMPLE 1 IN FIGURE 6.6, $\overline{AN}$, $\overline{CM}$ AND $\overline{BL}$ ARE MEDIANS OF $\triangle ABC$. IF $AN = 12$ CM, $OM = 5$ CM AND $BO = 6$ CM, FIND BL, ON AND OL.

SOLUTION:

BY THEOREM 6.1,

$$BO = \frac{2}{3}\ BL \text{ AND } AO = \frac{2}{3}\ AN$$

SUBSTITUTING $6 = \frac{2}{3}\ BL$ AND $AO = \frac{2}{3} \times 12$

SO $BL = 9$ CM AND $AO = 8$ CM.

SINCE $BL = BO + OL$,

$OL = BL - BO = 9 - 6 = 3$ CM.

NOW, $AN = AO + ON$ GIVES

$ON = AN - AO = 12 - 8 = 4$ CM

$\therefore\ BL = 9$ CM, $OL = 3$ CM AND $ON = 4$ CM

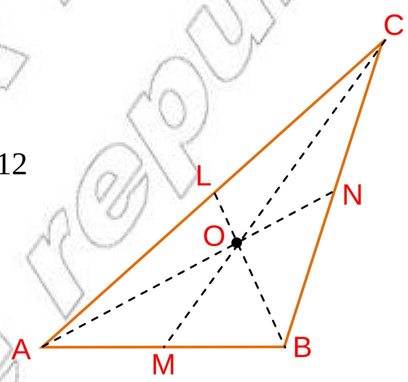


Figure 6.6

Note: THE POINT OF INTERSECTION OF THE MEDIANS OF A TRIANGLE IS CALLED THE ***centroid*** OF THE TRIANGLE.

Altitude of a triangle

THE ***altitude*** OF A TRIANGLE IS A LINE SEGMENT DRAWN FROM A VERTEX, PERPENDICULAR TO THE OPPOSITE SIDE, OR TO THE OPPOSITE SIDE PRODUCED.

THE ***altitudes*** THROUGH B AND A FOR THE TRIANGLES ARE SHOWN IN FIGURE 6.7.

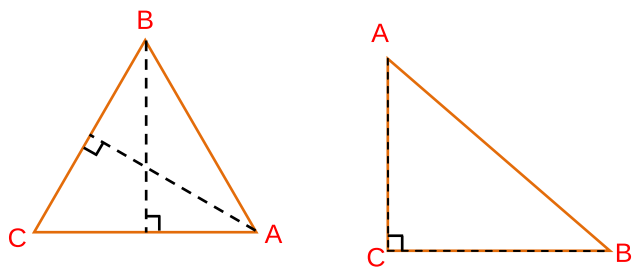


Figure 6.7

ACTIVITY 6.3

1. WHAT IS MEANT BY AN ANGLE BISECTOR?
2. ANY SIDE OF A TRIANGLE MAY BE DESIGNATED AS A BASE. HOW MANY BASES MAY A TRIANGLE HAVE?
3. HOW MANY ALTITUDES CAN A TRIANGLE HAVE?
4. BY DRAWING THE FOLLOWING TYPES OF TRIANGLE WITH THEIR RESPECTIVE ALTITUDES, DETERMINE WHETHER THE ALTITUDES INTERSECT INSIDE OR OUTSIDE THE TRIANGLE.
   - A AN ACUTE-ANGLED TRIANGLE; B AN OBTUSE-ANGLED TRIANGLE;
   - C A RIGHT-ANGLED TRIANGLE.
5. DRAW THE PERPENDICULAR BISECTORS OF THE SIDES OF THE FOLLOWING TRIANGLES, AND NOTE WHERE THE PERPENDICULAR BISECTORS INTERSECT.
   - A AN ACUTE-ANGLED TRIANGLE; B AN OBTUSE-ANGLED TRIANGLE;
   - C A RIGHT-ANGLED TRIANGLE.
6. DRAW ANY $\triangle ABC$. CONSTRUCT THE PERPENDICULAR BISECTORS OF THE SIDES $\overline{AB}$ AND $\overline{CB}$. LABEL THEIR INTERSECTION AS POINT O.
   - A WHY IS POINT O EQUIDISTANT FROM A AND B?
   - B WHY IS POINT O EQUIDISTANT FROM B AND C?
   - C DO YOU THINK THAT THE PERPENDICULAR BISECTOR OF THE SIDE $\overline{AC}$ PASSES THROUGH THE POINT O? (WHY?)

ACTIVITY 6.3 CAN HELP YOU TO STATE THE FOLLOWING THEOREM:

Theorem 6.2

The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

LET ΔABC BE GIVEN AND CONSTRUCT PERPENDICULAR BISECTORS ON ANY TWO OF THE SIDES. PERPENDICULAR BISECTORS OF $\overline{AB}$ AND $\overline{AC}$ ARE SHOWN IN FIGURE 6.8A. THESE PERPENDICULAR BISECTORS INTERSECT AT A POINT O, THEY CANNOT BE PARALLEL. (WHY?)

USING A RULER, FIND THE LENGTHS AO, BO AND CO. OBSERVE THAT THE INTERSECTION POINT O IS EQUIDISTANT FROM EACH VERTEX OF THE TRIANGLE.

NOTE THAT THE PERPENDICULAR BISECTOR OF THE REMAINING SIDE $\overline{BC}$ MUST PASS THROUGH THE POINT O. THEREFORE, THE POINT OF INTERSECTION OF THE THREE PERPENDICULAR BISECTORS IS EQUIDISTANT FROM THE THREE VERTICES OF ΔABC.

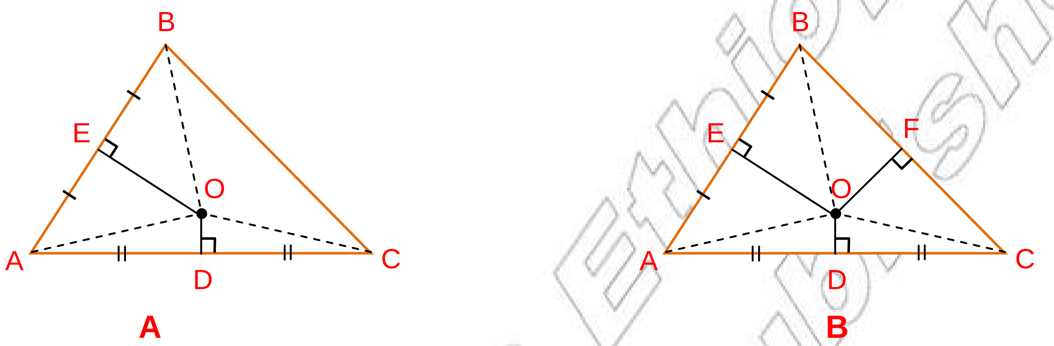


Figure 6.8

LET US TRY TO PROVE THIS RESULT.

WITH O THE POINT WHERE THE PERPENDICULAR BISECTORS OF $\overline{AB}$ AND $\overline{AC}$ MEET, AS SHOWN IN FIGURE 6.8B, $\Delta AOD \equiv \Delta COD$ BY SAS AND HENCE $\overline{AO} \equiv \overline{CO}$.

SIMILARLY, $\Delta AOE \equiv \Delta BOE$ BY SAS AND HENCE $\overline{AO} \equiv \overline{BO}$.

THUS, $\overline{AO} \equiv \overline{BO} \equiv \overline{CO}$. IT FOLLOWS THAT O IS EQUIDISTANT FROM THE VERTICES OF ΔABC.

NEXT, LET F BE THE FOOT OF THE PERPENDICULAR FROM O TO $\overline{BC}$. THEN, $\overline{OF}$ IS THE PERPENDICULAR BISECTOR OF $\overline{BC}$ BECAUSE ΔBOC IS AN ISOSCELES TRIANGLE.

THEREFORE, THE PERPENDICULAR BISECTORS OF THE SIDES OF ΔABC ARE CONCURRENT.

Note: THE POINT OF INTERSECTION OF THE PERPENDICULAR BISECTORS OF A TRIANGLE IS CALLED ***circumcentre*** OF THE TRIANGLE.

Theorem 6.3

The altitudes of a triangle are concurrent.

TO SHOW THAT THE THREE ALTITUDES OF ΔABC MEET AT A SINGLE POINT, CONSTRUCT $\Delta A'B'C'$ (SHOWN IN FIGURE 6.9) SO THAT THE THREE SIDES OF $\Delta A'B'C'$ ARE PARALLEL RESPECTIVELY TO THE THREE SIDES OF ΔABC:

LET $\overline{EA}$, $\overline{BF}$ AND $\overline{CD}$ BE THE ALTITUDES OF ΔABC.

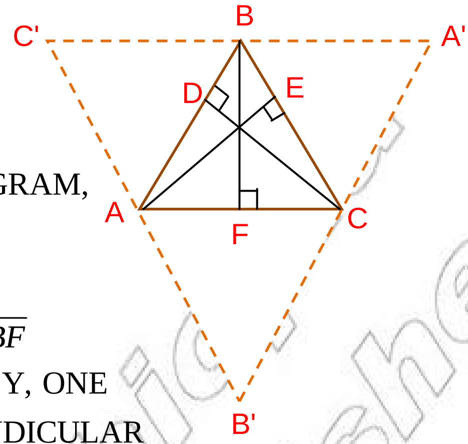


Figure 6.9

THE QUADRILATERALS $ABA'C$, $ABCB'$ AND $AC'BC$ ARE PARALLELOGRAMS. (WHY?)

SINCE $ABA'C$ IS A PARALLELOGRAM, $AC = BA'$. (WHY?) AGAIN, SINCE $ACBC'$ IS A PARALLELOGRAM, $AC = BC'$. THEREFORE, $BC' = BA'$ (WHY?) AND $\overline{BF}$ BISECTS $\overline{A'C'}$.

ACCORDINGLY, $\overline{BF}$ IS PERPENDICULAR TO $\overline{AC}$ AND SO $\overline{BF}$ IS THE PERPENDICULAR BISECTOR OF $\overline{A'C'}$. SIMILARLY, ONE CAN SHOW THAT $\overline{CD}$ AND $\overline{AE}$ ARE PERPENDICULAR BISECTORS OF $\overline{A'B'}$ AND $\overline{B'C'}$ RESPECTIVELY.

THEREFORE, THE ALTITUDES OF ΔABC ARE THE SAME AS THE PERPENDICULAR BISECTORS OF SIDES OF $\Delta A'B'C'$. SINCE THE PERPENDICULAR BISECTORS OF ANY TRIANGLE ARE CON (THEOREM 6.1), IT IS THEREFORE, TRUE THAT THE ALTITUDES OF ΔABC ARE CONCU

Note: THE POINT OF INTERSECTION OF THE ALTITUDES IS CALLED THE ***orthocentre*** OF THE TRIANGLE.

Angle bisector of a triangle

Theorem 6.4

The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

TO SHOW THAT THE ANGLE BISECTORS OF ΔABC MEET AT A SINGLE POINT, DRAW THE BISECTORS O $\angle A$ AND $\angle C$, INTERSECTING EACH OTHER AT O (FIGURE 6.10).

CONSTRUCT THE PERPENDICULARS $\overline{OA'}$, $\overline{OB'}$ AND $\overline{OC'}$.

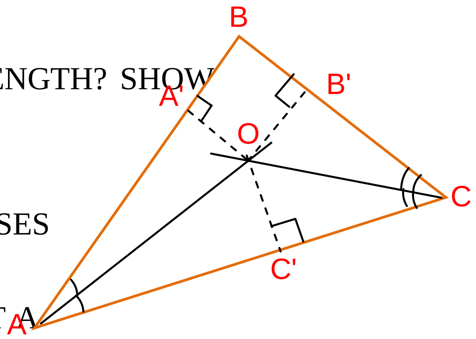


Figure 6.10

DO THESE SEGMENTS HAVE THE SAME LENGTH? SHOW THAT $\Delta OBB' \equiv \Delta OBA'$ AND CONCLUDE THAT

$\angle OBB' \equiv \angle OBA'$.

THEREFORE, THE BISECTOR OF $\angle B$ ALSO PASSES THROUGH THE POINT O.

THEREFORE, THE ANGLE BISECTORS OF ΔABC MEET AT A SINGLE POINT. ALSO THEIR POINT OF INTERSECTION IS EQUIDISTANT FROM THE THREE SIDES OF ΔABC.

Note: THE POINT OF INTERSECTION OF THE BISECTORS OF THE ANGLES OF A TRIANGLE IS CALLED THE ***incentre*** OF THE TRIANGLE.

EXAMPLE 2 IN A RIGHT ANGLE TRIANGLE ABC, $\angle C$ IS A RIGHT ANGLE, $AB = 8$ CM AND $CA = 6$ CM. FIND THE LENGTH OF $\overline{CO}$ WHERE O IS THE POINT OF INTERSECTION OF THE PERPENDICULAR BISECTORS OF AC AND BC.

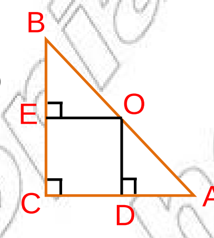


Figure 6.11

SOLUTION: THE PERPENDICULAR BISECTOR OF $\overline{AC}$ IS PARALLEL TO $\overline{CB}$.

HENCE, O IS ON $\overline{AB}$.

THEREFORE, $AO = 4$. (BY THEOREM 6.2, $AO = BO$)

BY THEOREM 6.2, O IS EQUIDISTANT FROM A, B AND C.

THEREFORE, $CO = AO = 4$ CM.

Group Work 6.1

WORK IN A SMALL GROUP ON ONE OR MORE OF THE FOLLOWING STATEMENTS. THERE WILL BE A CLASS DISCUSSION ON THESE, SO EACH ONE SHOULD BE ATTEMPTED BY AT LEAST ONE GROUP.

Task: CHECK THAT THE FOLLOWING STATEMENTS HOLD TRUE FOR ANY TRIANGLE BY CARRYING OUT THE CONSTRUCTION CAREFULLY.

Materials required: RULER, PROTRACTOR AND COMPASSES

Method: CONSTRUCTION AND MEASUREMENT

1. THE MEDIANS OF ANY TRIANGLE ARE CONCURRENT.
2. THE MEDIANS OF A TRIANGLE ARE CONCURRENT AT A POINT $\frac{2}{3}$ THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSTIE SIDE.
3. THE ALTITUDES OF ANY TRIANGLE ARE CONCURRENT.
4. THE PERPENDICULAR BISECTORS OF THE SIDES OF ANY TRIANGLE ARE CONCURRENT AT A POINT WHICH IS EQUIDISTANT FROM THE VERTICES OF THE TRIANGLE.
5. THE ANGLE BISECTORS OF ANY TRIANGLE ARE CONCURRENT AT A POINT WHICH IS EQUIDISTANT FROM THE SIDES OF THE TRIANGLE.
6. GIVEN ANY TRIANGLE, EXPLAIN HOW YOU CAN FIND THE CENTRE OF THE CIRCLE:
   - **A** INSCRIBED IN THE TRIANGLE (INCENTRE).
   - **B** CIRCUMSCRIBED ABOUT THE TRIANGLE (CIRCUMCENTRE).

Altitude theorem

THE ALTITUDE THEOREM IS STATED HERE FOR A RIGHT ANGLED TRIANGLE. IT RELATES THE LE ALTITUDE TO THE HYPOTENUSE OF A RIGHT ANGLED TRIANGLE, TO THE LENGTHS OF TH HYPOTENUSE.

Theorem 6.5 Altitude theorem

IN A RIGHT ANGLED TRIANGLE ABC WITH ALTITUDE $\overline{CD}$ TO THE HYPOTENUSE $\overline{AB}$,

$$\frac{AD}{DC} = \frac{CD}{DB}$$

Proof:-

CONSIDER ΔABC AS SHOWN IN FIGURE 6.12. $\Delta ABC \sim \Delta ACD$ . . . AA SIMILARITY

SO, $\angle ABC \equiv \angle ACD$

SIMILARLY, $\Delta ABC \sim \Delta CBD$ . . . AA SIMILARITY

SO, $\angle ABC \equiv \angle CBD$.

IT FOLLOWS THAT $\angle ACD \equiv \angle CBD$.

BY AA SIMILARITY, $\Delta ACD \sim \Delta CBD$.

HENCE, $\frac{AD}{CD} = \frac{CD}{BD}$ … (*)

EQUIVALENTLY, $\frac{AD}{DC} = \frac{CD}{DB}$

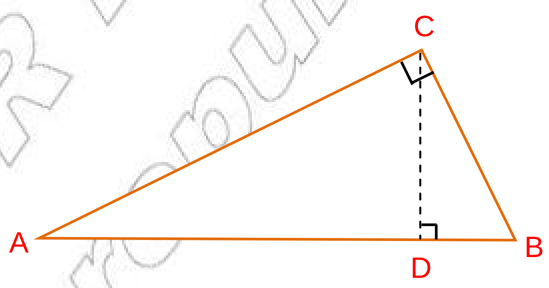


Figure 6.12

THE FOLLOWING ARE SOME FORMS OF THE ALTITUDE THEOREM.

FROM (*), $(CD)^2 = (AD)(BD)$

OR $(AD)(DB) = (CD)(DC)$

THIS CAN BE STATED AS:

THE SQUARE OF THE LENGTH OF THE ALTITUDE IS THE PRODUCT OF THE LEN SEGMENTS OF THE HYPOTENUSE.

EXAMPLE 3 IN ΔABC, $\overline{CD}$ IS THE ALTITUDE TO THE HYPOTENUSE $\overline{AB}$, $AD = 9$ CM AND $BD = 4$ CM. HOW LONG IS THE ALTITUDE $\overline{CD}$? See FIGURE 6.12.

SOLUTION LET $h = CD$. FROM THE ALTITUDE THEOREM, $(CD)^2 = (AD)(BD)$

SUBSTITUTING, $h^2 = 9 \times 4 = 36$ CM^2

SO, $h = 6$ CM.

THE LENGTH OF THE ALTITUDE IS 6 CM.

Menelaus' theorem

Menelaus' theorem WAS KNOWN TO THE ANCIENT GREEKS ALMOST TWO THOUSAND YEARS AGO. IT WAS NAMED IN HONOUR OF THE GREEK MATHEMATICIAN AND ASTRONOMER ***Menelaus*** ***(70 - 140 AD)***.

> **Theorem 6.6 Menelaus' theorem**
>
> If points *D, E* and *F* on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively of ΔABC (or their extensions) are collinear, then $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. Conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$, then the points *D*, *E* and *F* are collinear.

Note: **1** FOR A LINE SEGMENT AB, WE USE THE CONVENTION $AB = -BA$.

2 IF F IS IN $\overline{AB}$, THEN $\frac{AF}{FB} = r > 0$.

IN FIGURE 6.13, LET D DIVIDE $\overline{BC}$ IN THE RATIO r, E DIVIDE $\overline{CA}$ IN THE RATIO s AND F' DIVIDES $\overline{AB}$ IN THE RATIO t,

$$\text{I.E., } r = \frac{BD}{DC},\ s = \frac{CE}{EA} \text{ AND } t = \frac{AF}{FB}.$$

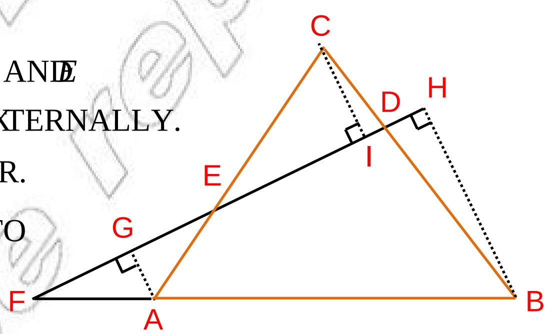


Figure 6.13

WE SEE FROM THE FIGURE THAT D DIVIDES $\overline{BC}$ AND E DIVIDES $\overline{CA}$ INTERNALLY, BUT F DIVIDES $\overline{AB}$ EXTERNALLY.

ASSUME THAT D, E AND F ARE COLLINEAR.

DRAW $\overline{AG}$, $\overline{BH}$, $\overline{CI}$ PERPENDICULAR TO $\overleftrightarrow{DF}$.

THEN, $\Delta CEI \sim \Delta AEG$ (WHY?),

$$\text{SO, } \frac{CE}{AE} = \frac{CI}{AG} \Rightarrow -\frac{CE}{EA} = \frac{CI}{AG}.$$

SIMILARLY, $\Delta AFG \sim \Delta BFH$ AND $\Delta BDH \sim \Delta CDI$

$$\text{SO, } \frac{AF}{BF} = \frac{AG}{BH},\ \frac{BD}{CD} = \frac{BH}{CI} \Rightarrow -\frac{AF}{FB} = \frac{AG}{BH},\ -\frac{BD}{DC} = \frac{BH}{CI}.$$

$$\text{HENCE, } rst = \left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right) = \left(\frac{-BH}{CI}\right)\left(\frac{-CI}{AG}\right)\left(\frac{-AG}{BH}\right) = -1$$

$$\text{THEREFORE, } \left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right) = -1$$

IT IS ALSO POSSIBLE FOR ALL THREE OF D, E AND F TO DIVIDE THEIR RESPECTIVE SIDES EXTERNALLY, AS YOU CAN SEE BY DRAWING A FIGURE. IN THIS CASE, r, s, t ARE ALL NEGATIVE. OTHERWISE THE PRECEDING PROOF WILL REMAIN UNCHANGED.

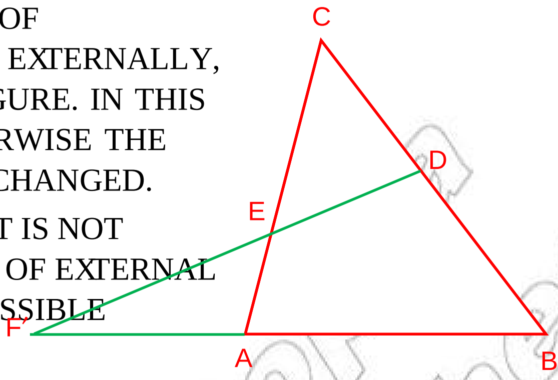


Figure 6.14

THEREFORE, rst = –1 IN THIS CASE ALSO. IT IS NOT POSSIBLE TO HAVE AN EVEN NUMBER OF EXTERNAL DIVISIONS, SO rst = –1 IN EACH OF THE POSSIBLE CASES.

TO PROVE THE CONVERSE OF MENELAUS' THEOREM, ASSUME THAT rst = –1.

EXTEND $\overline{DE}$ UNTIL IT INTERSECTS $\overleftrightarrow{AB}$ SAY AT A POINT F'. LET r' BE THE RATIO IN WHICH F' DIVIDES $\overline{AB}$, THEN $r'st$ = –1 (WHY?).

HENCE, $r' = r$ (WHY?)

SINCE F IS THE ONLY POINT THAT DIVIDES $\overline{AB}$ IN THE RATIO r, $F = F'$. THIS IMPLIES THAT D, E AND F ARE COLLINEAR.

Exercise 6.1

1 IN FIGURE 6.15, $\overline{AD} \equiv \overline{DC}$, $\overline{AE} \equiv \overline{EB}$, F IS THE INTERSECTION OF $\overline{CE}$ AND $\overline{BD}$. PROVE THAT $EF = \frac{1}{3}EC$.

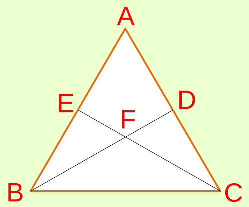


Figure 6.15

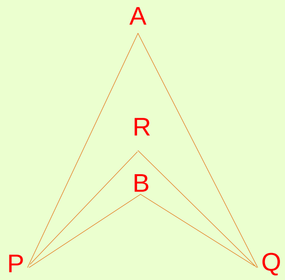


Figure 6.16

2 IN FIGURE 6.16, $\overline{RP}$ AND $\overline{RQ}$ ARE THE BISECTORS OF THE ANGLES APB AND AQB, RESPECTIVELY. IF $RP \equiv RQ$, PROVE THAT A, R, B LIE ON A STRAIGHT LINE.

Hint: JOIN P AND Q.

3 IF TWO MEDIANS OF A TRIANGLE ARE EQUAL, PROVE THAT THE TRIANGLE FORMED BY A SEGMENT OF EACH MEDIAN AND THE THIRD SIDE IS AN ISOSCELES TRIANGLE.

4 PROVE THAT THE SEGMENT JOINING THE MID-POINTS OF TWO SIDES OF A TRIANGLE IS PARALLEL TO THE THIRD SIDE AND IS HALF AS LONG AS THE THIRD SIDE.

5 **A** LET A (0, 0), B (6, 0) AND C (0, 4) BE VERTICES OF ΔABC.

I FIND THE POINT OF INTERSECTION OF THE MEDIANS OF ΔABC.

II SHOW THAT THE POINT OBTAINED IN **I** IS $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

B REPEAT 5A FOR ΔDEF WHERE D (0, 0), E (4, 0) AND F (2, 4) ARE THE VERTICES.

6 IN RIGHT ANGLED TRIANGLE ABC SHOWN IN FIGURE 6.17, $\overline{CD}$ IS ALTITUDE TO THE HYPOTENUSE $\overline{AB}$. IF AC = 5 UNITS AND AD = 4 UNITS, FIND THE LENGTH OF

A $\overline{BD}$ **B** $\overline{BC}$

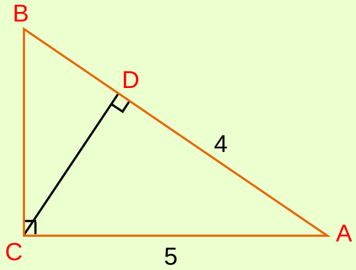


Figure 6.17

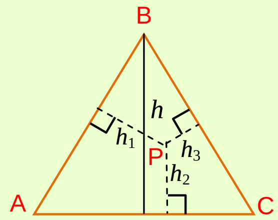


Figure 6.18

7 ***Altitude triangle for equilateral triangle:*** IN FIGURES 6.18, ΔABC IS AN EQUILATERAL TRIANGLE WITH ALTITUDE OF LENGTH h AND P IS AN INTERIOR POINT. THREE ALTITUDES OF LENGTHS h_1, h_2 AND h_3 ARE DRAWN FROM P TO THE SIDES OF THE TRIANGLE. SHOW THAT $h = h_1 + h_2 + h_3$.

Hint: COMPARE THE AREA OF ΔABC WITH THE SUM OF THE AREAS OF ΔAPC, ΔAPB AND ΔBPC.

IN PROBLEMS 8 – 10, THE LETTERS A, B, C, D, E, F, R, S, T HAVE THE SAME MEANING WHICH THEY HAVE IN THE STATEMENT OF MENELAUS' THEOREM.

8 IN FIGURE 6.19, D AND D' ARE SYMMETRICAL ABOUT THE MID-POINT OF $\overline{BC}$. E, E' AND F, F' ARE ALSO SYMMETRICAL ABOUT THE MID-POINTS OF THEIR CORRESPONDING SIDES. SHOW THAT D', E' AND F' ARE COLLINEAR IF D, E AND F ARE COLLINEAR.

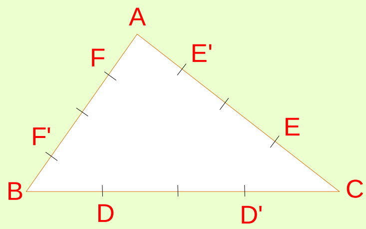


Figure 6.19

9 IN THE PROOF OF THE CONVERSE PART OF MENELAUS' THEOREM, ASSUME THAT $\overleftrightarrow{DE}$ MEETS $\overleftrightarrow{AB}$ AT SOME POINT F'.

A PROVE THAT IF $\overleftrightarrow{DE} // \overleftrightarrow{AB}$, THEN $rs = 1$.

B PROVE THAT IF $rst = -1$, THEN $\overleftrightarrow{DE}$ IS NOT PARALLEL TO $\overleftrightarrow{AB}$.

C PROVE THAT IF $rs = 1$, THEN $\overleftrightarrow{DE} // \overleftrightarrow{AB}$.

10 IN FIGURE 6.20 BELOW, D DIVIDES $\overline{BC}$ IN THE RATIO r AND D' DIVIDES $\overline{CB}$ IN THE SAME RATIO r. E IS THE MID-POINT OF $\overline{CA}$. D, E, F ARE COLLINEAR AND D', E, F' ARE ALSO COLLINEAR. SHOW THAT $AF = BF'$.

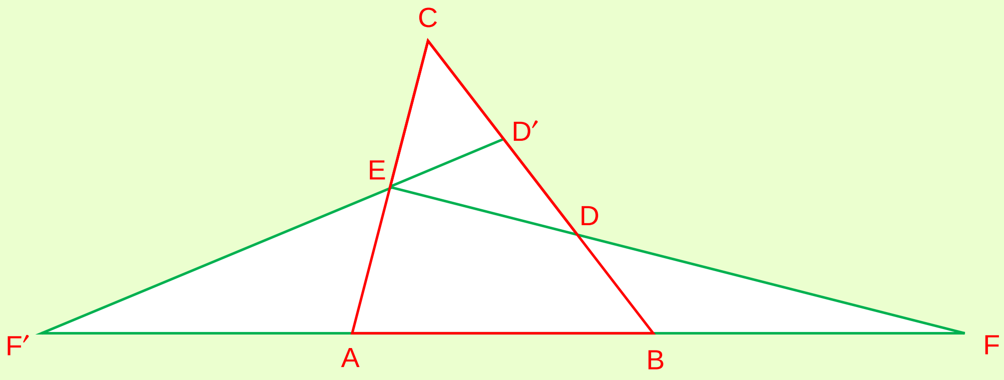


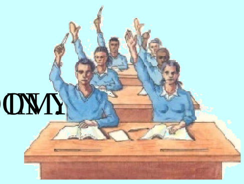
Figure 6.20

6.2 SPECIAL QUADRILATERALS

IN THIS SECTION, WE CONSIDER THE FOLLOWING SPECIAL QUADRILATERALS: ***trapezium***, ***parallelogram***, ***rectangle***, ***rhombus*** AND ***square***.

KEEP IN MIND THE MATHEMATICAL DEFINITIONS OF EACH OF THE ABOVE QUADRILATERALS.

ACTIVITY 6.4



1 DISCUSS PARALLEL LINES BASED ON WHAT YOU SEE IN YOUR CLASSROOM.

2 STATE THE PARALLEL LINES POSTULATE.

3 DISCUSS WHAT IS MEANT BY "EQUIANGULAR QUADRILATERAL" AND "EQUILATERAL QUADRILATERAL"?

4 DEFINE THE FOLLOWING QUADRILATERALS IN YOUR OWN TERMS.

A PARALLELOGRAM **B** RECTANGLE **C** SQUARE

5 WHAT IS AN ALTITUDE OF A PARALLELOGRAM?

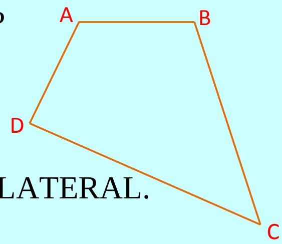


Figure 6.21

6 IN FIGURE 6.21,

I INDICATE A PAIR OF ADJACENT SIDES.

II INDICATE OPPOSITE VERTICES OF THE QUADRILATERAL.

III JOIN TWO OPPOSITE VERTICES.

WHAT DO YOU CALL THIS LINE SEGMENT?

7 WHAT IS A DIAGONAL OF A QUADRILATERAL? HOW MANY DIAGONALS DOES A PARALLELOGRAM OR RECTANGLE HAVE?

Trapezium

Definition 6.1

A ***trapezium*** is a quadrilateral where only two of the sides are parallel.

IN FIGURE 6.22, THE QUADRILATERAL *ABCD* IS A TRAPEZIUM. THE SIDES $\overline{AD}$ AND $\overline{BC}$ ARE NON-PARALLEL SIDES OF THE TRAPEZIUM *ABCD*.

Figure 6.22

NOTE THAT IF THE SIDES $\overline{AD}$ AND $\overline{BC}$ OF TRAPEZIUM *ABCD* ARE CONGRUENT, THEN THE TRAPEZIUM IS CALLED AN ***isosceles trapezium***.

Parallelogram

Definition 6.2

A ***parallelogram*** is a quadrilateral in which both pairs of opposite sides are parallel.

IN FIGURE 6.23, THE QUADRILATERAL *ABCD* IS A PARALLELOGRAM.

$\overline{AB} // \overline{DC}$ AND $\overline{AD} // \overline{BC}$

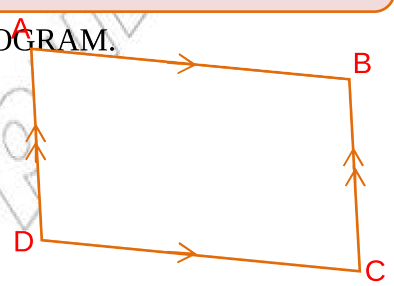


Figure 6.23

ACTIVITY 6.5

1 DRAW A QUADRILATERAL *ABCD*. LET *P*, *Q*, *R* AND *S* BE THE MID-POINTS OF ITS SIDES. CHECK, BY CONSTRUCTION AND MEASUREMENT THAT *PQRS* IS A PARALLELOGRAM.

2 DRAW A TRAPEZIUM *ABCD* WITH *AB* = 2 CM, *BC* = *DA* = 3 CM AND *DC* = 4 CM.

A INDICATE AND MEASURE THE BASE ANGLES OF TRAPEZIUM *ABCD*.

B DRAW THE DIAGONALS $\overline{DB}$ AND $\overline{AC}$ AND THEN MEASURE THEIR LENGTHS. ALSO, COMPARE THE LENGTHS OF THE TWO DIAGONALS.

3 DRAW A PARALLELOGRAM *ABCD* WITH *AB* = 3 CM AND *BC* = 8 CM.

A MARK POINTS ON $\overline{AB}$ THAT DIVIDE IT INTO THREE CONGRUENT PARTS. THROUGH THESE POINTS, DRAW LINES ACROSS *ABCD* PARALLEL TO $\overline{BC}$. WHY DO THESE LINES DIVIDE *ABCD* INTO THREE SMALLER PARALLELOGRAMS?

B MARK POINTS ON $\overline{BC}$ THAT DIVIDE IT INTO FOUR CONGRUENT SEGMENTS. THROUG THESE POINTS, DRAW LINES ACROSS $ABCD$ PARALLEL TO $\overline{AB}$. HOW MANY SMALL PARALLELOGRAMS DOES THIS MAKE?

C DRAW THE DIAGONALS OF ALL THE SMALLER PARALLELOGRAMS AND SHOW THAT THESE DIAGONALS ALSO FORM PARALLELOGRAMS.

PROPERTIES OF A PARALLELOGRAM AND TESTS FOR A QUADRILATERAL TO BE A PA STATED IN THE FOLLOWING THEOREM:

Theorem 6.7

A The opposite sides of a parallelogram are congruent.

B The opposite angles of a parallelogram are congruent.

C The diagonals of a parallelogram bisect each other.

D If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

E If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

F If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Proof of A and B:-

Given: PARALLELOGRAM $ABCD$

To prove: $\overline{AB} \equiv \overline{CD}$ AND $\overline{BC} \equiv \overline{DA}$

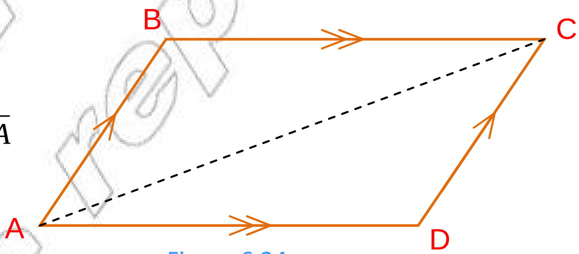


Figure 6.24

	Statement		Reason
1	DRAW DIAGONAL $\overline{AC}$	1	THROUGH TWO POINTS THERE IS EXACTLY ONE STRAIGHT LINE.
2	$\overline{AC} \equiv \overline{CA}$	2	COMMON SIDE.
3	$\angle CAB \equiv \angle ACD$ AND $\angle ACB \equiv \angle CAD$	3	ALTERNATE INTERIOR ANGLES OF PARALLEL LINES.
4	$\Delta ABC \equiv \Delta CDA$	4	ASA POSTULATE.
5	$\overline{AB} \equiv \overline{CD}$ AND $\overline{BC} \equiv \overline{DA}$, AND $\angle ABC \equiv \angle CDA$	5	CORRESPONDING PARTS OF CONGRUENT TRIANGLES

Can you show that $\angle BAD \equiv \angle DCB$?

Proof of C:-

Given: PARALLELOGRAM $ABCD$ WITH DIAGONALS $\overline{AC}$ AND $\overline{BD}$ INTERSECTING AT O.

To prove: $\overline{AO} \equiv \overline{OC}$ AND $\overline{BO} \equiv \overline{DO}$.

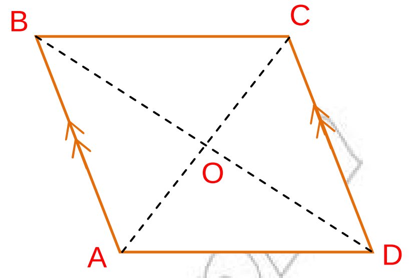


Figure 6.25

	Statement		Reason
1	$\overline{AB} \equiv \overline{CD}$	1	THEOREM 6.7A
2	$\angle CAB \equiv \angle ACD$ AND $\angle ABD \equiv \angle CDB$ HENCE, $\angle OAB \equiv \angle OCD$ AND $\angle ABO \equiv \angle CDO$	2	ALTERNATE INTERIOR ANGLES
3	$\Delta AOB \equiv \Delta COD$	3	ASA POSTULATE
4	$\overline{AO} \equiv \overline{CO}$ AND $\overline{BO} \equiv \overline{DO}$	4	CORRESPONDING PARTS OF CONGRUENT TRIANGLES.

Proof of F:-

Given: A QUADRILATERAL $ABCD$ WITH $\angle A \equiv \angle C$ AND $\angle B \equiv \angle D$.

To prove: $ABCD$ IS A PARALLELOGRAM

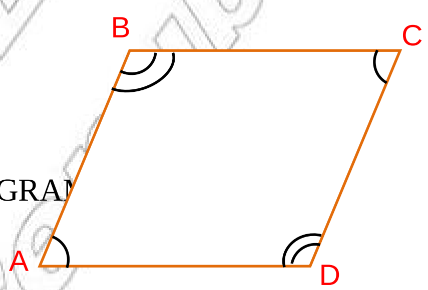


Figure 6.26

	Statement		Reason
1	$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^O$	1	THE SUM OF THE INTERIOR ANGLES OF A QUADRILATERAL IS 360^O
2	$m(\angle A) = m(\angle C)$ AND $m(\angle B) = m(\angle D)$	2	GIVEN
3	$2m(\angle A) + 2m(\angle D) = 360^O$	3	STEPS 1 AND 2
4	$m(\angle A) + m(\angle D) = 180^O$	4	SIMPLIFICATION
5	THEREFORE, $\overline{AB} // \overline{DC}$	5	$\angle A$ AND $\angle D$ ARE INTERIOR ANGLES ON THE SAME SIDE OF TRANSVERSAL $\overline{AD}$.
6	$m(\angle A) + m(\angle B) = 180^O$	6	STEP 2 AND 4.
7	THEREFORE, $\overline{AD} // \overline{BC}$	7	$\angle A$ AND $\angle B$ ARE INTERIOR ANGLES ON THE SAME SIDE OF TRANSVERSAL $\overline{AB}$.
8	$ABCD$ IS A PARALLELOGRAM	8	DEFINITION OF A PARALLELOGRAM STEPS 5 AND 7.

Rectangle

Definition 6.3

A ***rectangle*** is a parallelogram in which one of its angles is a right angle.

IN FIGURE 6.27, THE PARALLELOGRAM $ABCD$ IS A RECTANGLE WHOSE $\angle D$ IS A RIGHT ANGLE. WHAT IS THE MEASURE OF EACH OF THE OTHER ANGLES OF THE RECTANGLE $ABCD$?

Figure 6.27

Some properties of a rectangle

I A RECTANGLE HAS ALL PROPERTIES OF A PA

II EACH INTERIOR ANGLE OF A RECTANGLE IS A

III THE DIAGONALS OF A RECTANGLE ARE CONG

Rhombus

Definition 6.4

A ***rhombus*** is a parallelogram which has two congruent adjacent sides.

IN FIGURE 6.28, THE PARALLELOGRAM $ABCD$ IS A RHOMBUS.

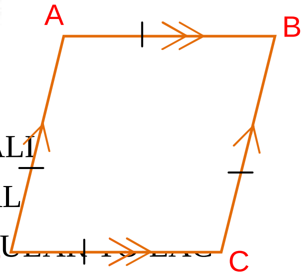


Figure 6.28

Some properties of a rhombus

i A RHOMBUS HAS ALL THE PROPERTIES OF A PARALL

ii A RHOMBUS IS AN EQUILATERAL QUADRILATERAL

iii THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR TO EACH OTHER.

iv THE DIAGONALS OF A RHOMBUS BISECT ITS ANGLES.

Square

Definition 6.5

A ***square*** is a rectangle which has congruent adjacent sides.

IN FIGURE 6.29, THE RECTANGLE $ABCD$ IS A SQUARE.

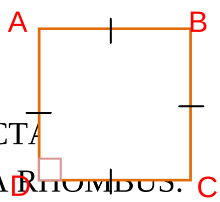


Figure 6.29

Some properties of a square

I A SQUARE HAS THE PROPERTIES OF A RECTA

II A SQUARE HAS ALL THE PROPERTIES OF A RHOMBUS.

Group Work 6.2

1. WHAT ARE SOME SIMILARITIES AND DIFFERENCES B… PARALLELOGRAM, A RECTANGLE AND A SQUARE?
2. IF $ABCD$ IS A PARALLELOGRAM WITH $AB = 3x + 4$, $BC = 2x + 7$ AND $CD = x + 18$, WHAT TYPE OF PARALLELOGRAM IS $ABCD$?
3. DISCUSS THE RELATIONSHIP AMONG THE FOUR TRIANGLES FORMED BY THE DIAGONALS OF A RHOMBUS.

Theorem 6.8

If the diagonals of a quadrilateral are congruent and are perpendicular bisectors of each other, then the quadrilateral is a square.

Proof:-

Given: $\overline{AC} \equiv \overline{BD}$; $\overline{AC}$ AND $\overline{BD}$ ARE PERPENDICULAR BISECTORS OF EACH O

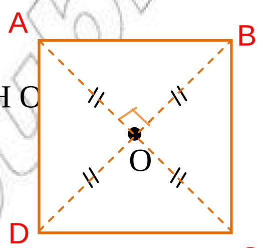


Figure 6.30

To prove: $ABCD$ IS A SQUARE.

LET O BE THE POINT OF INTERSECTION OF $\overline{AC}$ AND $\overline{BD}$.

	Statement		Reason
1	$\overline{AC} \equiv \overline{BD}$, $\overline{AC}$ AND $\overline{BD}$ A PERPENDICULAR BISECTORS OF	1	GIVEN
2	$\overline{AO} \equiv \overline{BO} \equiv \overline{CO} \equiv \overline{DO}$	2	STEP 1
3	$\angle AOB \equiv \angle BOC \equiv \angle COD \equiv \angle DOA$	3	ALL RIGHT ANGLES ARE CONGRUENT
4	$\Delta AOB \equiv \Delta BOC \equiv \Delta COD \equiv \Delta DOA$	4	SAS POSTULATE
5	$\angle CBD \equiv \angle ADB$ AND $\angle DCA \equiv \angle BAC$	5	CORRESPONDING ANGLES OF CONGRUENT TRIANGLES
6	$\overline{BC} // \overline{AD}$ AND $\overline{AB} // \overline{CD}$	6	ALTERNATE INTERIOR ANGLES ARE CONGRUENT
7	$ABCD$ IS A PARALLELOGRAM	7	DEFINITION OF A PARALLELOGRAM
8	$ABCD$ IS A RECTANGLE	8	DIAGONALS ARE CONGRUENT
9	$ABCD$ IS A SQUARE	9	DEFINITION OF A SQUARE, $\overline{AB} \equiv \overline{CD}$ AND STEP 4

Exercise 6.2

1 *ABCD* IS A PARALLELOGRAM; *P* IS THE MID-POINT OF $\overline{AB}$ AND *Q* IS THE MID-POINT OF $\overline{CD}$. PROVE THAT *APCQ* IS A PARALLELOGRAM.

2 THE MID-POINTS OF THE SIDES OF A RECTANGLE ARE THE VERTICES OF A QUADRILATERAL. WHAT KIND OF QUADRILATERAL IS IT? PROVE YOUR ANSWER.

3 THE MID-POINTS OF THE SIDES OF A PARALLELOGRAM ARE VERTICES OF A QUADRILATERAL. WHAT KIND OF QUADRILATERAL IS IT? PROVE YOUR ANSWER.

4 PROVE EACH OF THE FOLLOWING:

A IF THE DIAGONALS OF A PARALLELOGRAM ARE CONGRUENT, THEN THE PARALLELOGRAM IS A RECTANGLE.

B IF THE DIAGONALS OF A QUADRILATERAL BISECT EACH OTHER AND ONE ANGLE OF THE QUADRILATERAL IS A RIGHT ANGLE, THEN THE QUADRILATERAL IS A RECTAN

C IF ALL THE FOUR SIDES OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A RHOMBUS.

D THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR TO EACH OTHER.

5 IN EACH OF THE FOLLOWING STATEMENTS, CONDITIONS SUFFICIENT TO BE A PARALLELOGRAM ARE STATED. PROVE THIS IN EACH CASE.

A IF THE OPPOSITE SIDES OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A PARALLELOGRAM.

B IF ONE PAIR OF OPPOSITE SIDES OF A QUADRILATERAL IS CONGRUENT AND PARALLEL, THEN THE QUADRILATERAL IS A PARALLELOGRAM.

C IF THE DIAGONALS OF A QUADRILATERAL BISECT EACH OTHER, THEN THE QUADRILATERAL IS A PARALLELOGRAM.

6 DRAW A PARALLELOGRAM *ABCD*. EXTEND $\overline{AB}$ THROUGH *B* TO *P* SO THAT *AB* = *BP*; EXTEND $\overline{AD}$ THROUGH *D* TO *Q* SO THAT *AD* = *DQ*. PROVE THAT *C*, *P* AND *Q* ALL LIE ON ONE STRAIGHT LINE. (HINT: DRAW $\overline{BD}$)

7 M IS THE MID-POINT OF THE SIDE $\overline{BC}$ OF A PARALLELOGRAM *ABCD*. $\overline{DM}$ AND $\overline{AB}$ PRODUCED MEET AT N. PROVE THAT $\overline{AB} \equiv \overline{BN}$.

8 IF *ABCD* IS A PARALLELOGRAM WITH *M* AND *N* THE MID-POINTS OF $\overline{DC}$ AND $\overline{AB}$ RESPECTIVELY, PROVE THAT $\overline{AM} \equiv \overline{CN}$.

9 *ABCD* IS A PARALLELOGRAM WITH $\overline{AD}$ PRODUCED TO *F* AND $\overline{CB}$ PRODUCED TO *E* SUCH THAT $\overline{DF} \equiv \overline{BE}$. PROVE THAT *AECF* IS A PARALLELOGRAM.

6.3 MORE ON CIRCLES

IN THIS SECTION, YOU ARE GOING TO STUDY CIRCLES AND THE LINES AND ANGLES A THEM. OF ALL SIMPLE GEOMETRIC FIGURES, A CIRCLE IS PERHAPS THE MOST APPEALIN EVER CONSIDERED HOW USEFUL A CIRCLE IS? WITHOUT CIRCLES THERE WOULD BE WAGONS, AUTOMOBILES, STEAMSHIPS, ELECTRICITY OR MANY OTHER MODERN CONVEN

RECALL THAT A ***circle*** IS A PLANE FIGURE, ALL POINTS OF WHICH ARE EQUIDISTANT FROM A GIVEN POINT CALLED THE ***centre*** OF THE CIRCLE.

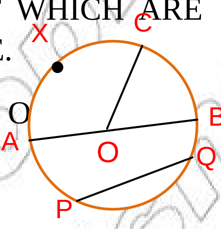


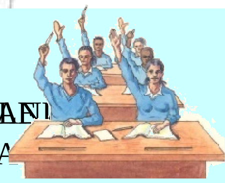
Figure 6.31

AS YOU RECALL FROM GRADE 9, IN FIGURE 6.31, $\overline{PQ}$ IS A CHORD OF THE CIRCLE WITH CENTRE O. $\overline{AB}$ IS A ***chord*** (DIAMETER). $\overset{\frown}{AXC}$ IS AN ***arc*** OF THE CIRCLE.

IF A AND C ARE NOT END-POINTS OF A DIAMETER, $\overset{\frown}{AXC}$ IS A MINOR ARC.

$\angle BOC$ IS A ***central angle***. $\overset{\frown}{AXC}$ OR ARC AXC IS SAID TO ***subtend*** $\angle AOC$ OR $\angle AOC$ ***intercepts*** ARC AXC.

ACTIVITY 6.6



1 DRAW A CIRCLE AND A LINE INTERSECTING THE CIRCLE AT TWO POINTS. DRAW ANOTHER LINE INTERSECTING AT ONE POINT. DRAW A LINE THAT DOES NOT INTERSECT THE CIRCLE.

2 IF THE LENGTH OF A RADIUS OF A CIRCLE IS r, THEN WHAT IS THE LENGTH OF ITS DIAMETER?

3 REFERRING TO FIGURE 6.32, ANSWER EACH OF THE FOLLOWING QUESTIONS:

A NAME AT LEAST THREE CHORDS, TWO SECANTS AND TWO TANGENTS.

B NAME THREE ANGLES FORMED BY TWO INTERSECTING CHORDS.

C NAME AN ANGLE FORMED BY TWO INTERSECTING TANGENTS.

D NAME AN ANGLE FORMED BY TWO INTERSECTING SECANTS.

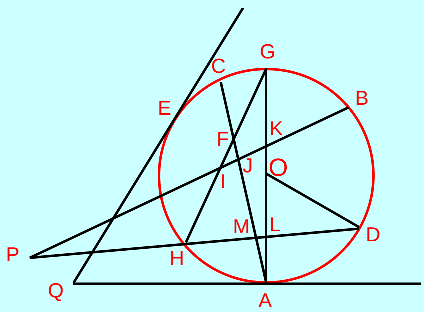


Figure 6.32

4 CONSTRUCT:

A A CENTRAL ANGLE OF 75^{o} IN A CIRCLE. **B** A CENTRAL ANGLE OF 120^{o} IN A CIRCLE.

5 HOW LARGE IS A CENTRAL ANGLE THAT IS SUBTENDED BY AN ARC OF LENGTH 3 CM IN A CIRCLE OF RADIUS 3 CM?

6 WHAT IS THE MEASURE OF A SEMI-CIRCLE AS AN ARC?

7 IS THE STATEMENT 'THE MEASURE OF AN ARC IS EQUAL TO THE MEASURE OF THE CORRESPONDING CENTRAL ANGLE' TRUE OR FALSE?

6.3.1 Angles and Arcs Determined by Lines Intersecting Inside and On a Circle

WE NOW EXTEND THE DISCUSSION TO ANGLES WHOSE VERTICES DO NOT NECESSARILY LIE AT THE CENTRE OF THE CIRCLE.

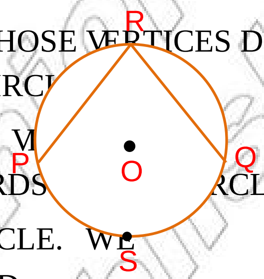


Figure 6.33

IN A CIRCLE, ***an inscribed angle*** IS AN ANGLE WHOSE VERTEX LIES ON THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE.

IN FIGURE 6.33, ANGLE PRQ IS INSCRIBED IN THE CIRCLE. WE ALSO SAY THAT $\angle PRQ$ IS INSCRIBED IN THE ARC PRQ AND $\angle PRQ$ IS ***subtended*** BY ARC PSQ (OR $\widehat{PSQ}$).

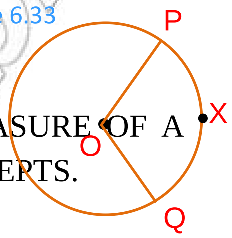


Figure 6.34

MEASURE OF A CENTRAL ANGLE: NOTE THAT THE MEASURE OF A CENTRAL ANGLE IS THE MEASURE OF THE ARC IT INTERCEPTS.

SO, $m(\angle POQ) = m(\widehat{PXQ})$.

Theorem 6.9

THE MEASURE OF AN ANGLE INSCRIBED IN A CIRCLE IS HALF THE MEASURE OF THE ARC SUBTENDING IT.

Proof:-

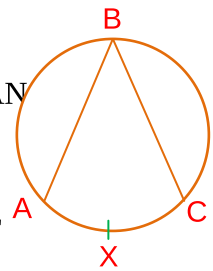


Figure 6.35

Given: CIRCLE O WITH $\angle B$ AN INSCRIBED ANGLE INTERCEPTING $\widehat{AC}$.

To prove: $m(\angle ABC) = \frac{1}{2} m(\widehat{AXC})$, WHERE X IS A POINT AS SHOWN IN FIGURE 6.35.

TO PROVE THEOREM 6.9, WE CONSIDER THREE CASES.

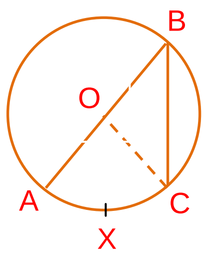


Figure 6.36

Case 1: SUPPOSE THAT ONE SIDE OF $\angle ABC$ IS A DIAMETER OF THE CIRCLE WITH CENTRE O.

Statement		Reason	
1	DRAW RADIUS $\overline{OB}$	1	CONSTRUCTION.
2	$\overline{OC} \equiv \overline{OB}$	2	RADII OF THE SAME CIRCLE.
3	$\angle OBC \equiv \angle OCB$	3	BASE ANGLES OF AN ISOSCELES TRIANGLE.
4	$\angle AOC \equiv \angle OCB + \angle OBC$	4	AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO TH SUM OF THE TWO OPPOSITE INTERIOR ANGLES.
5	$m(\angle AOC) = 2m(\angle ABC)$	5	SUBSTITUTION.
6	BUT $m(\angle AOC) = m(\overset{\frown}{AXC})$	6	$\angle AOC$ IS A CENTRAL ANGLE.
7	$2m(\angle ABC) = m(\overset{\frown}{AXC})$	7	SUBSTITUTION.
8	$m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$	8	DIVISION OF BOTH SIDES BY 2.

THEREFORE, $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$

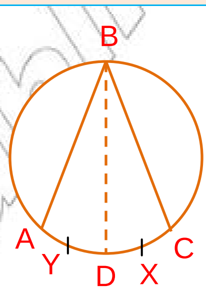


Figure 6.37

Case 2: SUPPOSE THAT A AND C ARE ON OPPOSITE SIDES OF THE DIAMETER THROUGH B, AS SHOWN IN FIGURE 6.37.

Statement		Reason	
1	$m(\angle ABD) = \frac{1}{2} m(\overset{\frown}{AYD})$	1	CASE 1
2	$m(\angle DBC) = \frac{1}{2} m(\overset{\frown}{DXC})$	2	CASE 1
3	$m(\angle ABD) + m(\angle DBC) = \frac{1}{2}(\overset{\frown}{AYD}) + \frac{1}{2} m(\overset{\frown}{DXC})$	3	ADDITION
4	$\therefore m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$	4	SUBSTITUTION

THEREFORE, $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$

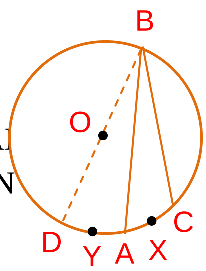


Figure 6.38

Case 3: SUPPOSE THAT A AND C ARE ON THE SA SIDE OF THE DIAMETER THROUGH B, AS SHOWN IN FIGURE 6.38.

	Statement		Reason
1	$m(\angle DBC) = \frac{1}{2}m(\overset{\frown}{DAC})$	1	CASE 1
2	$m(\angle DBA) = \frac{1}{2}m(\overset{\frown}{DYA})$	2	CASE 1
3	$m(\angle DBC) - m(\angle DBA) = \frac{1}{2}m(\overset{\frown}{DAC}) - \frac{1}{2}m(\overset{\frown}{DYA})$	3	ADDITION
4	$\therefore m(\angle ABC) = \frac{1}{2}m(\overset{\frown}{AXC})$	4	SUBSTITUTION

THEREFORE, $m(\angle ABC) = \frac{1}{2}m(\overset{\frown}{AXC})$ IN ALL CASES AND THE THEOREM HOLDS.

EXAMPLE 1 IN FIGURE 6.39, $m(\overset{\frown}{PXQ}) = 110^{o}$. FIND THE MEASURE OF $\angle PRQ$.

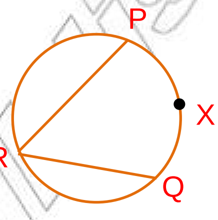


Figure 6.39

SOLUTION: BY THEOREM 6.9, WE HAVE

$$m(\angle PRQ) = \frac{1}{2}m(\overset{\frown}{PXQ}) = \frac{1}{2}(110^{o}) = 55^{o}$$

Corollary 6.9.1

An angle inscribed in a semi-circle is a right angle.

Proof:-

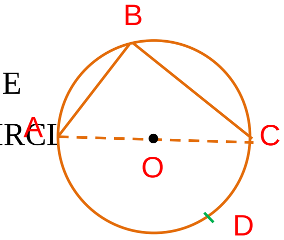


Figure 6.40

IN FIGURE 6.40, $\angle ABC$ IS INSCRIBED IN SEMI-CIRCLE ABC.

$\angle ABC$ IS SUBTENDED BY ARC ADC, WHICH IS A SEMI-CIRCLE.

THE MEASURE OF $\overset{\frown}{ADC}$ IS 180^{o} OR π RADIANS.

BY THEOREM 6.9, $m(\angle ABC) = \frac{1}{2}m(\overset{\frown}{ADC})$

$$= \frac{1}{2}(180^{o}) = 90^{o} \text{ OR } \frac{\pi}{2} \text{ RADIANS.}$$

Corollary 6.9.2

An angle inscribed in an arc less than a semi-circle is obtuse.

Proof:-

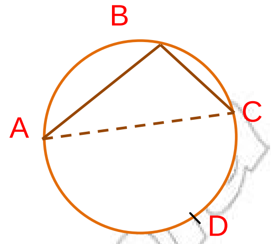


Figure 6.41

$$m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{ADC})$$

BUT $m(\overset{\frown}{ABC})$ < LENGTH OF A SEMI-CIRCLE

$m(\overset{\frown}{ABC}) < 180^{O}$

THEREFORE, $m(\overset{\frown}{ADC}) > 180^{O}$

$$m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{ADC}) > \frac{1}{2}(180^{O})$$

$m(\angle ABC) > 90^{O}$. SO, $\angle ABC$ IS AN OBTUSE ANGLE.

Corollary 6.9.3

An angle inscribed in an arc greater than a semi-circle is ***acute***.

Theorem 6.10

Two parallel lines intercept congruent arcs on the same circle.

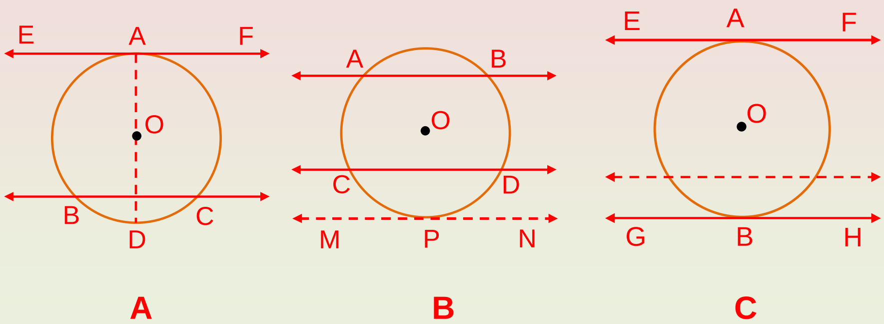


Figure 6.42

Proof:-

TO PROVE THIS FACT, YOU HAVE TO CONSIDER THE FOLLOWING THREE POSSIBLE CASES:

A WHEN ONE OF THE PARALLEL LINES $\overleftrightarrow{EF}$ IS A TANGENT LINE AND THE OTHER $\overleftrightarrow{BC}$ IS A SECANT LINE AS SHOWN IN FIGURE 6.42A.

B WHEN BOTH PARALLEL LINES $\overleftrightarrow{AB}$ AND $\overleftrightarrow{CD}$ ARE SECANTS AS SHOWN IN FIGURE 6.42B.

C WHEN BOTH PARALLEL LINES $\overleftrightarrow{EF}$ AND $\overleftrightarrow{GH}$ ARE TANGENTS AS SHOWN IN FIGURE 6.42C.

Case a:

Given: A CIRCLE WITH CENTRE O, $\overleftrightarrow{EF}$ AND $\overleftrightarrow{BC}$ ARE TWO PARALLEL LINES SUCH THAT $\overleftrightarrow{EF}$ IS A TANGENT TO THE CIRCLE AND $\overleftrightarrow{BC}$ IS A SECANT.

To prove: $\overset{\frown}{AB} \equiv \overset{\frown}{AC}$

	Statement		Reason
1	DRAW DIAMETER $\overline{AE}$	**1**	CONSTRUCTION.
2	$\overline{AD} \perp \overline{EF}$ AND $\overline{AD} \perp \overline{BC}$	**2**	A TANGENT IS PERPENDICULAR TO THE DIAMETER DRAWN TO THE POINT OF TANGENCY AND $\overline{EF} \parallel \overline{BC}$ IS ALSO GIVEN.
3	$\overset{\frown}{BD} \equiv \overset{\frown}{CD}$	**3**	ANY PERPENDICULAR FROM THE CENTRE OF A CIRCLE TO A CHORD BISECTS THE CHORD AND THE ARC SUBTENDED BY IT.
4	$\overset{\frown}{AB} \equiv \overset{\frown}{AC}$	**4**	$\overset{\frown}{ABD} \equiv \overset{\frown}{ACD}$ (SEMICIRCLES) AND STEP 3.

PROOFS OF CASE B AND CASE C ARE LEFT AS EXERCISES.

Theorem 6.11

An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.

Given: CIRCLE O WITH $\angle ABC$ FORMED BY TANGENT t AND CHORD $\overline{AB}$ AT B, THE POINT OF CONTACT.

To prove: $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXB})$

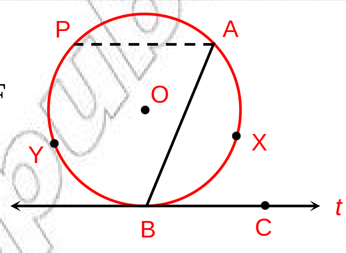


Figure 6.43

	Statement		Reason
1	DRAW $\overline{AP}$ PARALLEL TO t	**1**	CONSTRUCTION.
2	$\angle PAB \equiv \angle ABC$	**2**	ALTERNATE INTERIOR ANGLES OF PARALLEL LINES
3	$m(\angle PAB) = \frac{1}{2} m(\overset{\frown}{PYB})$	**3**	THEOREM 6.9.
4	BUT $\overset{\frown}{PYB} \equiv \overset{\frown}{AXB}$	**4**	THEOREM 6.10.
5	$\therefore m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXB})$	**5**	SUBSTITUTION FROM STEPS 2 – 4.

Theorem 6.12

The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

Proof:-

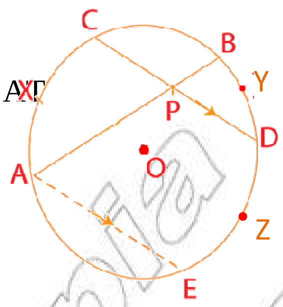


Figure 6.44

Given: TWO LINES $\overleftrightarrow{AB}$ AND $\overleftrightarrow{CD}$ INTERSECTING AT P INSIDE THE CIRCLE.

To prove: $m(\angle BPD) = \frac{1}{2}m(\overparen{AXC}) + \frac{1}{2}m(\overparen{BYD})$.

Statement		Reason	
1	DRAW A LINE THROUGH A SUCH THAT $\overline{AE} \parallel \overline{CD}$	1	CONSTRUCTION
2	$m(\angle BPD) = m(\angle BAE)$	2	CORRESPONDING ANGLES FORMED BY TWO PARALLEL LINES AND A TRANSVERSAL LINE.
3	$m(\angle BAE) = \frac{1}{2}m(\overparen{BDE})$	3	THEOREM 6.9
4	$\overparen{AXC} \equiv \overparen{DZE}$	4	THEOREM 6.10
5	$\therefore m(\angle BPD) = \frac{1}{2}m(\overparen{BDE})$ $= \frac{1}{2}m(\overparen{BYD}) + \frac{1}{2}m(\overparen{DZE})$	5	THEOREM 6.11
6	$m(\angle BPD) = \frac{1}{2}m(\overparen{BYD}) + \frac{1}{2}m(\overparen{AXC})$	6	SUBSTITUTION AND STEP 4.

THEREFORE, $m(\angle BPD) = \frac{1}{2}\left[m(\overparen{AXC}) + m(\overparen{BYD})\right]$

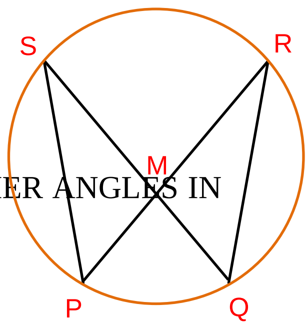


Figure 6.45

EXAMPLE 2 IN FIGURE 6.45, $m(\angle MRQ) = 30^O$, AND $m(\angle MQR) = 40^O$.

WRITE DOWN THE MEASURE OF ALL THE OTHER ANGLES IN THE TWO TRIANGLES ΔPSM AND ΔQMR. WHAT DO YOU NOTICE ABOUT THE TWO TRIANGLES?

SOLUTION: $m(\angle QMR) = 180^O - (30^O + 40^O)$ (WHY?)

$= 180^O - 70^O = 110^O$

$m(\angle RQS) = \frac{1}{2}m(\overparen{RS})$

THEREFORE, $40^O = \frac{1}{2}m(\overparen{RS})$

$\therefore m(\overparen{RS}) = 80^O$

$$m(\angle PRQ) = \frac{1}{2} m(\overset{\frown}{PQ})$$

HENCE, $30^{O} = \frac{1}{2} m(\overset{\frown}{PQ})$

$\therefore m(\overset{\frown}{PQ}) = 60^{O}$

$$m(\angle PSQ) = \frac{1}{2} m(\overset{\frown}{PQ}) = \frac{1}{2}(60^{O}) = 30^{O}$$

$$m(\angle RPS) = \frac{1}{2} m(\overset{\frown}{RS}) = \frac{1}{2}(80^{O}) = 40^{O}$$

THE TWO TRIANGLES ARE SIMILAR BY AA SIMILARITY.

EXAMPLE 3 AN ANGLE FORMED BY TWO CHORDS INTERSECTING WITHIN A CIRCLE IS 48^{O} AND ONE OF THE INTERCEPTED ARCS MEASURES 42^{O}. FIND THE MEASURES OF THE OTHER INTERCEPTED ARC.

SOLUTION: CONSIDER FIGURE 6.46.

$$m(\angle PRB) = \frac{1}{2} m(\overset{\frown}{PB}) + \frac{1}{2} m(\overset{\frown}{AQ}) \text{ (by THEOREM 6.11)}$$

$$48^{O} = \frac{1}{2}(42^{O}) + \frac{1}{2}(\overset{\frown}{AQ})$$

$$\Rightarrow 96^{O} = 42^{O} + m(\overset{\frown}{AQ})$$

$$\therefore 54^{O} = m(\overset{\frown}{AQ})$$

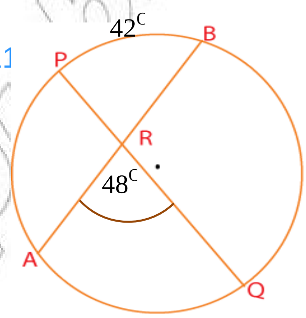


Figure 6.46

Remark: THE FOLLOWING RESULT IS SOMETIMES CALLED THE ***product of rectangle property of a circle***.

IF TWO CHORDS INTERSECT IN A CIRCLE AS SHOWN IN FIGURE 6.47, THEN $(AP)(PB) = (XP)(PY)$.

HINT FOR PROOF:

1	$\angle XAP \equiv \angle BYP$ AND $\angle AXP \equiv \angle YBP$	(WHY?)
2	$\Delta PAX \sim \Delta PYB$	(WHY?)
3	$\frac{AP}{YP} = \frac{PX}{PB}$	(WHY?)
4	$\therefore (AP)(PB) = (YP)(PX)$	(WHY?)

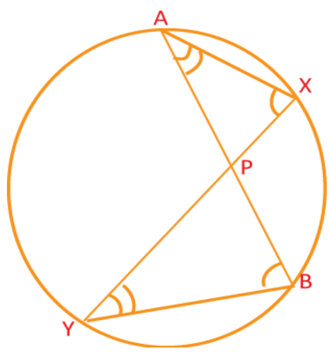


Figure 6.47

EXAMPLE 4 IN FIGURE 6.48, CALCULATE THE RADIUS OF THE CIRCLE.

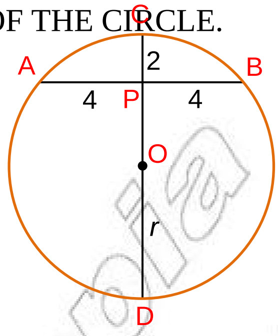


Figure 6.48

SOLUTION: LET THE RADIUS OF THE CIRCLE BE r UNITS LONG.

THEN, $OD = r$ AND $PD = 2r - 2$.

SINCE $(AP)(PB) = (CP)(PD)$, YOU HAVE

$$4 \times 4 = 2(2r - 2)$$

$$16 = 4r - 4$$

$$r = 5$$

Group Work 6.3

1 IN FIGURE 6.49, $\overline{AB}$ AND $\overline{PQ}$ ARE PARALLEL, $m(\angle BOQ) = 70^{O}$ AND O IS THE CENTRE OF THE CIRCLE. WHAT IS THE MEASURE OF $\angle AOP$?

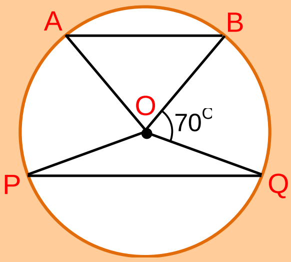


Figure 6.49

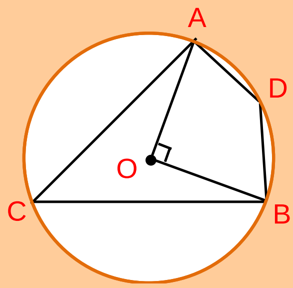


Figure 6.50

2 IN FIGURE 6.49, IF $PO = 5$ UNITS AND $m(\widehat{PQ}) = 120^{o}$, FIND THE LENGTH OF $\overline{PQ}$.

3 IN FIGURE 6.50, IF CENTRAL ANGLE AOB IS A RIGHT ANGLE.

A WHAT ARE THE DEGREE MEASURES OF $\angle ACB$ AND $\angle ADB$?

B FIND THE DEGREE MEASURE OF $\angle CBO$, IF $m(\angle CAO) = 20°$.

Exercise 6.3

1 IN FIGURE 6.51, $\overline{AB}$ IS A DIAMETER, O IS THE CENTRE OF THE CIRCLE, $\overline{OC} \parallel \overline{BD}$ AND $m(\angle ABD) = 60^{O}$, FIND $m(\angle OCD)$.

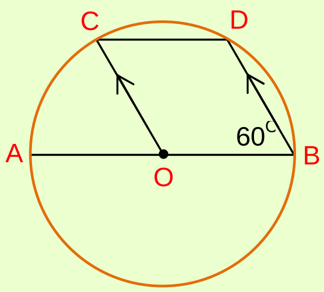


Figure 6.51

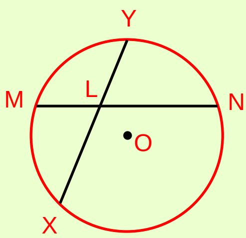


Figure 6.52

2 Prove that, if an angle inscribed in an arc of a circle is a right angle, then the arc is a semicircle.

3 In Figure 6.52, $\overset{\frown}{MX}$ is an arc of 28^{O}, and $\overset{\frown}{YN}$ is an arc of 50^{O}.

A What is the degree measure of $\angle YLN$?

B If $ML = 4$ units, $LX = 5$ units and $LN = 7$ units, find YL

4 In Figure 6.52 of Question 3, would it be possible for MLX to be a 30^{O} angle and for the measure of $\overset{\frown}{MX}$ to be 40^{O}? If so, what would be the measure of $\overset{\frown}{YN}$?

5 In Figure 6.53, O is the centre of the circle. If $m(\angle AOB) = 40^{O}$ and $m(\angle COD) = 60^{O}$, find

A $m(\angle AQB)$ **B** $m(\angle APB)$?

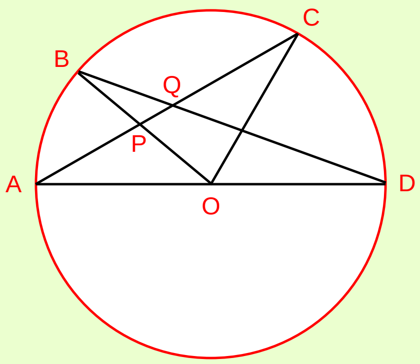


Figure 6.53

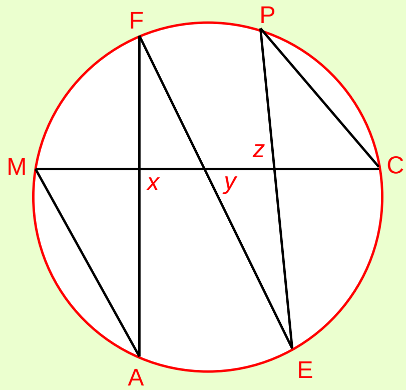


Figure 6.54

6 In Figure 6.54, if $m(\angle FAM) = 40^{O}$ and $m(\angle CPE) = 50^{O}$, what is the degree measure of $\angle EYC$?

7 **A** In Figure 6.55, the vertices of quadrilateral $ABCD$ lie on the circle O. Such a quadrilateral is called ***cyclic quadrilateral***.

I What is the sum of the measure of arcs ABC and ADC ?

II Prove that opposite angles of a cyclic quadrilateral are supplementary.

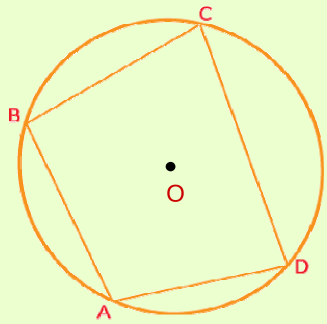


Figure 6.55

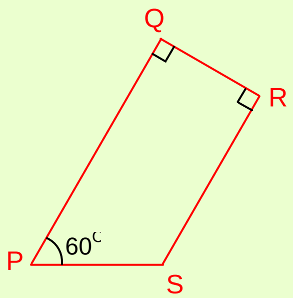


Figure 6.56

B In Figure 6.56, is there a circle containing P, Q, R and S?

8 IN FIGURE 6.57, FIND THE VALUES OF x AND y GIVEN THAT O IS THE CENTRE OF THE CIRCLE AND $m(\angle AOC) = 160^{O}$

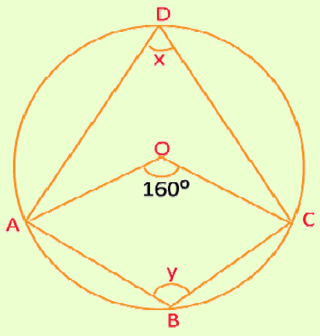


Figure 6.57

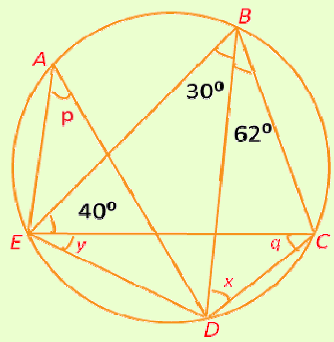


Figure 6.58

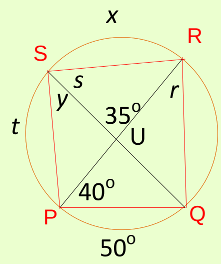


Figure 6.59

9 IN FIGURE 6.58, CALCULATE THE ANGLES MARKED p, q, x AND y.

10 FIND THE VALUES OF THE ANGLES MARKED x, y, s, r AND t AS SHOWN IN FIGURE 6.59.

6.3.2 Angles and Arcs Determined by Lines Intersecting Outside a Circle

WHAT HAPPENS IF TWO SECANT LINES INTERSECT OUTSIDE A CIRCLE? IN FIGURE 6.60, $\overleftrightarrow{AB}$ AND $\overleftrightarrow{XY}$ INTERSECT AT P OUTSIDE THE CIRCLE. THEY INTERCEPT ARCS BY AND AX. DRAW THE CHORD $\overline{AC}$ PARALLEL TO $\overleftrightarrow{XY}$. CAN YOU SEE THAT THE MEASURE OF $\angle APX$ IS HALF THE DIFFERENCE BETWEEN THE MEASURES OF ARCS BY AND AX? CAN YOU PROVE IT?

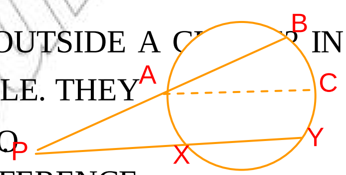


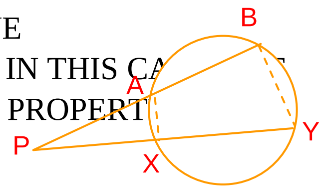
Figure 6.60

THIS IS STATED IN THEOREM 6.13.

Theorem 6.13

The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept.

THE PRODUCT PROPERTY $(PA)(PB) = (PX)(PY)$ IS ALSO TRUE WHEN TWO CHORDS INTERSECT OUTSIDE A CIRCLE. IN THIS CASE THE PROOF IS SIMILAR TO THE PROOF OF THE PRODUCT PROPERTY IN SECTION 6.3.1



DRAW $\overline{AX}$ AND $\overline{BY}$. TWO SIMILAR TRIANGLES ARE FORMED.

Figure 6.61

BY CONSIDERING CORRESPONDING SIDES, WE SEE THAT

$$(PA)(PB) = (PX)(PY).$$

Can you point out the similar triangles, in Figure 6.61 and put in the other details?

Theorem 6.14

The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

Proof:-

Given: SECANT $\overline{PBA}$ AND TANGENT $\overline{PD}$ INTERSECTING AT P

To prove: $m(\angle P) = \frac{1}{2}[m(\overset{\frown}{AXD}) - m(\overset{\frown}{BD})]$

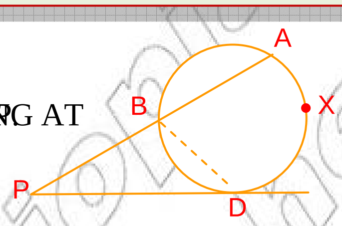


Figure 6.62

	Statement		Reason
1	DRAW $\overline{BD}$	1	CONSTRUCTION.
2	$\angle ABD \equiv \angle BDP + \angle DPA$	2	AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO THE SUM OF THE TWO OPPOSITE INTERIOR ANGLES OF A TRIANGLE.
3	$\angle ABD - \angle BDP \equiv \angle DPA \equiv \angle P$	3	SUBTRACTION.
4	$m(\angle ABD) = \frac{1}{2} m(\overset{\frown}{AXD})$ AND $m(\angle PDB) = \frac{1}{2} m(\overset{\frown}{BD})$	4	THEOREM 6.9 AND THEOREM 6.11.
5	$m(\angle ABD) - m(\angle BDP) = \frac{1}{2} m(\overset{\frown}{AXD}) - \frac{1}{2} m(\overset{\frown}{BD})$	5	SUBSTITUTION.
6	$\therefore m(\angle P) = \frac{1}{2} m(\overset{\frown}{AXD}) - \frac{1}{2} m(\overset{\frown}{BD})$	6	SUBSTITUTION.

Theorem 6.15

If a secant and a tangent are drawn from a point outside a circle, then the square of the length of the tangent is equal to the product of the lengths of line segments given by

$$(PA)^2 = (PB)(PC).$$

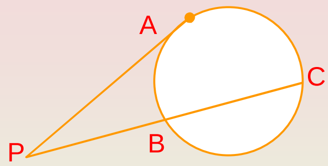


Figure 6.63

Proof:-

Given: A CIRCLE WITH SECANT $\overline{PC}$ AND TANGENT $\overline{PA}$ AS IN FIGURE 6.64

To prove: $(PA)^2 = (PB)\,(PC)$

DRAW $\overline{AB}$ AND $\overline{CA}$. THEN $\Delta PCA \sim \Delta PAB$ (SHOW!)

HENCE, $\frac{PC}{PA} = \frac{PA}{PB}$ AND $(PA)^2 = (PB)\,(PC)$

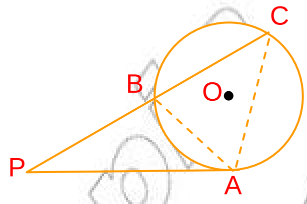


Figure 6.64

EXAMPLE 5 IN FIGURE 6.65, FROM P SECANTS $\overline{PA}$ AND $\overline{PC}$ ARE DRAWN SO THAT $m(\angle APC) = 30^O$; CHORDS $\overline{AB}$ AND $\overline{CD}$ INTERSECT AT F SUCH THAT $m(\angle AFC) = 85^O$. FIND THE MEASURE OF ARC AC, MEASURE OF ARC BD AND MEASURE OF $\angle ABC$.

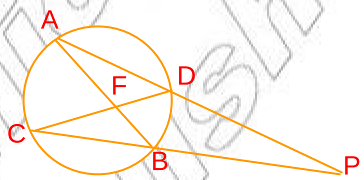


Figure 6.65

SOLUTION: LET $m(\overset{\frown}{AC}) = x$ AND $m(\overset{\frown}{DB}) = y$

SINCE $m(\angle AFC) = \frac{1}{2} m(\overset{\frown}{AC}) + \frac{1}{2} m(\overset{\frown}{BD})$

$$85^O = \frac{1}{2}(x + y)$$

$$x + y = 170^O \text{ . . . . (1)}$$

AGAIN AS $m(\angle APC) = \frac{1}{2} m(\overset{\frown}{AC}) - \frac{1}{2} m(\overset{\frown}{BD})$

$$30^O = \frac{1}{2}(x - y)$$

$$x - y = 60^O \text{ . . . . (2)}$$

SOLVING *equation 1* AND *equation 2* SIMULTANEOUSLY, WE GET

$$\begin{cases} x + y = 170^O \\ x - y = 60^O \end{cases}$$

$$2x = 230^O$$

$$x = 115^O$$

SUBSTITUTING x IN EQUATION 2,

$$115^O - y = 60^O$$

$$y = 55^O$$

THEREFORE, $m(\overset{\frown}{AC}) = 115^O$ AND $m(\overset{\frown}{DB}) = 55^O$.

$$m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AC}) = \frac{1}{2}(115^O) = 57.5^O.$$

Group Work 6.4

1 $\overline{AB}$ IS A DIAMETER OF A CIRCLE CENTRED AT O. C IS A POINT ON THE CIRCUMFERENCE. D IS A POINT ON $\overline{AC}$ SUCH THAT $\overline{OD}$ BISECTS $\angle AOC$. PROVE THAT $\overline{OD}$ IS PARALLEL TO $\overline{BC}$.

2 IN FIGURE 6.66, SUPPOSE LINES $\overleftrightarrow{PA}$ AND $\overleftrightarrow{PX}$ ARE TANGENTS TO A CIRCLE. PROVE THAT

$$m(\angle APX) = \frac{1}{2}(\text{MEASURE OF MAJOR ARC } AX) - \frac{1}{2}(\text{MEASURE OF MINOR ARC } AX)$$

OR $m(\angle P) = \frac{1}{2}m(\widehat{ACX}) - \frac{1}{2}m(\widehat{ABX})$

Hint: DRAW A LINE THROUGH A PARALLEL TO $\overline{PX}$

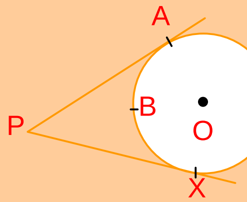


Figure 6.66

3 SUPPOSE A GEOSTATIONARY SATELLITE S ORBITS 35,000 KM ABOVE EARTH, ROTATING SO THAT IT APPEARS TO HOVER DIRECTLY OVER THE EQUATOR. USE FIGURE 6.67 TO DETERMINE THE MEASURE OF THE ARC ON THE EQUATOR VISIBLE TO THIS GEOSTATIONARY SATELL

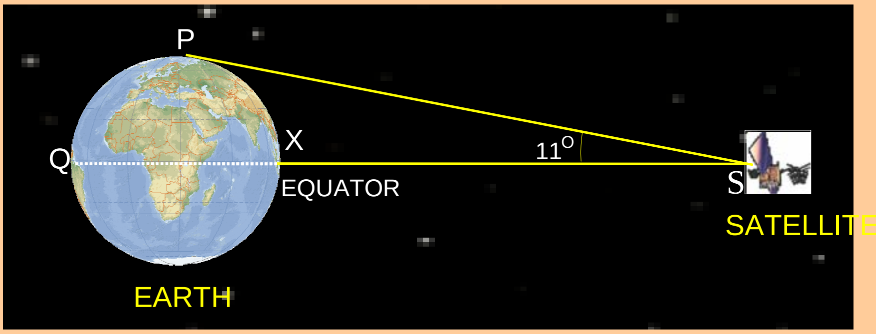


Figure 6.67

Exercise 6.4

1 IF THE MEASURE OF ARC AQ IS 30^{o} AND THE MEASURE OF ARC BR IS 60^{o}, WHAT IS THE MEASURE OF $\angle P$? REFER TO FIGURE 6.68.

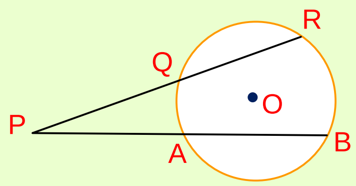


Figure 6.68

2 In Figure 6.69, $\overleftrightarrow{AP}$ is a tangent to the circle. Prove that $\angle CAP \cong \angle ABC$.

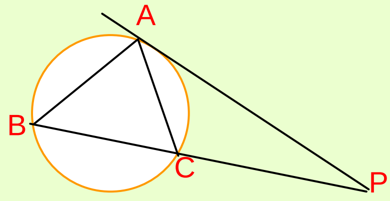


Figure 6.69

3 In Figure 6.70, $\overline{CD}$ is a diameter and $\overline{AB}$ is bisected by $\overline{CD}$ at P. A square with side $\overline{AP}$ and a rectangle with sides $\overline{CP}$ and $\overline{PD}$ are drawn. Prove that the areas of the square and the rectangle are equal.

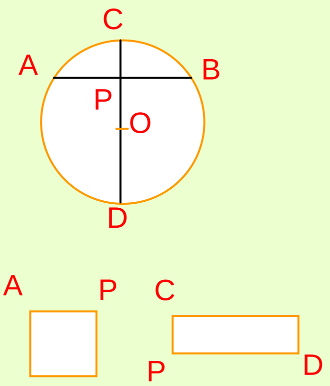


Figure 6.70

4 In Figure 6.71, $\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EG}$ are tangents to the circle with centre O, at B, D and F respectively. Prove that $CB + EF = CE$.

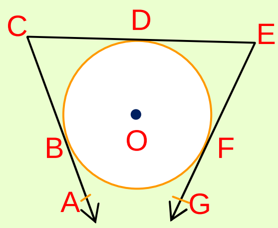


Figure 6.71

5 Use the circle in Figure 6.72 with tangent $\overline{PT}$, secants $\overline{PE}$, $\overline{PC}$ and chord $\overline{BD}$ to find the lengths of $\overline{CH}$, $\overline{EF}$ and $\overline{PT}$, if $CG = 4$ units, $GA = 6$ units, $DG = 3$ units, $PF = 9$ units and $PB = 8$ units.

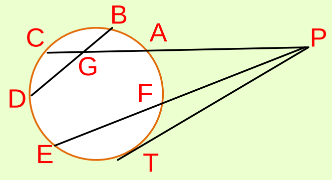


Figure 6.72

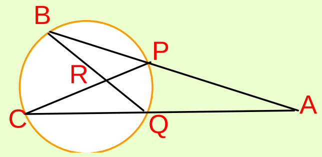


Figure 6.73

6 IN FIGURE 6.73, $m(\angle BPC) = 48^{O}$, $m(\angle BRC) = 68^{O}$ AND $m(\angle BCR) = 62^{O}$. CALCULATE THE MEASURES OF ANGLES OF $\triangle ABC$.

7 THE DIAGONALS $\overline{AC}$ AND $\overline{BD}$ OF THE PARALLELOGRAM $ABCD$ ARE OF LENGTHS 20 CM AND 12 CM RESPECTIVELY. IF THE CIRCLE BCD CUTS $\overline{CA}$ AT F, FIND THE LENGTH OF $\overline{AF}$.

8 IN FIGURE 6.74, BP = 6 CM, DC = 10 CM AND CP = 8 CM. CALCULATE THE LENGTHS OF THE CHORD $\overline{AB}$ AND THE TANGENT $\overline{PT}$.

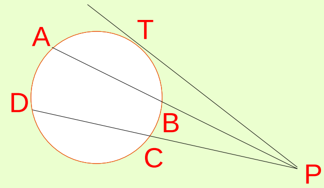


Figure 6.74

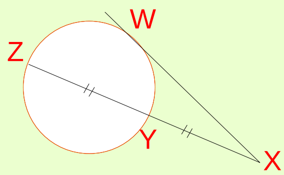


Figure 6.75

9 IN FIGURE 6.75, Y IS THE MID-POINT OF $\overline{XZ}$ AND $\overleftrightarrow{WX}$ IS TANGENT TO THE CIRCLE. FIND WX IN TERMS OF XY. EXPLAIN YOUR REASONING.

6.4 REGULAR POLYGONS

A POLYGON WHOSE VERTICES ARE ON A CIRCLE IS SAID TO BE ***inscribed*** IN THE CIRCLE. THE CIRCLE IS ***circumscribed*** ABOUT THE POLYGON.

IN FIGURE 6.76, THE POLYGON $ABCDE$ IS INSCRIBED IN THE CIRCLE OR THE CIRCLE IS CIRCUMSCRIBED ABOUT THE POLYGON.

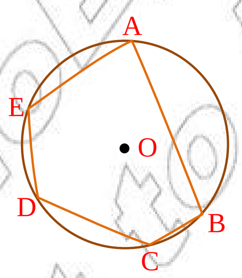


Figure 6.76

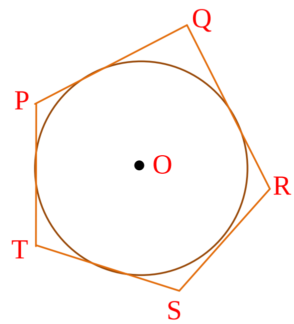


Figure 6.77

A POLYGON WHOSE SIDES ARE TANGENT TO A CIRCLE IS CIRCUMSCRIBED ABOUT THE CIRCLE. IN FIGURE 6.77, THE PENTAGON $PQRST$ IS CIRCUMSCRIBED ABOUT THE CIRCLE. THE CIRCLE IS INSCRIBED IN THE PENTAGON.

ACTIVITY 6.7

1 WHAT IS A REGULAR POLYGON? GIVE EXAMPLES.

2 DRAW THREE CIRCLES OF RADIUS 5 CM. CIRCUMSCRIBE A SQUARE ABOUT THE FIRST CIRCLE, A TRIANGLE ABOUT THE SECOND, AND A 7-SIDED POLYGON ABOUT THE THIRD.

3 CIRCUMSCRIBE A CIRCLE ABOUT A SQUARE.

4 DRAW A CIRCLE SUCH THAT THREE OF THE SIDES OF A RECTANGLE ARE TANGENT TO IT. GIVE REASONS WHY A CIRCLE CANNOT BE INSCRIBED IN THE RECTANGLE OF UNEQUAL S

5 SHOW THAT A CIRCLE CAN ALWAYS BE CIRCUMSCRIBED ABOUT A QUADRILATERAL IF TWO OPPOSITE ANGLES ARE RIGHT ANGLES.

6 SHOW THAT, IF A CIRCLE CAN BE CIRCUMSCRIBED ABOUT A PARALLELOGRAM, THEN THE PARALLELOGRAM IS A RECTANGLE.

7 WHAT IS THE MEASURE OF AN ANGLE BETWEEN THE BISECTORS OF TWO ADJACENT ANGLES IN A REGULAR POLYGON OF n SIDES, 3, 5, 10,

8 WHAT IS THE MEASURE OF AN ANGLE BETWEEN THE PERPENDICULAR BISECTORS OF TWO ADJACENT SIDES OF A REGULAR POLYGON OF n SIDES 3, 7, 10,

9 DRAW A SQUARE WITH SIDE 5 CM. DRAW THE INSCRIBED AND CIRCUMSCRIBING CIRCLES.

6.4.1 Perimeter of a Regular Polygon

YOU HAVE STUDIED HOW TO FIND THE LENGTH OF A SIDE (S) AND PERIMETER (P) OF A RE POLYGON WITH RADIUS "r" AND "THE NUMBER OF SIDES n" IN GRADE 9. THE FOLLOWING EXAMPLE IS GIVEN TO REFRESH YOUR MEMORY.

EXAMPLE 1 THE PERIMETER OF A REGULAR POLYGON WITH 9 SIDES IS GIVEN BY:

$$P = 9 \times 2r \text{ SIN}\frac{180^{\circ}}{9} = 9d \text{ SIN}\frac{180^{\circ}}{9}, \text{ WHERE } d = 2r \text{ IS DIAMETER}$$

$$= 9d \text{ SIN } 20^{\circ} \approx 3.0782d$$

EXAMPLE 2 FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR QUADRILATERAL WITH RADIUS 5 UNITS.

SOLUTION:

$$s = 2r \text{ SIN}\frac{180^{\circ}}{n} \qquad P = 2nr \text{ SIN}\frac{180^{\circ}}{n}$$

$$s = 2 \times 5 \text{ SIN}\frac{180^{\circ}}{4} = 10 \text{ SIN } 45^{\circ} \qquad P = 2 \times 4 \times 5 \text{ SIN}\frac{180^{\circ}}{4} = 40 \text{ SIN } 45^{\circ}$$

$$= 10 \times \frac{\sqrt{2}}{2} \qquad = 40 \times \frac{\sqrt{2}}{2}$$

$$\therefore s = 5\sqrt{2} \text{ UNITS.} \qquad \therefore P = 20\sqrt{2} \text{ UNITS.}$$

6.4.2 Area of a Regular Polygon

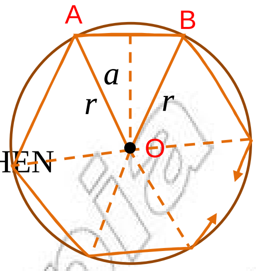


Figure 6.78

DRAW A CIRCLE WITH CENTRE O AND RADIUS r. INSCRIBE IN IT A REGULAR POLYGON WITH n SIDES AS SHOWN IN FIGURE 6.78.

JOIN O TO EACH VERTEX. THE POLYGONAL REGION IS THEN DIVIDED INTO n TRIANGLES. $\triangle AOB$ IS ONE OF THEM.

$\angle AOB$ HAS DEGREE MEASURE $\frac{360^{\circ}}{n}$

RECALL THAT THE FORMULA FOR THE AREA A OF A TRIANGLE WITH SIDES a AND b UNITS LONG AND $\angle C$ INCLUDED BETWEEN THESE SIDES IS:

$$A = \frac{1}{2}\, ab \sin(\angle C)$$

HENCE, AREA OF $\triangle AOB$ IS

$$A = \frac{1}{2}\, r \times r \sin(\angle AOB) = \frac{1}{2} r^2 \sin\frac{360^{\circ}}{n}$$

THEREFORE, THE AREA A OF THE POLYGON IS GIVEN BY

$$A = \frac{1}{2} nr^2 \sin\frac{360^{\circ}}{n} \quad \text{(WHY?)}$$

Theorem 6.16

The area A of a regular polygon with n sides and radius r is

$$A = \frac{1}{2} nr^2 \sin\frac{360^{o}}{n}.$$

THIS FORMULA FOR THE AREA OF A REGULAR POLYGON CAN BE USED TO FIND THE AREA (
NUMBER OF SIDES INCREASES, THE AREA OF THE POLYGON BECOMES CLOSER TO THE ARE

ACTIVITY 6.8

SQUARE $ABCD$ IS INSCRIBED IN A CIRCLE OF RA

A WHAT IS THE MEASURE OF ANGLE AOB?

B FIND THE AREA OF THE SQUARE $ABCD$.

C FIND THE AREA OF THE SQUARE IF r = 10 CM, L

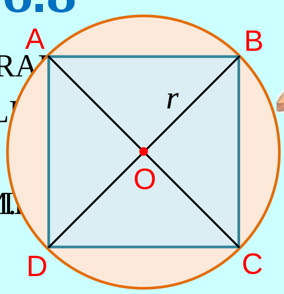


Figure 6.79

EXAMPLE 3 SHOW THAT THE AREA A OF A REGULAR HEXAGON INSCRIBED IN A CIRCLE WITH RADIUS r IS $\frac{3\sqrt{3}}{2} r^2$.

SOLUTION: $A = \frac{1}{2}nr^2 \sin\frac{360^o}{n} = \frac{1}{2}\times 6\times r^2 \sin\frac{360^o}{6} = 3r^2 \sin 60^o$

$A = 3r^2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}r^2}{2}$ SQ UNITS.

Exercise 6.5

1 FIND THE AREA OF A REGULAR NINE-SIDED POLYGON WITH RADIUS 6 UNITS.

2 FIND THE AREA OF A REGULAR TWELVE-SIDED POLYGON WITH RADIUS 3 UNITS.

3 PROVE THAT THE AREA A OF AN EQUILATERAL TRIANGLE INSCRIBED IN A CIRCLE WITH RADIUS r IS $A = \frac{3\sqrt{3}r^2}{4}$. USE THIS FORMULA TO FIND THE AREA OF AN EQUILATERAL TRIANGL IN A CIRCLE WITH RADIUS:

A 2 CM **B** 3 CM **C** $\sqrt{2}$ CM **D** $2\sqrt{3}$ CM.

4 PROVE THAT THE AREA A OF A SQUARE INSCRIBED IN A CIRCLE WITH RADIUS r IS $A = 2r^2$. USE THIS FORMULA TO FIND THE AREA OF A SQUARE INSCRIBED IN A CIRCLE WITH RAD

A 3 CM **B** 2 CM **C** $\sqrt{3}$ CM **D** 4 CM.

5 SHOW THAT ALL THE DISTANCES FROM THE CENTRE OF A REGULAR POLYGON TO THE SIDES ARE EQUAL.

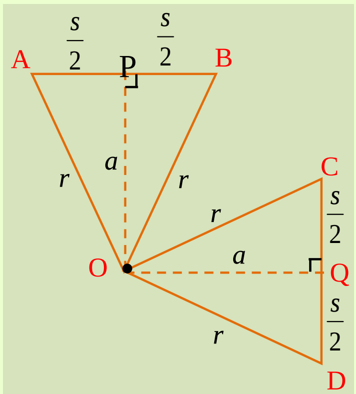


Figure 6.80

6 USE FIGURE 6.80 GIVEN ABOVE TO PROVE THE FORMULA FOR THE APOTHEM A:

$$a = r\cos\frac{180^o}{n}.$$

7 USE THE FORMULA $a = r\cos\frac{180^o}{n}$ TO CALCULATE THE APOTHEMS OF THE FOLLOWING REG POLYGONS INSCRIBED IN A CIRCLE OF RADIUS 12 CM:

A TRIANGLE **B** QUADRILATERAL **C** HEXAGON **D** NONAGON.

8 SHOW THAT A FORMULA FOR A OF THE REGULAR POLYGON OF n SIDES, APOTHEM a AND PERIMETER P IS $A = \frac{1}{2} aP$.

USE THIS FORMULA TO CALCULATE THE AREA OF A REGULAR;

A TRIANGLE **B** QUADRILATERAL **C** HEXAGON **D** OCTAGON.

GIVE YOUR ANSWER IN TERMS OF ITS RADIUS.

9 **A** SHOW THAT ANOTHER FORMULA FOR THE AREA A OF A REGULAR POLYGON OF n SIDES, WITH RADIUS r AND PERIMETER P IS

$$A = \frac{1}{2} Pr \cos\frac{180^{\circ}}{n}.$$

B SHOW THAT THE RATIO OF THE AREA OF TWO REGULAR n-SIDED POLYGONS IS THE SQUARE OF THE RATIO OF THEIR RADII.

C USE THE FORMULA FOR THE APOTHEM $a = r \cos\frac{180^{o}}{n}$ TO SHOW THAT THE RATIO OF THE AREAS OF TWO REGULAR POLYGONS WITH THE SAME NUMBER OF SIDES I OF THE SQUARES OF THE LENGTHS OF CORRESPONDING SIDES.

D CAN YOU PROVE THE RESULT IN C ABOVE WITHOUT USING ANY OF THE FORMULAE OF THIS SECTION?

10 A CIRCULAR TIN IS PLACED ON A SQUARE. IF THE SIDE OF THE SQUARE IS CONGRUENT TO THE DIAMETER OF THE TIN, CALCULATE THE PERCENTAGE OF THE SQUARE WHI UNCOVERED. GIVE YOUR ANSWER CORRECT TO 2 DECIMAL PLACES.

Key Terms

altitude
apothem
arc
bisector
central angle
centroid
chord
circle
circumcentre
circumcircle
collinear points
concurrent lines
Euclidean Eeometry
incentre
incircle
inscribed angle
major arc
median
minor arc
orthocenter
parallelogram
perpendicular
plane geometry
product property
quadrilateral
rectangle
regular polygon
rhombus
semi-circle
square
trapezium

Summary

1 THE MEDIANS OF A TRIANGLE ARE CONCURRENT AT A POINT $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

2 THE PERPENDICULAR BISECTORS OF THE SIDES OF ANY TRIANGLE ARE CONCURRENT AT A POINT CALLED CIRCUMCENTER WHICH IS EQUIDISTANT FROM THE VERTICES OF THE TRIANGLE.

3 THE ALTITUDES OF A TRIANGLE ARE CONCURRENT AT THE ORTHOCENTRE OF THE TRIANGLE. IF POINTS D, E AND F ON THE SIDES $\overline{BC}$, $\overline{CA}$ AND $\overline{AB}$ RESPECTIVELY OF $\triangle ABC$ (OR THEIR EXTENSIONS) ARE COLLINEAR, THEN $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=-1$. CONVERSELY, IF $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=-1$, THEN THE POINTS D, E AND F ARE COLLINEAR.

4 A TRAPEZIUM IS A QUADRILATERAL THAT HAS EXACTLY TWO SIDES PARALLEL.

5 A PARALLELOGRAM IS A QUADRILATERAL IN WHICH BOTH PAIRS OF OPPOSITE SIDES ARE PARALLEL.

6 A RECTANGLE IS A PARALLELOGRAM IN WHICH ONE OF ITS ANGLES IS A RIGHT ANGLE.

7 A RHOMBUS IS A PARALLELOGRAM WHICH HAS TWO ADJACENT SIDES CONGRUENT.

8 A SQUARE IS A RECTANGLE WHICH HAS CONGRUENT ADJACENT SIDES.

9 IN A CIRCLE, AN INSCRIBED ANGLE IS AN ANGLE WHOSE VERTEX IS ON THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE.

10 IN FIGURE 6.81, $m(\angle APB) = \frac{1}{2} m(\overset{\frown}{AXB})$

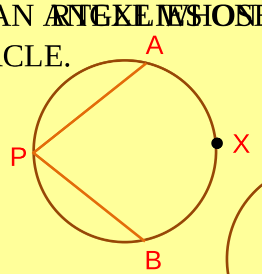


Figure 6.81

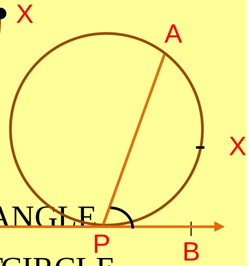


Figure 6.82

11 AN ANGLE INSCRIBED IN A SEMI-CIRCLE IS A RIGHT ANGLE.

12 AN ANGLE INSCRIBED IN AN ARC LESS THAN A SEMI-CIRCLE IS OBTUSE.

13 AN ANGLE INSCRIBED IN AN ARC GREATER THAN A SEMI-CIRCLE IS ACUTE.

14 IN FIGURE 6.82, $m(\angle APB) = \frac{1}{2} m(\overset{\frown}{AXP})$.

15 IN FIGURE 6.83, $m(\angle BPD) = \frac{1}{2}m(\overset{\frown}{AXC})+\frac{1}{2}m(\overset{\frown}{BYD})$

AND $(AP)(PB) = (CP)(PD)$

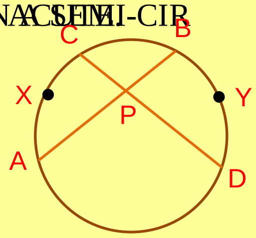


Figure 6.83

16 IN FIGURE 6.84:

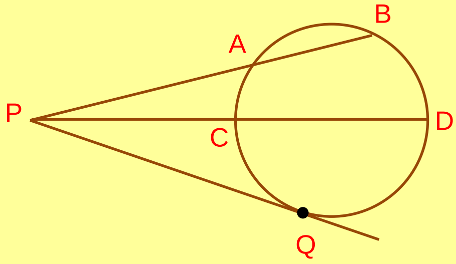


Figure 6.84

A $m\,(BPD) = \frac{1}{2} m\,(\widehat{BD}) - \frac{1}{2} m\,(\widehat{AC})$

B $m\,(DPQ) = \frac{1}{2} m\,(\widehat{DQ}) - \frac{1}{2} m\,(\widehat{QC})$

C $(PA)\,(PB) = (PC)\,(PD)$

D $(PQ)^2 = (PC)\,(PD)$

17 THE LENGTH OF A SIDE s AND PERIMETER P OF A REGULAR POLYGON WITH n SIDES AND RADIUS r ARE:

$$s = 2r \text{ SIN } \frac{180^{O}}{n} \qquad P = 2n\,r \text{ SIN } \frac{180^{O}}{n} \qquad P = ns$$

18 THE AREA A OF A REGULAR POLYGON WITH n SIDES AND RADIUS r IS

$$A = \frac{1}{2} nr^2 \text{ SIN } \frac{360^{O}}{n}.$$

Review Exercises on Unit 6

1 THE POINTS E AND F ARE THE MID-POINTS OF SIDES $\overline{AB}$ AND $\overline{AD}$ OF PARALLELOGRAM $ABCD$. PROVE THAT AREA (ECF) = $\frac{1}{2}$ AREA ($ABCD$). (*See* FIGURE 6.85)

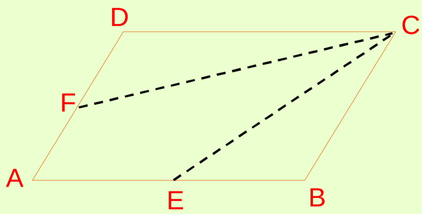


Figure 6.85

2 TWO CHORDS $\overline{AB}$ AND $\overline{CD}$ OF A CIRCLE INTERSECT AT RIGHT ANGLES AT A POINT INS… CIRCLE. IF $m(\angle BAC) = 35^{O}$, FIND THE MEASURES OF $\angle ABD$, $\widehat{CB}$ AND $\widehat{AD}$.

3 IN FIGURE 6.86, O IS THE CENTRE OF THE CIRCLE. CALCULATE x AND y.

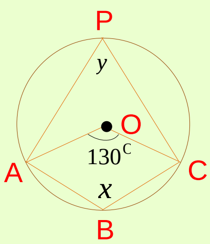


Figure 6.86

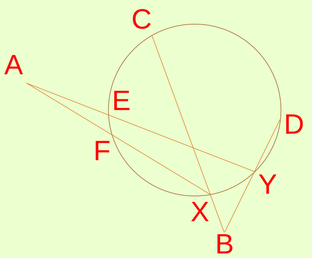


Figure 6.87

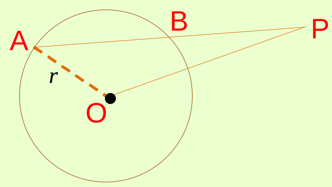


Figure 6.88

4 IN FIGURE 6.87, IF $m(\angle A) = 10^{O}$, $m(\overset{\frown}{EF}) = 15^{\circ}$ AND $m(\overset{\frown}{CD}) = 95^{O}$, FIND $m(\angle B)$.

5 FROM ANY POINT OUTSIDE A CIRCLE WITH CENTRE O AND RADIUS r, A LINE IS DRAWN CUTTING THE CIRCLE AT A AND B. PROVE THAT $(PA)(PB) = (PO)^2 - r^2$, AS SHOWN IN FIGURE 6. 88

6 TWO CHORDS $\overline{AB}$ AND $\overline{CD}$ OF A CIRCLE INTERSECT WHEN PRODUCED AT A POINT P OUTSIDE THE CIRCLE. $\overline{PT}$ IS TANGENT FROM P TO THE CIRCLE.

PROVE THAT $(PA)(PB) = (PC)(PD) = (PT)^2$.

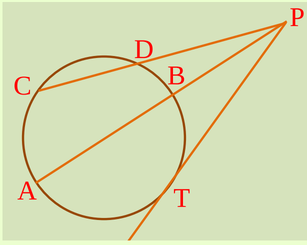


Figure 6.89

7 A CHORD OF A CIRCLE OF RADIUS 6 CM IS 8 CM LONG. FIND THE DISTANCE OF THE CHORD FROM THE CENTRE.

8 $\overline{MN}$ IS A DIAMETER AND $\overline{QR}$ IS A CHORD OF A CIRCLE, SUCH THAT $\overline{MN} \perp \overline{QR}$ AT L (AS SHOWN IN FIGURE 6.90). PROVE THAT $(QL)^2 = (ML).(LN)$.

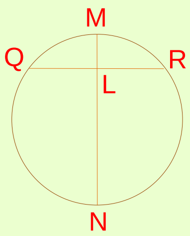


Figure 6.90

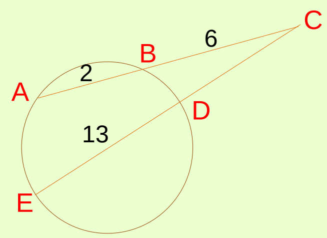


Figure 6.91

9 SECANTS $\overleftrightarrow{CA}$ AND $\overleftrightarrow{CE}$ INTERSECT A CIRCLE AT A, B, D AND E AS SHOWN IN FIGURE 6.91. IF THE LENGTHS OF THE SEGMENTS ARE AS SHOWN, FIND THE LENGTH OF $\overline{CD}$.

10 $\overleftrightarrow{AOB}$, $\overleftrightarrow{COD}$ ARE TWO STRAIGHT LINES SUCH THAT $AB = 26$ CM, $CD = 19$ CM, $AO = 6$ CM, $CO = 7$ CM. PROVE THAT $ACBD$ IS A CYCLIC QUADRILATERAL.

11 $ABXY$ IS A PARALLELOGRAM OF AREA 18 CM^2, $AB = 6$ CM, $AY = 4$ CM AND C IS A POINT ON $\overline{YX}$ OR EXTENDED SUCH THAT $BC = 5$ CM. FIND:

A THE AREA OF $\triangle ABC$

B THE DISTANCE FROM B TO $\overline{AY}$.

C THE DISTANCE FROM A TO $\overline{CB}$

Unit 7

THE PYRAMIDS AT GIZA IN EGYPT ARE AMONG THE BEST KNOWN PIECES OF ARCHITECTURE IN THE WORLD. THE PYRAMID OF KHAFRE WAS BUILT AS THE FINAL RESTING PLACE OF THE PHARAOH KHAFRE AND IS ABOUT 136 M HIGH.

MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- *solve problems involving surface area and volume of solid figures.*
- *know basic facts about frustums of cones and pyramids.*

Main Contents

7.1 Revision on Surface Areas and Volumes of Prisms and Cylinders

7.2 Pyramids, Cones and Spheres

7.3 Frustums of Pyramids and Cones

7.4 Surface Areas and Volumes of Composite Solids

Key Terms

Summary

Review Exercises

INTRODUCTION

RECALL THAT GEOMETRICAL FIGURES THAT HAVE THREE DIMENSIONS (LENGTH, WIDTH CALLED solid figures. FOR EXAMPLE, CUBES, PRISMS, CYLINDERS, CONES AND PYRAMIDS THREE DIMENSIONAL SOLID FIGURES. IN YOUR LOWER GRADES, YOU HAVE LEARNT HO SURFACE AREAS AND VOLUMES OF SOLID FIGURES LIKE CYLINDERS AND PRISMS. IN T WILL LEARN MORE ABOUT SURFACE AREAS AND VOLUMES OF OTHER SOLID FIGURES. STUDY ABOUT SURFACE AREAS AND VOLUMES OF COMPOSED SOLIDS AND FRUSTUMS AND CONES.

OPENING PROBLEM

ATO NIGATU DECIDED TO BUILD A GARAGE AND BEGAN BY CALCULATING THE NUM REQUIRED. THE FLOOR OF THE GARAGE IS RECTANGULAR WITH LENGTHS 6 M AND 4 M. T THE BUILDING IS 4 M. EACH BRICKUSED TO CONSTRUCT THE BUILDING MEASURES 22 CM BY 7 CM.

A HOW MANY BRICKS MIGHT BE NEEDED TO CONSTRUCT THE GARAGE?

B FIND THE AREA OF EACH SIDE OF THE BUILDING.

C WHAT MORE INFORMATION DO YOU NEED TO FIND THE EXACT NUMBER OF BRI REQUIRED?

7.1 REVISION ON SURFACE AREAS AND VOLUMES OF PRISMS AND CYLINDERS

THERE ARE MANY THINGS AROUND US WHICH ARE EITHER prismatic OR cylindrical IN SHAPE. IN THIS SUB-UNIT, YOU WILL CLOSELY LOOK AT THE GEOMETRICAL SOLIDS CALLED prisms AND cylinders AND THEIR SURFACE AREAS AND VOLUMES.

LET E_1 AND E_2 BE TWO PARALLEL PLANES, ℓ A LINE INTERSECTING BOTH PLANES, AND R A REGION IN E_1. FOR EACH POINT P OF R, LET P' BE THE POINT IN E_2 SUCH THAT $\overline{PP'}$ IS PARALLEL TO ℓ.

THE UNION OF ALL POINTS P' IS A REGION R' IN E_2 CORRESPONDING TO THE REGION R IN E_1. THE UNION OF ALL THE SEGMENTS $\overline{PP'}$ IS CALLED A solid region D. THIS SOLID REGION IS KNOWN AS A cylinder. SEE FIGURE 7.2

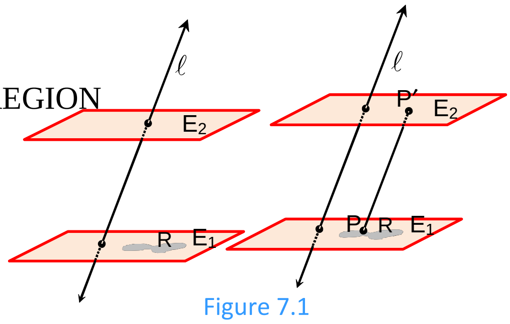


Figure 7.1

Some important terms

FOR THE CYLINDER D, THE REGION R IS CALLED ITS LOWER BASE OR SIMPLY base AND R' IS ITS upper base.

THE LINE ℓ IS CALLED ITS directrix AND THE PERPENDICULAR DISTANCE BETWEEN E_1 AND E_2 IS THE altitude of D. IF ℓ IS PERPENDICULAR TO E_1, THEN D IS CALLED A right cylinder, OTHERWISE IT IS AN oblique cylinder. IF R IS A CIRCULAR REGION, THEN D IS CALLED A circular cylinder.

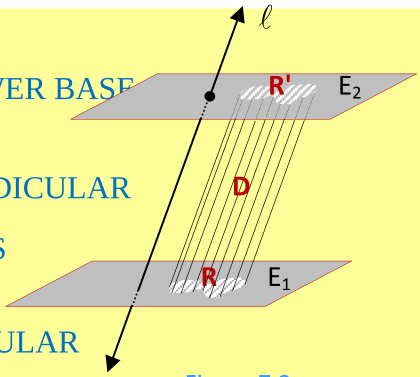


Figure 7.2

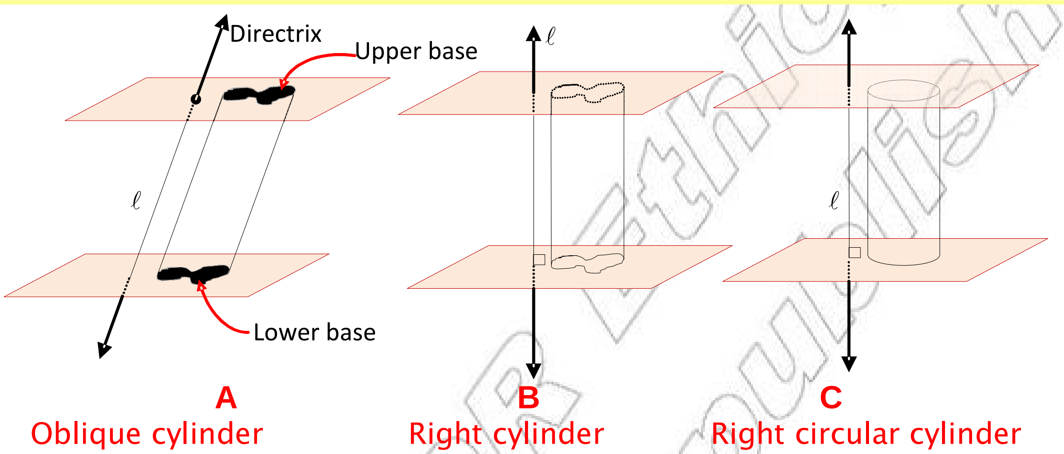


Figure 7.3

LET C BE THE BOUNDING CURVE OF THE BASE REGION R. THE UNION OF ALL THE ELEMENTS $\overline{PP'}$ FOR WHICH P BELONGS TO C IS CALLED THE lateral surface OF THE CYLINDER. THE total surface IS THE UNION OF THE LATERAL SURFACE AND THE BASES OF THE CYLINDER.

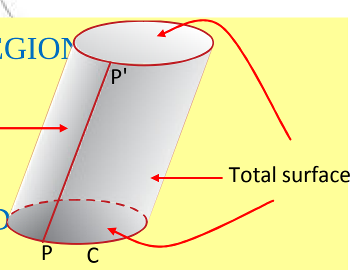


Figure 7.4

THERE ARE OTHER FAMILIAR SOLID FIGURES THAT ARE SPECIAL CASES OF THE GENERAL CYLINDER SOLID FIGURE D DESCRIBED ABOVE IN FIGURE 7.2.

Definition 1.1

If R is a polygonal region, then D is called a prism.
If R is a parallelogram region, then D is a parallelepiped.
If R is a triangular region, then D is a triangular prism.
If R is a square region, then D is a square prism.
A cube is a square right prism whose altitude is equal to the length of the edge of the base.

Note:

IN THE PRISM SHOWN IN FIGURE 7.5

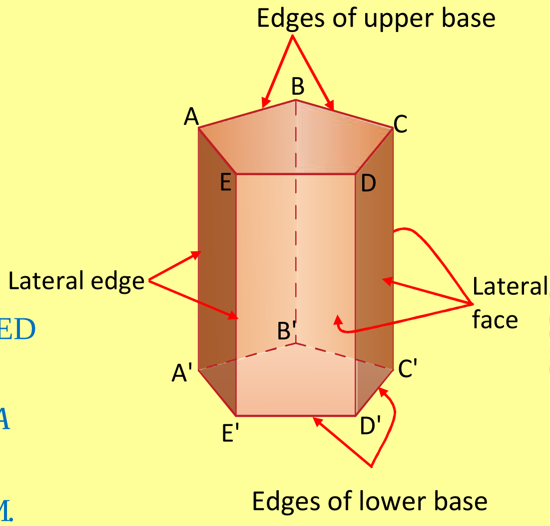


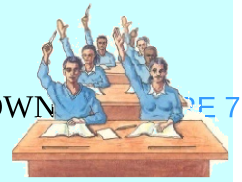
Figure 7.5

1. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$ ARE edges OF THE UPPER BASE.

   $\overline{A'B'}$, $\overline{B'C'}$, $\overline{C'D'}$, $\overline{D'E'}$, $\overline{E'A'}$ ARE edges OF THE LOWER BASE.
2. $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, $\overline{DD'}$, $\overline{EE'}$ ARE CALLED lateral edges OF THE PRISM.
3. THE PARALLELOGRAM REGIONS $ABB'A'$, $BCC'B'$, $AEE'A'$, $DCC'D'$, $EDD'E$

   ARE CALLED lateral faces OF THE PRISM.
4. THE UNION OF THE LATERAL FACES OF A PRISM IS CALLED ITS lateral surface
5. THE UNION OF ITS LATERAL FACES AND ITS TWO BASES IS CALLED ITS total surface OR SIMPLY ITS surface.

ACTIVITY 7.1



1. HOW MANY EDGES DOES THE BASE OF THE PRISM SHOWN IN FIGURE 7.5 HAVE? NAME THEM.
2. IDENTIFY EACH OF THE SOLIDS IN FIGURE 7.6 AS PRISM, CYLINDER, TRIANGULAR PRISM, RIGHT PRISM, PARALLELEPIPED, RECTANGULAR PARALLELEPIPED AND CUBE.

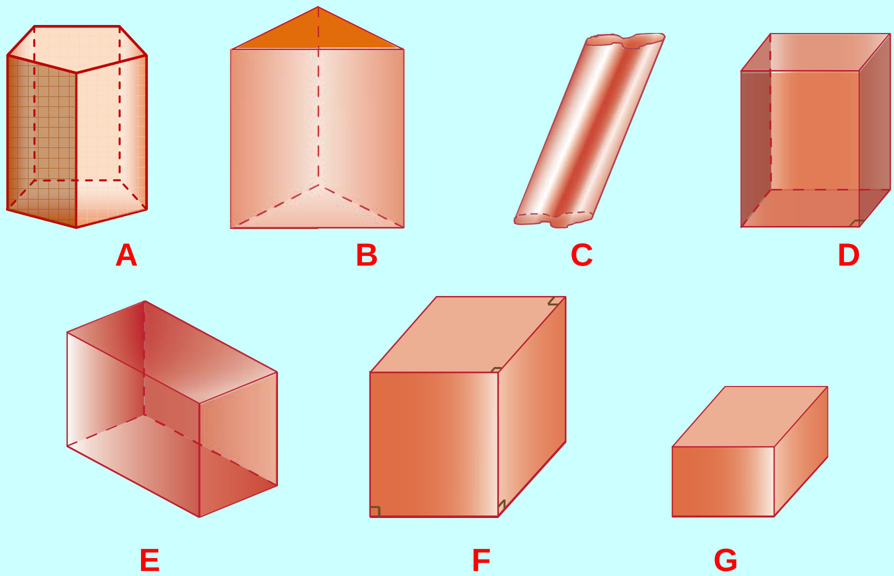


Figure 7.6

3. ARE THE LATERAL EDGES OF A PRISM EQUAL AND PARALLEL?

4 USING FIGURE 7.7, COMPLETE THE FOLLOWING BLANK SPACES TO MAKE TRUE STATEMENTS:

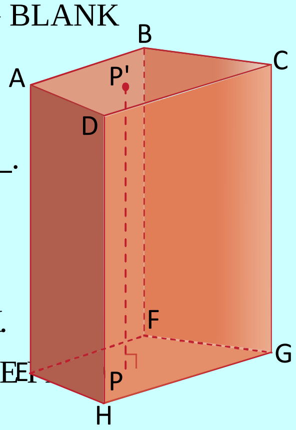


Figure 7.7

- **A** THE FIGURE IS CALLED A ______.
- **B** THE REGION $ABCD$ IS CALLED A __________.
- **C** $\overline{AE}$ AND $\overline{CG}$ ARE CALLED ______.
- **D** THE REGION $EHAD$ IS CALLED A ______ .
- **E** ______ IS THE ALTITUDE OF THE PRISM.
- **F** IF $ABCD$ WERE A PARALLELOGRAM, THEN THE PRISM WOULD BE CALLED A ______.
- **G** IF $\overline{AE}$ WERE PERPENDICULAR TO THE PLANE OF THE QUADRILATERAL $EFGH$, THEN THE PRISM WOULD BE CALLED ______.

5 CONSIDER A RECTANGULAR PRISM WITH DIMENSIONS l AND w OF ITS BASE AND HEIGHT h. DETERMINE:

A THE BASE AREA **B** LATERAL SURFACE AREA **C** TOTAL SURFACE AREA

IF WE DENOTE THE LATERAL SURFACE AREA OF THE PRISM BY A_L, THE AREA OF THE BASE BY A_B, ALTITUDE h AND THE TOTAL SURFACE AREA BY A_T, THEN

$A_L = Ph$; WHERE P IS THE PERIMETER OF THE BASE AND h IS THE HEIGHT OF THE PRISM.

$A_T = 2A_B + A_L$

EXAMPLE 1 FIND THE LATERAL SURFACE AREA OF EACH OF THE FOLLOWING RIGHT PRISMS.

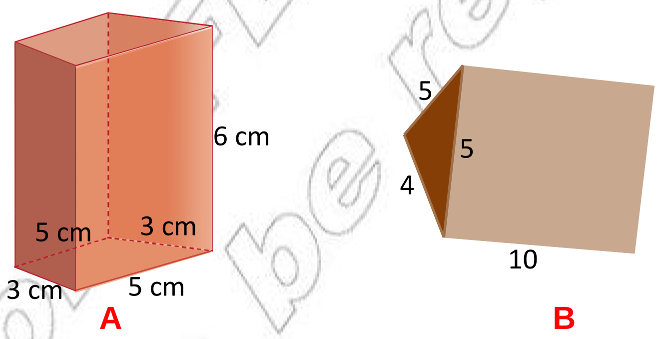


Figure 7.8

SOLUTION:

A $A_L = Ph = (3 + 5 + 3 + 5)\text{ cm} \times 6\text{ cm} = 16\text{ cm} \times 6\text{ cm} = 96\text{ cm}^2$

B $A_L = Ph = (5 + 5 + 4) \times 10 = 14 \times 10 = 140\text{ units}^2$

SIMILARLY, THE LATERAL SURFACE AREA (A_L) OF A RIGHT CIRCULAR CYLINDER IS EQUAL TO THE PRODUCT OF THE CIRCUMFERENCE OF THE BASE AND THE ALTITUDE (h) OF THE CYLINDER. THAT IS,

$A_L = 2\pi rh$, WHERE r IS THE RADIUS OF THE BASE OF THE CYLINDER.

THE TOTAL SURFACE AREA, A_T IS EQUAL TO THE SUM OF THE AREAS OF THE BASES AND THE LATERAL SURFACE AREA. THAT IS,

$$A_T = A_L + 2A_B$$

$$A_T = 2\ r\,h + 2\ r^2 = 2\ r\,(h + r)$$

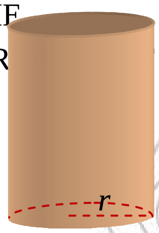


Figure 7.9

EXAMPLE 2 THE TOTAL SURFACE AREA OF A CIRCULAR CYLINDER IS 12 CM^2 AND THE ALTITUDE IS 1CM. FIND THE RADIUS OF THE BASE.

SOLUTION: $A_T = 2\ r\,(h + r) \Rightarrow 12\ = 2\ r\,(1 + r) \Rightarrow 6 = r + r^2$

$r^2 + r - 6 = 0 \Rightarrow (r + 3)(r - 2) = 0 \Rightarrow r + 3 = 0$ OR $r - 2 = 0$

$\Rightarrow r = -3$ OR $r = 2$.

THEREFORE, THE RADIUS OF THE BASE IS 2 CM. (WHY?)

THE MEASUREMENT OF SPACE COMPLETELY ENCLOSED BY THE BOUNDING SURFACE OF A SOLID IS CALLED ITS **volume**.

THE VOLUME (V) OF ANY PRISM EQUALS THE PRODUCT OF ITS BASE AREA (A_B) AND ALTITUDE (h). THAT IS,

$$V = A_B h$$

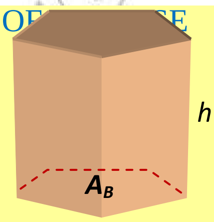


Figure 7.10

EXAMPLE 3 FIND THE TOTAL SURFACE AREA AND VOLUME OF THE FOLLOWING PRISM.

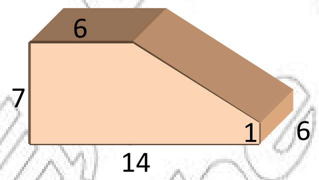


Figure 7.11

SOLUTION: TAKING THE BASE OF THE PRISM TO BE, AS SHADED IN THE FOLLOWING FIGURE, WE HAVE

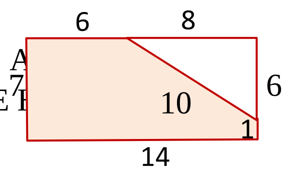


Figure 7.12

$$A_B = (7 \times 14) - \left(\frac{1}{2}\times 8 \times 6\right)$$

$= 98 - 24 = 74$ $UNITS^2$

$A_L = Ph = (7 + 6 + 10 + 14 + 1) \times 6$

$= 38 \times 6 = 228$ $UNITS^2$

$A_T = A_L + 2A_B = 228 + 2 \times 74 = 376$ $UNITS^2$

$V = A_B\,h = 74 \times 6 = 444$ $UNITS^3$

VOLUME OF A RIGHT CIRCULAR CYLINDER

THE VOLUME V OF A CIRCULAR CYLINDER IS EQUAL TO THE P... OF THE BASE AREA (A_B) AND ITS ALTITUDE (h). THAT IS,

$V = A_B h$

$V = r^2 h$, WHERE r IS THE RADIUS OF THE BASE.

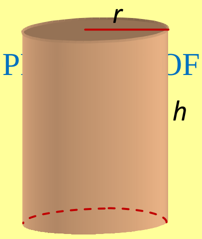


Figure 7.13

EXAMPLE 4 FIND THE VOLUME OF THE CYLINDER WHOSE BASE CIRCUMFERENCE IS 1 AND WHOSE LATERAL AREA IS 288 CM^2.

SOLUTION: $C = 2 r \Rightarrow 12 = 2 r \Rightarrow r = 6$ CM

$A_L = 2 r h$

$288 \text{ CM}^2 = 2 \times 6 \text{ CM} \times h \Rightarrow 288 \text{ CM}^2 = 12 \text{ CM} \times h \Rightarrow h = 24$ CM

THEREFORE, $V = r^2 h = (6 \text{ CM})^2 \times 24 \text{ CM} = 36 \text{ CM}^2 \times 24 \text{ CM} = 864 \text{ CM}^3$

Exercise 7.1

1 THE ALTITUDE OF A RECTANGULAR PRISM IS 4 UNITS AND THE WIDTH AND LENGTH OF ITS BASE ARE 3 UNITS AND 2 UNITS RESPECTIVELY. FIND:

A THE LATERAL SURFACE AREA **B** THE TOTAL SURFACE AREA **C** THE VOLUME

2 THE ALTITUDE OF THE RIGHT PENTAGONAL PRISM SHOWN IN FIGURE 7.14 IS 5 UNITS AND THE LENGTHS OF THE EDGES OF ITS BASE ARE 3, 4, 5, 6 AND 4 UNITS. FIND THE LATERA AREA OF THE PRISM.

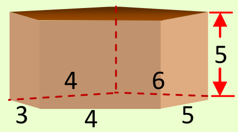


Figure 7.14

3 A LATERAL EDGE OF A RIGHT PRISM IS 6 CM AND THE PERIMETER OF ITS BASE IS 20 C THE AREA OF ITS LATERAL SURFACE.

4 FIND THE LATERAL SURFACE AREA OF EACH OF THE SOLID FIGURES GIVEN IN FIGU

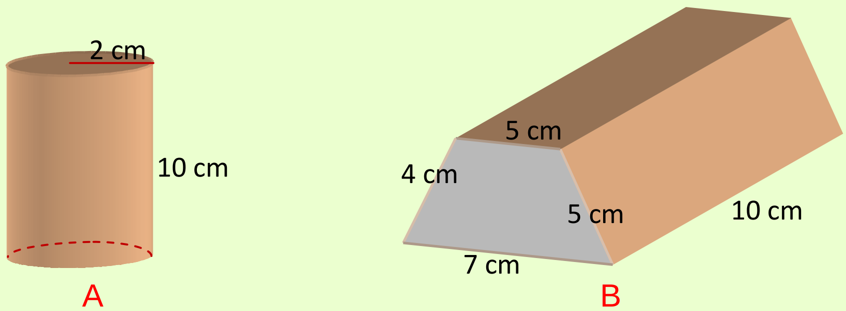


Figure 7.15

5 FIND THE PERIMETER OF THE BASE OF A RIGHT PRISM FOR WHICH THE AREA OF THE LATERAL SURFACE IS 180 UNITS^2 AND THE ALTITUDE IS 4 UNITS.

6 THE BASE OF A RIGHT PRISM IS AN EQUILATERAL TRIANGLE OF LENGTH 3 CM AND ITS LATERAL SURFACES ARE RECTANGULAR REGIONS. IF ITS ALTITUDE IS 8 CM, THEN FIND:

A THE TOTAL SURFACE AREA OF THE PRISM. **B** THE VOLUME OF THE PRISM.

7 IF THE DIMENSIONS OF A RIGHT RECTANGULAR PRISM ARE 7 CM, 9 CM AND 3 CM, TH

A ITS TOTAL SURFACE AREA **B** ITS VOLUME

C THE LENGTH OF ITS DIAGONAL.

8 FIND THE TOTAL SURFACE AREA AND THE VOLUME OF EACH OF THE FOLLOWING SO

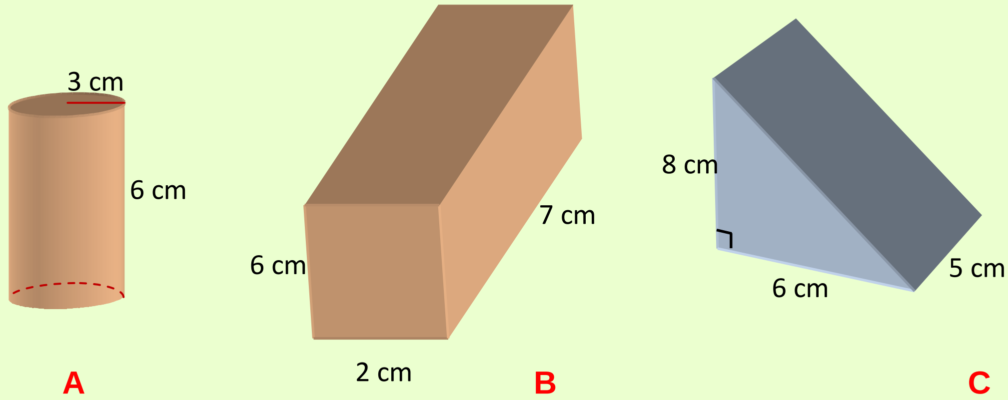


Figure 7.16

9 IF THE DIAGONAL OF A CUBE IS $\sqrt{12}$ CM, FIND THE AREA OF ITS LATERAL SURFACE.

10 THE RADIUS OF THE BASE OF A RIGHT CIRCULAR CYLINDER IS 2 CM AND ITS ALTITU FIND:

A THE AREA OF ITS LATERAL SURFACE **B** THE TOTAL SURFACE AREA

C THE VOLUME.

11 SHOW THAT THE AREA OF THE LATERAL SURFACE OF A RIGHT CIRCULAR CYLINDER WHOSE ALTITUDE IS h AND WHOSE BASE HAS RADIUS r IS $2\pi rh$.

12 IMAGINE A CYLINDRICAL CONTAINER IN WHICH THE HEIGHT AND THE DIAMETER ARE EQUAL. FIND EXPRESSIONS, IN TERMS OF ITS HEIGHT, FOR ITS

A TOTAL SURFACE AREA **B** VOLUME.

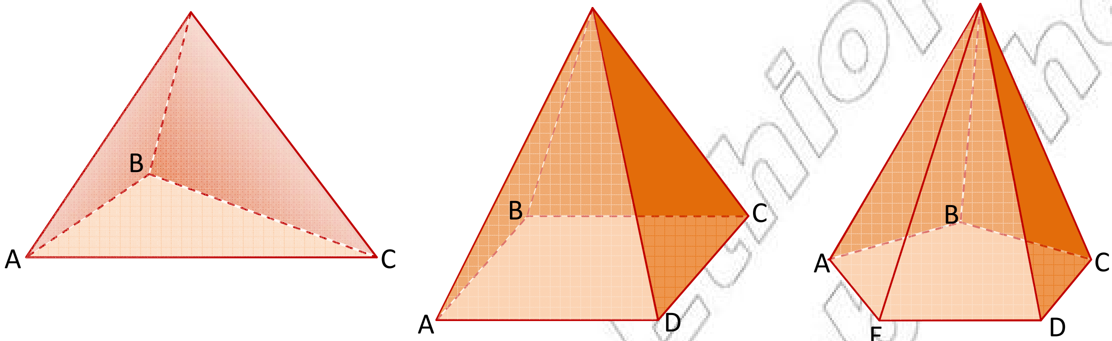
13 A CIRCULAR HOLE OF RADIUS 5 CM IS DRILLED THROUGH THE CENTRE OF A RIGHT CIRCULAR CYLINDER WHOSE BASE HAS RADIUS 6 CM AND WHOSE ALTITUDE IS 8 CM. FIND THE TOTAL SURFACE AREA AND VOLUME OF THE RESULTING SOLID FIGURE.

7.2 PYRAMIDS, CONES AND SPHERES

DO YOU REMEMBER WHAT YOU LEARNT ABOUT PYRAMIDS, CONES AND SPHERES IN YO GRADES? CAN YOU GIVE SOME EXAMPLES OF PYRAMIDS, CONES AND SPHERES FROM REA

Definition 7.2

A pyramid is a solid figure formed when each vertex of a polygon is joined to the same point not in the plane of the polygon (See FIGURE 17).



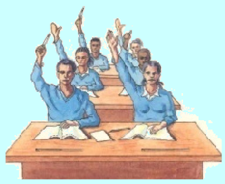
TRIANGULAR PYRAMID
A

QUADRILATERAL PYRAMID
B

ENTAGONAL PYRAMID
C

Figure 7.17

ACTIVITY 7.2



1. WHAT IS A REGULAR PYRAMID?
2. WHAT IS A TETRAHEDRON?
3. DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE:
   - A THE LATERAL FACES OF A PYRAMID ARE TRIANGULAR REGIONS.
   - B THE NUMBER OF TRIANGULAR FACES OF A PYRAMID HAVING SAME VERTEX IS THE NUMBER OF EDGES OF THE BASE.
   - C THE ALTITUDE OF A CONE IS THE PERPENDICULAR DISTANCE FROM THE BASE VERTEX OF THE CONE.
4. USING FIGURE 7.18, COMPLETE THE FOLLOWING TO MAKE TRUE STATEMENTS.
   - A THE FIGURE IS CALLED A ______.
   - B THE REGION *EDV* IS CALLED A _______.
   - C THE REGION *ABCDEF* IS CALLED ______.
   - D ______ IS THE ALTITUDE OF THE PYRAMID.
   - E $\overline{VE}$ AND $\overline{VF}$ ARE CALLED ______.

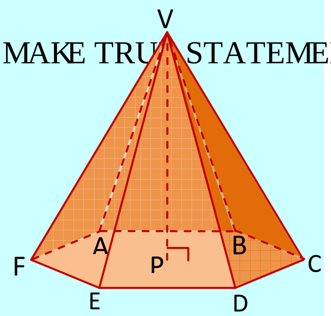


Figure 7.18

F SINCE *ABCDEF* IS A HEXAGONAL REGION, THE PYRAMID IS CALLED A ________.

5 DRAW A CONE AND INDICATE:

A ITS SLANT HEIGHT B ITS BASE C ITS LATERAL SURFACE.

THE altitude OF A PYRAMID IS THE LENGTH OF THE PERPENDICULAR FROM THE VERTEX TO CONTAINING THE BASE.

THE slant height OF A REGULAR PYRAMID IS THE ALTITUDE OF ANY OF ITS LATERAL FACE

Definition 7.3

The solid figure formed by joining all points of a circle to a point not on the plane of the circle is called a cone.

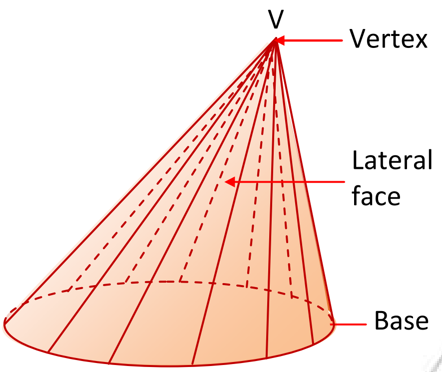


Figure 7.19

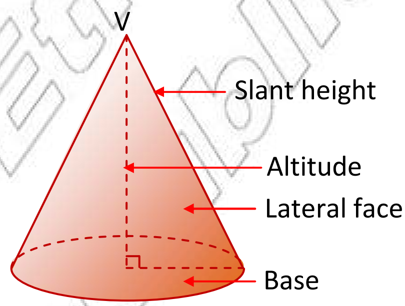


Figure 7.20

THE FIGURE SHOWN IN FIGURE 7.19 REPRESENTS A CONE. NOTE THAT THE CURVED SURFACE IS lateral surface OF THE CONE.

A right circular cone (SEE FIGURE 7.20) IS A CONE WITH THE FOOT OF ITS ALTITUDE AT TH CENTRE OF THE BASE. A LINE SEGMENT FROM THE VERTEX OF A CONE TO ANY POI BOUNDARY OF THE BASE (CIRCLE) IS CALLED THE slant height

ACTIVITY 7.3

1 CONSIDER A REGULAR SQUARE PYRAMID WITH BASE EDGE ... AND SLANT HEIGHT 5 CM.

A HOW MANY LATERAL FACES DOES IT HAVE?

B FIND THE AREA OF EACH LATERAL FACE.

C FIND THE LATERAL SURFACE AREA.

D FIND THE TOTAL SURFACE AREA.

2 TRY TO WRITE THE FORMULA FOR THE TOTAL SURFACE AREA OF A PYRAMID OR A C

Surface area

THE LATERAL SURFACE AREA OF A REGULAR PYRAMID IS EQUAL TO HALF THE PRODUCT OF ITS SLANT HEIGHT AND THE PERIMETER OF THE BASE. THAT IS,

$$A_L = \frac{1}{2}P\ell,$$

WHERE A_L DENOTES THE LATERAL SURFACE AREA;
P DENOTES THE PERIMETER OF THE BASE;
ℓ DENOTES THE SLANT HEIGHT.

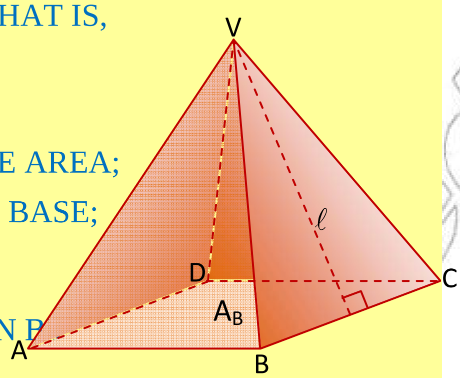


Figure 7.21

THE TOTAL SURFACE AREA (A_T) OF A PYRAMID IS GIVEN BY

$$A_T = A_B + A_L = A_B + \frac{1}{2}P\ell,$$

WHERE A_B IS AREA OF THE BASE.

EXAMPLE 1 A REGULAR PYRAMID HAS A SQUARE BASE WHOSE SIDE IS 4 CM LONG. THE LATERAL EDGES ARE 6 CM EACH.

A WHAT IS ITS SLANT HEIGHT? **B** WHAT IS THE LATERAL SURFACE AREA?

C WHAT IS THE TOTAL SURFACE AREA?

SOLUTION: CONSIDER FIGURE 7.22

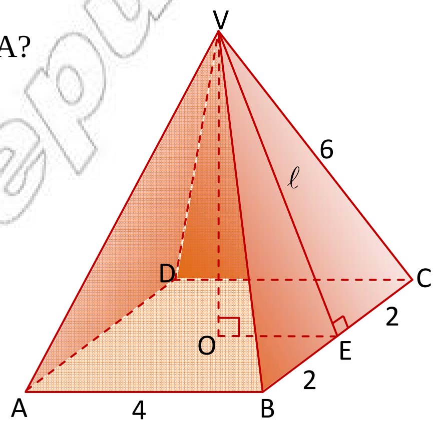


Figure 7.22

A $(VE)^2 + (EC)^2 = (VC)^2$

$\ell^2 + 2^2 = 6^2$

$\ell^2 = 32$

$\ell = 4\sqrt{2}$ CM

THEREFORE, THE SLANT HEIGHT IS $4\sqrt{2}$ CM

B THERE ARE 4 ISOSCELES TRIANGLES.

THEREFORE,

$$A_L = 4\times\frac{1}{2}BC \times VE = 4\left(\frac{1}{2} \times 4 \times 4\sqrt{2}\right) = 32\sqrt{2}\ \text{CM}^2$$

$$\text{OR}\quad A_L = \frac{1}{2}P\ell = \frac{1}{2}(4+4+4+4)\,4\sqrt{2} = 8\times 4\sqrt{2} = 32\sqrt{2}\ \text{CM}^2$$

C $A_T = A_L + A_B = 32\sqrt{2} + 4\times 4$

$= 32\sqrt{2} + 16 = 16(2\sqrt{2} + 1)\ \text{CM}^2$

THE LATERAL SURFACE AREA OF A RIGHT CIRCULAR CONE IS EQUAL TO HALF THE PRODUCT OF ITS SLANT HEIGHT AND THE CIRCUMFERENCE OF THE BASE. THAT IS,

$$A_L = \frac{1}{2}P\ell = \frac{1}{2}(2\ \ R)\ell = \ \ r\ell;$$

$$\ell = \sqrt{h^2 + r^2}$$

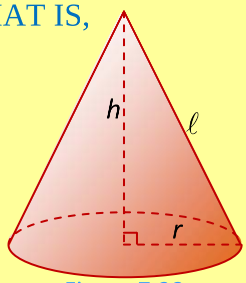


Figure 7.23

WHERE A_L DENOTES THE LATERAL SURFACE AREA, ℓ REPRESENTS THE SLANT HEIGHT, r STANDS FOR THE BASE RADIUS, AND h FOR THE ALTITUDE.

THE TOTAL SURFACE AREA (A_T) IS EQUAL TO THE SUM OF THE AREA OF THE BASE AND THE LATERAL SURFACE AREA. THAT IS,

$$A_T = A_L + A_B = \ \ r\ell + \ \ r^2 = \ \ r(\ell + r)$$

EXAMPLE 2 THE ALTITUDE OF A RIGHT CIRCULAR CONE IS 8 CM. IF THE RADIUS OF THE BASE IS 6 CM, THEN FIND ITS:

A SLANT HEIGHT **B** LATERAL SURFACE AREA **C** TOTAL SURFACE AREA.

SOLUTION: CONSIDER FIGURE 7.24

A $\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = \sqrt{100}$

$\ell = 10$ CM

B $A_L = \ \ r\ell = \ \ \times 6 \times 10 = 60\ \ $ CM^2

C $A_T = \ \ r(\ell + r) = \ \ \times 6\ (10 + 6) = 6\ \ \times 16$

$= 96\ \ $ CM^2

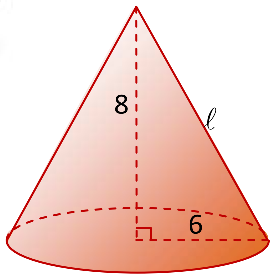


Figure 7.24

Volume

THE VOLUME OF ANY PYRAMID IS EQUAL TO ONE THIRD THE PRODUCT OF ITS BASE AREA AND ITS ALTITUDE. THAT IS,

$$V = \frac{1}{3}A_B h,$$

WHERE V DENOTES THE VOLUME, A_B THE AREA OF THE BASE AND h THE ALTITUDE.

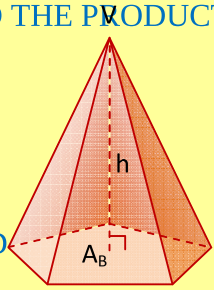


Figure 7.25

EXAMPLE 3 FIND THE VOLUME OF THE PYRAMID GIVEN IN EXAMPLE 1 ABOVE.

SOLUTION: HERE, WE NEED TO FIND THE ALTITUDE OF THE PYRAMID AS SHOWN BELOW

$$(VO)^2 + (OE)^2 = (VE)^2 \Rightarrow h^2 + 2^2 = (4\sqrt{2})^2$$
$$h^2 + 4 = 32$$
$$h^2 = 28 \Rightarrow h = 2\sqrt{7} \text{ CM}$$
$$V = \frac{1}{3} A_B h = \frac{1}{3} \times (4 \times 4) \times 2\sqrt{7} = \frac{32}{3}\sqrt{7} \text{ CM}^3$$

THE VOLUME OF A CIRCULAR CONE IS EQUAL TO ONE-THIRD OF THE PRODUCT OF ITS BASE AREA AND ITS ALTITUDE. THAT IS,

$$V = \frac{1}{3} A_B h = \frac{1}{3}\pi r^2 h$$

WHERE V DENOTES THE VOLUME, r THE RADIUS OF THE BASE AND h THE ALTITUDE.

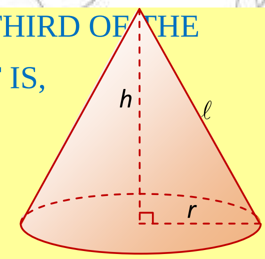
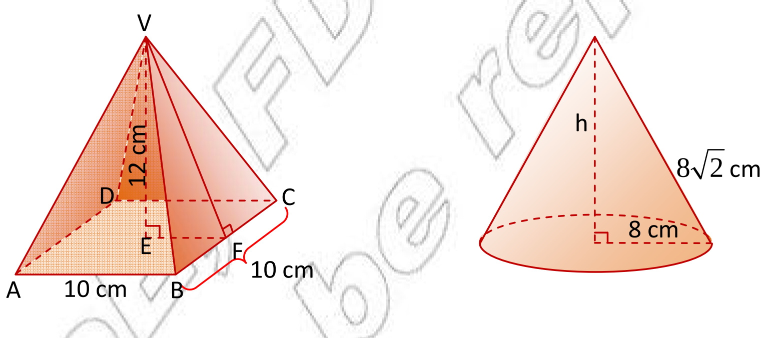


Figure 7.26

EXAMPLE 4 FIND THE VOLUME OF THE RIGHT CIRCULAR CONE GIVEN IN EXAMPLE 2 ABOVE.

SOLUTION: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (6)^2 \times 8 = 96\pi \text{ CM}^3$

EXAMPLE 5 FIND THE LATERAL SURFACE AREA, TOTAL SURFACE AREA AND THE VO[LUME OF] THE FOLLOWING REGULAR PYRAMID AND RIGHT CIRCULAR CONE.



A **B**

Figure 7.27

SOLUTION:

A TO FIND THE LATERAL SURFACE AREA, WE MUST FIND THE SLANT HEIGHT ℓ.

IN ΔVEF, WE HAVE,

$$(VE)^2 + (EF)^2 = (VF)^2 \Rightarrow 12^2 + 5^2 = (VF)^2$$
$$169 = (VF)^2 \Rightarrow VF = 13 \text{ CM}$$

THEREFORE, THE SLANT HEIGHT IS 13 CM.

NOW, $A_L = \frac{1}{2}P\ell = \frac{1}{2}(10 + 10 + 10 + 10)13 = 260\ \text{CM}^2$

$A_T = A_L + A_B = 260\ \text{CM}^2 + 100\ \text{CM}^2 = 360\ \text{CM}^2$

$V = \frac{1}{3}A_B h = \frac{1}{3}\times 100\times 12 = 400\ \text{CM}^3.$

B ALTITUDE : $h = \sqrt{\ell^2 - r^2} = \sqrt{(8\sqrt{2})^2 - 8^2} = \sqrt{128-64} = \sqrt{64} = 8\ \text{CM}$

$A_L = \ r\ell = \ \times 8\times 8\sqrt{2} = 64\sqrt{2}\ \ \text{CM}^2$

$A_T = \ r(\ell + r) = 8\ \ (8\sqrt{2} + 8) = 64\ \ (\sqrt{2} + 1)\ \text{CM}^2$

$V = \frac{1}{3}\ r^2 h = \frac{1}{3}\ (8)^2\times 8 = \frac{512}{3}\ \text{CM}^3$

Surface area and volume of a sphere

THE SPHERE IS ANOTHER SOLID FIGURE YOU STUDIED IN LOWER GRADES.

Definition 7.4

A sphere is a closed surface, all points of which are equidistant from a point called the centre.

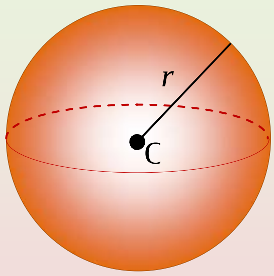


Figure 7.28

THE SURFACE AREA (A) AND THE VOLUME (V) OF A SPHERE OF RADIUS r ARE GIVEN BY

$A = 4\ r^2$

$V = \frac{4}{3}\ r^3$

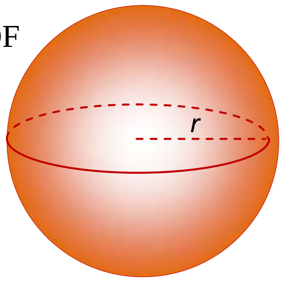


Figure 7.29

EXAMPLE 6 FIND THE SURFACE AREA AND VOLUME OF A SPHERICAL BALL WHICH HAS A DIAMETER OF 10 M.

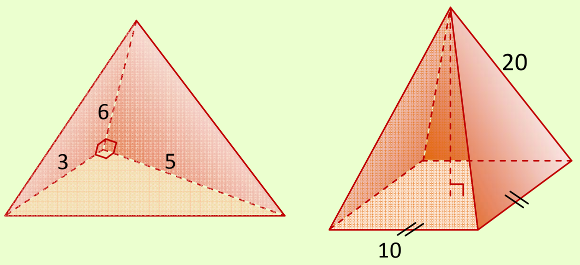
SOLUTION: WE KNOW THAT $d = 2r$ OR $r = \frac{d}{2}$ $\therefore r = \frac{10}{2} = 5$ M

$A = 4\ r^2 = 4\ (5)^2 = 100\ \ \text{M}^2$

$V = \frac{4}{3}\ r^3 = \frac{4}{3}\ (5)^3 = \frac{500}{3}\ \ \text{M}^3$

Exercise 7.2

1 CALCULATE THE VOLUME OF EACH OF THE FOLLOWING SOLID FIGURES:



A B C

Figure 7.30

2 ONE EDGE OF A RIGHT SQUARE PYRAMID IS 6 CM LONG. IF THE LATERAL EDGE IS 8 CM, THEN FIND:

A ITS TOTAL SURFACE AREA **B** ITS VOLUME.

3 THE ALTITUDE OF A RIGHT EQUILATERAL TRIANGULAR PYRAMID IS 6 CM. IF ONE EDGE OF THE BASE IS 6 CM, THEN FIND:

A ITS TOTAL SURFACE AREA **B** ITS VOLUME.

4 A REGULAR SQUARE PYRAMID HAS ALL ITS EDGES 7 CM LONG. FIND:

A ITS TOTAL SURFACE AREA **B** ITS VOLUME

5 THE ALTITUDE AND RADIUS OF A RIGHT CIRCULAR CONE ARE 12 CM AND 5 CM RESPECTIVELY. FIND:

A ITS TOTAL SURFACE AREA **B** ITS VOLUME.

6 THE VOLUME OF A PYRAMID IS 240 CM^3. THE PYRAMID HAS A RECTANGULAR BASE WITH SIDES 6 CM BY 4 CM. FIND THE ALTITUDE AND LATERAL SURFACE AREA OF THE PY PYRAMID HAS EQUAL LATERAL EDGES.

7 SHOW THAT THE VOLUME OF A REGULAR SQUARE PYRAMID WHOSE LATERAL FACES ARE EQUILATERAL TRIANGLES OF SIDE LENGTH s, IS $\frac{s^3\sqrt{2}}{6}$.

8 THE LATERAL EDGE OF A REGULAR TETRAHEDRON IS 8 CM. FIND ITS ALTITUDE.

9 FIND THE VOLUME OF A CONE OF HEIGHT 12 CM AND SLANT HEIGHT 13 CM.

10 FIND THE VOLUME AND SURFACE AREA OF A SPHERICAL FOOTBALL WITH A RADIUS

11 A GLASS IS IN THE FORM OF AN INVERTED CONE WHOSE BASE HAS A DIAMETER OF 20 CM. IF 0.1 LITRES OF WATER FILLS THE GLASS COMPLETELY, FIND THE DEPTH OF WATER $\left(\text{TAKE } \approx \frac{22}{7}\right)$

12 A SOLID METAL CYLINDER WITH A LENGTH OF 24 CM AND RADIUS 2 CM IS MELTED D FORM A SPHERE. WHAT IS THE RADIUS OF THE SPHERE?

7.3 FRUSTUMS OF PYRAMIDS AND CONES

IN THE PRECEDING SECTION, YOU HAVE STUDIED ABOUT PYRAMIDS AND CONES. YOU STUDY THE SOLID FIGURE OBTAINED WHEN A PYRAMID AND A CONE ARE CUT BY A PLA THE BASE AS SHOWN IN FIGURE 7.31.

LET E BE THE PLANE THAT CONTAINS THE BASE AND E' BE THE PLANE PARALLEL TO THE BASE THAT CU THE PYRAMID AND CONE.

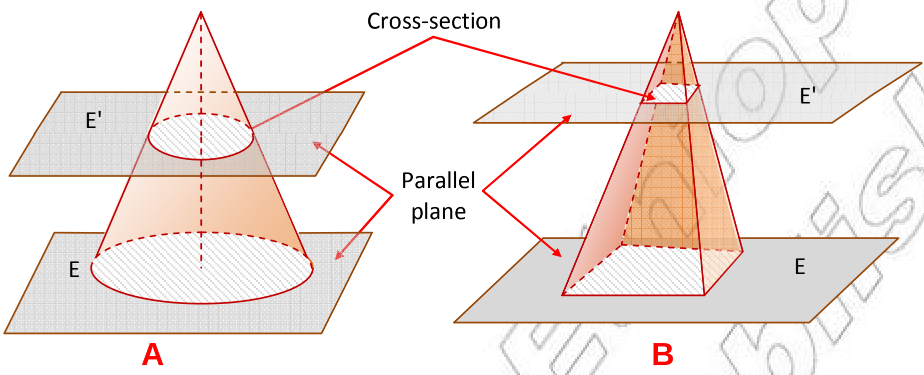


Figure 7.31

Definition 7.5

If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal cross-section of the pyramid (or the cone).

LET US NOW EXAMINE THE RELATIONSHIP BETWEEN THE BASE AND THE CROSS-SECTION

LET ΔABC BE THE BASE OF THE PYRAMID LYING IN THE PLANE E. LET h BE THE ALTITUDE OF THE PYRAMID AND LET $A'B'C'$ BE THE CROSS-SECTION AT A DISTANCE k UNITS FROM THE VERTEX.

LET D AND D' BE THE POINTS AT WHICH THE PERPENDICULAR FROM V MEETS E AND E', RESPECTIVELY.

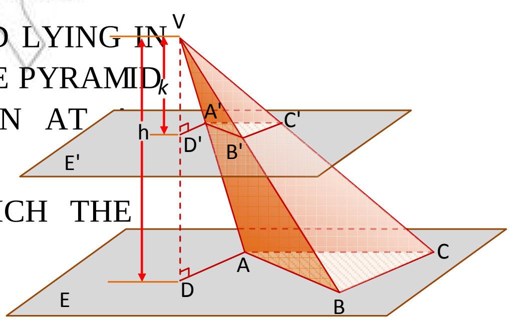


Figure 7.32

WE HAVE,

1 $\Delta VA'D' \sim \Delta VAD$.

THIS FOLLOWS FROM THE FACT THAT IF A PLANE INTERSECTS EACH OF TWO PARA INTERSECTS THEM IN TWO PARALLEL LINES, AND AN APPLICATION OF THE AA THEOREM. HENCE,

$$\frac{VA'}{VA} = \frac{VD'}{VD} = \frac{k}{h}$$

2 SIMILARLY, $\Delta VD'B' \sim \Delta VDB$ AND HENCE,

$$\frac{VB'}{VB} = \frac{VD'}{VD} = \frac{k}{h}$$

THEN, FROM 1 AND 2 AND THE SAS SIMILARITY THEOREM, WE GET,

3 $\Delta VA'B' \sim \Delta VAB$. THEREFORE, $\frac{A'B'}{AB} = \frac{VA'}{VA} = \frac{k}{h}$

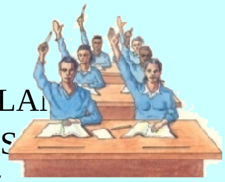
BY AN ARGUMENT SIMILAR TO THAT LEADING TO (3), WE HAVE

4 $\frac{B'C'}{BC} = \frac{k}{h}$ AND $\frac{A'C'}{AC} = \frac{k}{h}$

HENCE, BY THE SSS SIMILARITY THEOREM,

$$\Delta ABC \sim \Delta A'B'C'$$

ACTIVITY 7.4



IN THE PYRAMID SHOWN IN FIGURE 7.33, ΔABC IS EQUILATERAL. A PLANE PARALLEL TO THE BASE INTERSECTS THE LATERAL EDGES AT D, E AND F SUCH THAT $VE = \frac{1}{3}EB$.

A WHAT IS $\frac{VF}{VC}$?

B WHAT IS $\frac{EF}{BC}$?

C COMPARE THE AREAS OF ΔVEF AND ΔVBC AND OF ΔDEF AND ΔABC.

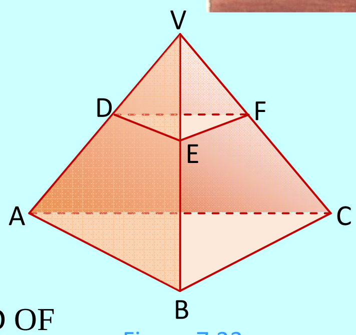
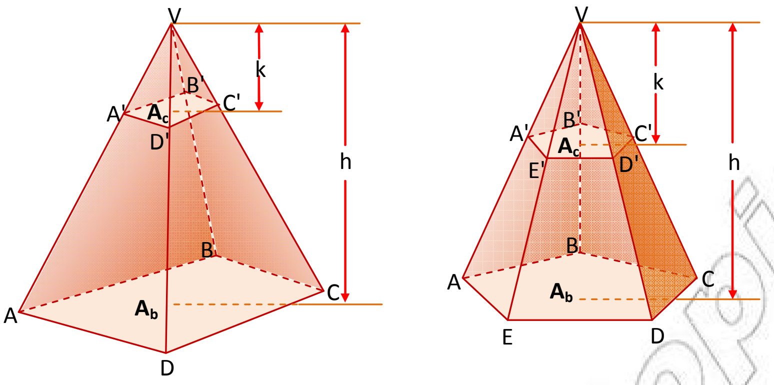


Figure 7.33

Theorem 7.1

In any pyramid, the ratio of the area of a cross-section to the area of the base is $\frac{k^2}{h^2}$ where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



$$\frac{A_c}{A_b} = \frac{\text{AREA}(A'B'C'D')}{\text{AREA}(ABCD)} = \frac{k^2}{h^2} \qquad \frac{\text{AREA}(A'B'C'D'E')}{\text{AREA}(ABCDE)} = \frac{k^2}{h^2}$$

Figure 7.34

EXAMPLE 1 THE AREA OF THE BASE OF A PYRAMID IS 90 CM^2. THE ALTITUDE OF THE PYRAMID IS 12 CM. WHAT IS THE AREA OF A HORIZONTAL CROSS-SECTION 4 CM FROM THE

SOLUTION: LET A_c BE THE AREA OF THE CROSS-SECTION AND A_b THE BASE AREA.

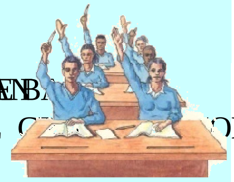
THEN, $\frac{A_c}{A_b} = \frac{k^2}{h^2} \Rightarrow \frac{A_c}{90} = \frac{4^2}{12^2}$

$\therefore \quad A_c = \frac{90 \times 16}{144}$ CM^2 = 10 CM^2

NOTE THAT SIMILAR PROPERTIES HOLD TRUE WHEN A CONE IS CUT BY A PLANE PARAL

Can you state them?

ACTIVITY 7.5



1 THE ALTITUDE OF A SQUARE PYRAMID IS 5 UNITS AND THE SIDE OF THE BA IS 4 UNITS LONG. FIND THE AREA OF A HORIZONTAL C ON AT A DISTANCE 2 UNITS ABOVE THE BASE.

2 THE AREA OF THE BASE OF A PYRAMID IS 64 CM^2. THE ALTITUDE OF THE PYRAMID IS 8 CM. WHAT IS THE AREA OF A CROSS-SECTION 2 CM FROM THE VERTEX?

3 THE RADIUS OF A CROSS-SECTION OF A CONE AT A DISTANCE 5 CM FROM THE BASE THE RADIUS OF THE BASE OF THE CONE IS 3 CM, FIND ITS ALTITUDE.

WHEN A PRISM IS CUT BY A PLANE PARALLEL TO THE BASE, EACH PART OF THE PRIS PRISM AS SHOWN IN FIGURE 7.35A

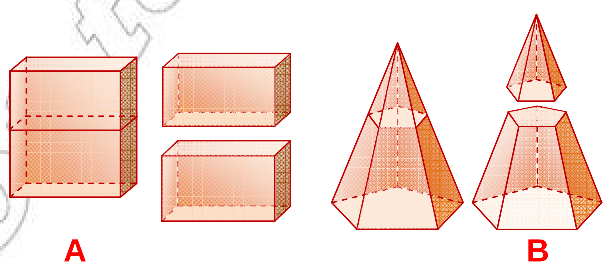


Figure 7.35

HOWEVER, WHEN A PYRAMID IS CUT BY A PLANE PARALLEL TO THE BASE, THE PART O BETWEEN THE VERTEX AND THE HORIZONTAL CROSS-SECTION IS AGAIN A PYRAMID WHI PART IS NOT A PYRAMID (AS SHOWN IN FIGURE 7.35B).

Frustum of a pyramid

Definition 7.6

A frustum of a pyramid is a part of the pyramid included between the base and a plane parallel to the base.

THE BASE OF THE PYRAMID AND THE CROSS-SECTION MADE BY THE PLANE PARALLEL T THE bases of the frustum. THE OTHER FACES ARE CALLED lateral faces. THE TOTAL SURFACE OF A FRUSTUM IS THE SUM OF THE LATERAL SURFACE AND THE BASES.

THE altitude OF A FRUSTUM OF A PYRAMID IS THE PERPENDICULAR DISTANCE BETWEEN T

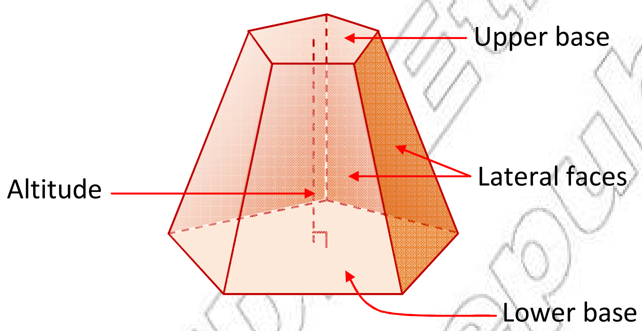


Figure 7.36

Note:

I THE LATERAL FACES OF A FRUSTUM OF A PYRAMID ARE TRAPEZIUMS.

II THE LATERAL FACES OF A FRUSTUM OF A REGULAR PYRAMID ARE CONGRUENT ISOSCELES TRAPEZIUMS.

III THE SLANT HEIGHT OF A FRUSTUM OF A REGULAR PYRAMID IS THE ALTITUDE OF ANY ONE OF THE LATERAL FACES.

IV THE LATERAL SURFACE AREA OF A FRUSTUM OF A PYRAMID IS THE SUM OF THE AREAS OF THE LATERAL FACES.

Frustum of a cone

Definition 7.7

A frustum of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.

FOR A FRUSTUM OF A CONE, THE BASES ARE THE ... OF THE CONE AND THE CROSS-SECTION PARALLEL T... ASE. THE **lateral surface** IS THE CURVED SURFACE T... MAKES UP THE FRUSTUM. THE ALTITUDE ... PERPENDICULAR DISTANCE BETWEEN THE BA...

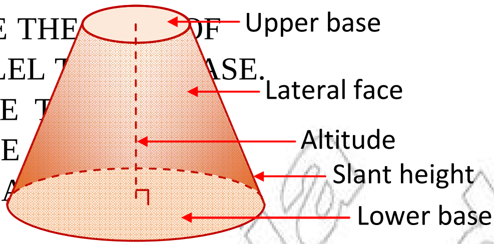


Figure 7.37

THE **slant height** OF A FRUSTUM OF A RIGHT CIRCULAR CONE IS THAT PART OF THE SLAN... THE CONE WHICH IS INCLUDED BETWEEN THE BASES.

CAN YOU NAME SOME OBJECTS WE USE IN REAL LIFE (AT HOME) THAT ARE FRUSTUMS... ARE A BUCKET AND A GLASS FRUSTUM OF CONES? DISCUSS.

EXAMPLE 2 THE LOWER BASE OF THE FRUSTUM OF A REGULAR PYRAMID IS A SQUARE 4 CM LONG. THE UPPER BASE IS 3 CM LONG. IF THE SLANT HEIGHT IS 6 CM, FIND ITS L... SURFACE AREA.

SOLUTION: AS SHOWN IN FIGURE 7.38, EACH LATERAL FACE IS A TRAPEZIUM, THE AREA O... EACH LATERAL FACE IS

$$A_L = \frac{1}{2}\times h(b_1+b_2) = \frac{1}{2}\times 6(3+4) = 21 \text{ CM}^2$$

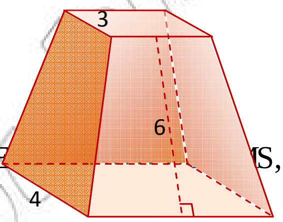


Figure 7.38

SINCE THE FOUR FACES ARE CONGRUENT ISOSCE...S, THE LATERAL SURFACE AREA IS

$$A_L = 4\times 21 \text{ CM}^2 = 84 \text{ CM}^2$$

EXAMPLE 3 THE LOWER BASE OF THE FRUSTUM OF A REGULAR PYRAMID IS A SQUARE OF SIDE s UNITS LONG. THE UPPER BASE IS s' UNITS LONG. IF THE SLA... HEIGHT OF THE FRUSTUM IS ℓ, THEN FIND THE LATERAL SURFACE AREA.

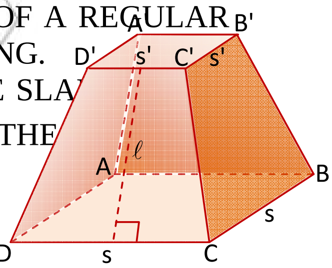


Figure 7.39

SOLUTION: FIGURE 7.39 REPRESENTS THE GIVEN PROBLEM. $ABCD$ IS A SQUARE s UNITS LONG. SIMILARLY $A'B'C'D'$ IS A SQUARE s' UNITS LONG.

LATERAL SURFACE AREA:

$$A_L = \text{AREA}(D'C'CD) + \text{AREA}(C'B'BC) + \text{AREA}(A'B'BA) + \text{AREA}(D'A'AD)$$

$$= \frac{1}{2}\ell(s+s') + \frac{1}{2}\ell(s+s') + \frac{1}{2}\ell(s+s') + \frac{1}{2}\ell(s+s')$$

$$A_L = \frac{1}{2}\ell(4s+4s') = 2\ell(s+s').$$

OBSERVE THAT $4s$ AND $4s'$ ARE THE PERIMETERS OF THE LOWER AND UPPER BASES, RESPECTIV...

IN GENERAL, WE HAVE THE FOLLOWING THEOREM:

Theorem 7.2

The lateral surface area (A_L) of a frustum of a regular pyramid is equal to half the product of the slant height (ℓ) and the sum of the perimeter (P) of the lower base and the perimeter (P') of the upper base. That is,

$$A_L = \frac{1}{2}\ell(P + P')$$

Group Work 7.1

CONSIDER THE FOLLOWING FIGURE.

1 FIND THE AREAS OF THE BASES.

2 FIND THE CIRCUMFERENCES OF THE BASES OF THE FRUSTUM, c_1 AND c_2.

3 FIND LATERAL SURFACE AREA OF THE BIGGER CONE.

4 FIND LATERAL SURFACE AREA OF THE SMALLER CONE.

5 FIND LATERAL SURFACE AREA OF THE FRUSTUM.

6 GIVE THE VOLUME OF THE FRUSTUM.

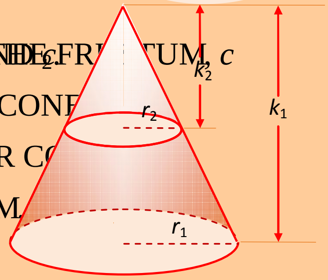


Figure 7.40

EXAMPLE 4 A FRUSTUM OF HEIGHT 4 CM IS FORMED FROM A RIGHT CIRCULAR CONE OF HEIGHT 8 CM AND BASE RADIUS 6 CM AS SHOWN IN FIGURE 7.41. CALCULATE THE LATERAL SURFACE AREA OF THE FRUSTUM.

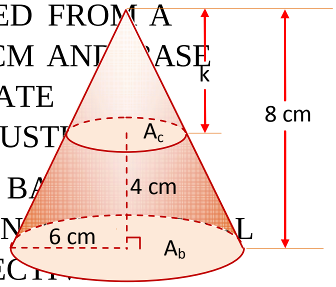


Figure 7.41

SOLUTION: LET A_b, A_c AND A_L STAND FOR AREA OF THE BASE OF THE CONE, AREA OF THE CROSS-SECTION AND LATERAL SURFACE AREA OF THE FRUSTUM, RESPECTIVELY.

$$\frac{\text{AREA OF CROSS-SECTION}}{\text{AREA OF THE BASE}} = \left(\frac{k}{h}\right)^2$$

$$\frac{A_c}{A_b} = \left(\frac{4}{8}\right)^2, \text{ SINCE } k = 8\text{ CM} - 4\text{ CM} = 4\text{ CM}$$

$$\frac{A_c}{36\pi} = \frac{1}{4} \quad (\text{AREA OF THE BASE} = \pi r^2 = \pi \times 6^2 = 36\pi)$$

$$A_c = \frac{1}{4} \times 36\pi = 9\pi \text{ CM}^2$$

$A_c = \pi(r')^2$, WHERE r' IS RADIUS OF THE CROSS-SECTION

$$\therefore \quad 9\pi = \pi(r')^2 \Rightarrow r' = 3\text{ CM}$$

SLANT HEIGHT OF THE BIGGER CONE IS:

$$\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10\ CM$$

SLANT HEIGHT OF THE SMALLER CONE IS:

$$\ell' = \sqrt{k^2 + (r')^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5\ CM$$

NOW THE LATERAL SURFACE AREA OF:

THE SMALLER CONE $= r\ell' = (3\ CM) \times 5\ CM = 15\ CM^2$

THE BIGGER CONE $= r\ell = (6\ CM) \times 10\ CM = 60\ CM^2$.

HENCE, THE AREA OF THE LATERAL SURFACE OF THE FRUSTUM IS

$A_L = 60\ CM^2 - 15\ CM^2 = 45\ CM^2$.

THE LATERAL SURFACE (CURVED SURFACE) OF A FRUSTUM OF A CIRCULAR CONE IS A T
PARALLEL SIDES (BASES) HAVE LENGTHS EQUAL TO THE CIRCUMFERENCE OF THE BASES
AND WHOSE HEIGHT IS EQUAL TO THE HEIGHT OF THE FRUSTUM.

Theorem 7.3

For a frustum of a right circular cone with altitude h and slant height ℓ, if the circumferences of the bases are c and c', then the lateral surface area of the frustum is given by

$$A_L = \frac{1}{2}\ell(c + c') = \frac{1}{2}\ell(2\ r + 2\ r') = \ell\ (r + r')$$

EXAMPLE 5 A FRUSTUM FORMED FROM A RIGHT CIRCULAR CONE HAS BASE RADII OF 8
12 CM AND SLANT HEIGHT OF 10 CM. FIND:

A THE AREA OF THE CURVED SURFACE

B THE AREA OF THE TOTAL SURFACE. (USE = 3.14).

SOLUTION:

A $A_L = \ell(r + r') = \times 10\ CM\ (8 + 12)\ CM = 10\ CM \times 20\ CM$

$= 200\ CM^2 = 200 \times 3.14\ CM^2 = 628\ CM^2$

B AREA OF BASES:

$A_B = A_c + A_b = (r')^2 + r^2 = (8\ CM)^2 + (12\ CM)^2 = 64\ CM^2 + 144\ CM^2$

$= 208\ CM^2 \approx 208 \times 3.14\ CM^2 \approx 653\ CM^2$

TOTAL SURFACE AREA OF THE FRUSTUM:

$A_T = A_L + A_B \approx 628\ CM^2 + 653\ CM^2 = 1281\ CM^2$

EXAMPLE 6 THE AREA OF THE UPPER AND LOWER BASES OF A FRUSTUM OF A PYRAMID ARE 25 CM^2 AND 36 CM^2 RESPECTIVELY. IF ITS ALTITUDE IS 2 CM, THEN FIND THE ALTITUDE OF THE PYRAMID.

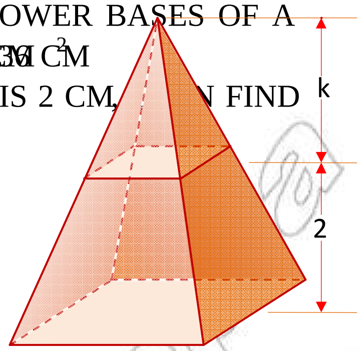


Figure 7.42

SOLUTION:

$$\frac{A_c}{A_b} = \left(\frac{k}{h}\right)^2 \Rightarrow \frac{25}{36} = \frac{k^2}{(2+k)^2}$$

$$\Rightarrow \frac{5}{6} = \frac{k}{2+k} \Rightarrow 6k = 5k + 10$$

$$\therefore \quad k = 10$$

THEREFORE, THE ALTITUDE OF THE PYRAMID IS 2 CM + 10 CM = 12 CM.

NOTE THAT THE UPPER AND LOWER BASES OF THE FRUSTUM OF A PYRAMID ARE SIMI AND THAT OF A CONE ARE SIMILAR CIRCLES.

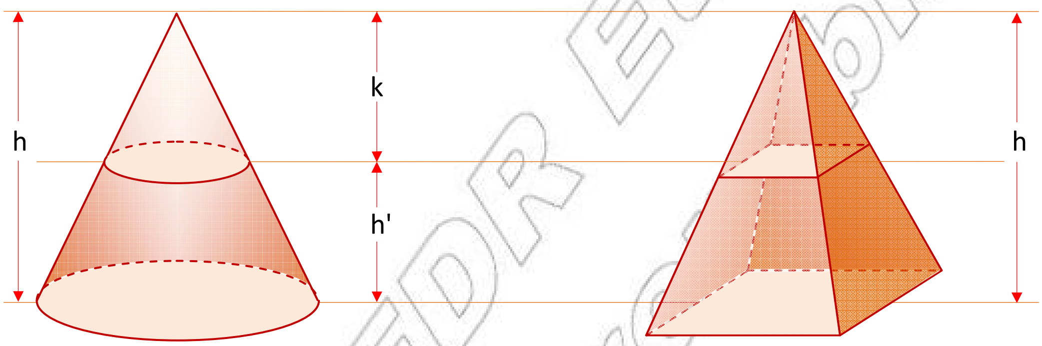


Figure 7.43

LET h = THE HEIGHT (ALTITUDE)OF THE COMPLETE CONE OR PYRAMID.

k = THE HEIGHT OF THE SMALLER CONE OR PYRAMID.

A = THE BASE AREA OF THE BIGGER CONE OR PYRAMID (LOWER BASE OF THE FRUST

A' = THE BASE AREA OF THE COMPLETING CONE OR PYRAMID (UPPER BASE OF THE F

$h' = h - k$ = THE HEIGHT OF THE FRUSTUM OF THE CONE OR PYRAMID.

V = THE VOLUME OF THE BIGGER CONE OR PYRAMID.

V' = THE VOLUME OF THE SMALLER CONE OR PYRAMID (UPPER PART).

V_f = THE VOLUME OF THE FRUSTUM

$V = \frac{1}{3}Ah$ AND $V' = \frac{1}{3}A'k$, CONSEQUENTLY THE VOLUME (V_f) OF THE FRUSTUM OF THE PYRAMID IS

$$V_f = V - V' = \frac{1}{3}Ah - \frac{1}{3}A'k = \frac{1}{3}(Ah - A'k)$$

USING THIS NOTION, WE SHALL GIVE THE FORMULA FOR FINDING THE VOLUME OF A FRUSTUM OF A CONE OR PYRAMID AS FOLLOWS:

$$V_f = \frac{h'}{3}\left(A + A' + \sqrt{AA'}\right)$$

WHERE A IS THE LOWER BASE AREA, A' IS THE UPPER BASE AREA AND h' IS THE HEIGHT OF A FRUSTUM OF A CONE OR PYRAMID.

FROM THIS, WE CAN GIVE THE FORMULA FOR FINDING THE VOLUME OF A FRUSTUM OF A CONE IN TERMS OF r AND r' AS FOLLOWS:

$$V_f = \frac{\pi}{3}h'\left(r^2 + (r')^2 + rr'\right)$$

WHERE r IS THE RADIUS OF THE BIGGER (THE LOWER BASE OF THE FRUSTUM) CONE AND r' IS THE RADIUS OF THE SMALLER CONE (UPPER BASE OF THE FRUSTUM).

EXAMPLE 7 A FRUSTUM OF A REGULAR SQUARE PYRAMID HAS HEIGHT 5 CM. THE UPPER BASE IS OF SIDE 2 CM AND THE LOWER BASE IS OF SIDE 6 CM. FIND THE VOLUME OF THE FRUSTUM.

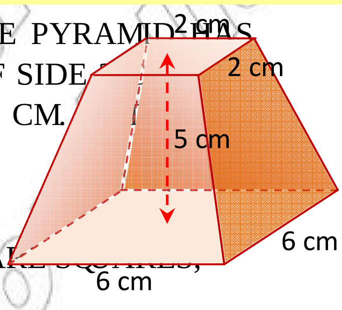


Figure 7.44

SOLUTION:

SINCE THE UPPER BASE AND LOWER BASE ARE SQUARES,

$A = (6 \text{ CM})^2 = 36 \text{ CM}^2$

$A' = (2 \text{ CM})^2 = 4 \text{ CM}^2$

$$V_f = \frac{h'}{3}\left(A + A' + \sqrt{AA'}\right) = \frac{5}{3}\left(36 + 4 + \sqrt{36\times 4}\right)\text{CM}^3$$

$$= \frac{5}{3}(40 + 12)\text{ CM}^3 = \frac{5}{3}\times 52 \text{ CM}^3 = \frac{260}{3} \text{ CM}^3$$

Exercise 7.3

1. THE LOWER BASE OF A FRUSTUM OF A REGULAR PYRAMID IS A SQUARE OF SIDE 6 CM, AND THE UPPER BASE HAS SIDE LENGTH 3 CM. IF THE SLANT HEIGHT IS 8 CM, FIND:
   - **A** ITS LATERAL SURFACE AREA
   - **B** ITS TOTAL SURFACE AREA.
2. A CIRCULAR CONE WITH ALTITUDE h AND BASE RADIUS r IS CUT AT A HEIGHT OF $\frac{2}{3}$ OF THE WAY FROM THE BASE TO FORM A FRUSTUM OF A CONE. FIND THE VOLUME OF THE FRUSTUM.
3. THE AREAS OF BASES OF A FRUSTUM OF A PYRAMID ARE 49 CM^2 AND 121 CM^2. IF ITS ALTITUDE IS 3 CM, FIND ITS VOLUME.

4 THE SLANT HEIGHT OF A FRUSTUM OF A CONE IS 10 CM. IF THE RADII OF THE BASES AND 3 CM, FIND

A THE LATERAL SURFACE AREA **B** THE TOTAL SURFACE AREA

C THE VOLUME OF THE FRUSTUM.

5 A FRUSTUM OF A REGULAR SQUARE PYRAMID WHOSE LATERAL FACES ARE EQUILAT OF SIDE 10 CM HAS ALTITUDE 5 CM. CALCULATE THE VOLUME OF THE FRUSTUM.

6 THE ALTITUDE OF A PYRAMID IS 10 CM. THE BASE IS A SQUARE WHOSE SIDES ARE E 6 CM LONG. IF A PLANE PARALLEL TO THE BASE CUTS THE PYRAMID AT A DISTANC FROM THE VERTEX, THEN FIND THE VOLUME OF THE FRUSTUM FORMED.

7 THE BUCKET SHOWN IN FIGURE 7.45 IS IN THE FORM OF A FRUSTUM OF RIGHT CIRCULAR CO THE RADII OF THE BASES ARE 12 CM AND 20 CM, AND THE VOLUME IS 6000 CM^2. FIND ITS

A HEIGHT **B** SLANT HEIGHT

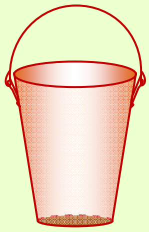


Figure 7.45

8 A FRUSTUM OF HEIGHT 12 CM IS FORMED FROM A RIGHT CIRCULAR CONE OF HEIGHT AND BASE RADIUS 8 CM. CALCULATE:

A THE LATERAL SURFACE AREA OF THE FRUSTUM

B THE TOTAL SURFACE AREA OF THE FRUSTUM

C THE VOLUME OF THE FRUSTUM.

9 A FRUSTUM IS FORMED FROM A REGULAR PYRAMID. LET THE PERIMETER OF THE LOWER BASE BE P, THE PERIMETER OF THE UPPER BASE BE P' AND THE SLANT HEIGHT BE ℓ. SHOW THAT THE LATERAL SURFACE AREA OF THE FRUSTUM IS

$$A_L = \frac{1}{2}\ell(P + P').$$

10 A FRUSTUM OF HEIGHT 5 CM IS FORMED FROM A RIGHT CIRCULAR CONE OF HEIGHT 10 CM AND BASE RADIUS 4 CM. CALCULATE:

A THE LATERAL SURFACE AREA **B** THE VOLUME OF THE FRUSTUM.

11 A FRUSTUM OF A REGULAR SQUARE PYRAMID HAS HEIGHT 2 CM. IF THE LATERAL FAC PYRAMID ARE EQUILATERAL TRIANGLES OF SIDE $2\sqrt{3}$ CM, FIND THE VOLUME OF THE FRUSTUM.

12 A CONTAINER IS IN THE SHAPE OF AN INVERTED FRUSTUM OF A RIGHT CIRCUL AR CONE AS SHOWN IN FIGURE 7.46. IT HAS A CIRCULAR BOTTOM OF RADIUS 20 CM, A CIRCULAR TO RADIUS 60 CM AND HEIGHT 40 CM. HOW MANY LITRES OF OIL COULD IT CONTAIN?

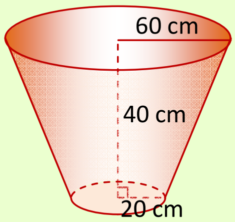


Figure 7.46

7.4 SURFACE AREAS AND VOLUMES OF COMPOSED SOLIDS

IN THE PRECEDING SECTIONS, YOU HAVE LEARNED HOW TO CALCULATE THE VOLUME AND SURFACE AREA OF CYLINDERS, PRISMS, CONES, PYRAMIDS, SPHERES AND FRUSTUMS. IN THIS SE WILL STUDY HOW TO FIND THE AREAS AND VOLUMES OF SOLIDS FORMED BY COMBINI SOLID FIGURES.

ACTIVITY 7.6

1 GIVE THE FORMULA USED FOR:

A FINDING THE LATERAL SURFACE AREA OF A

I CYLINDER **II** PRISM **III** CONE **IV** PYRAMID

V SPHERE **VI** FRUSTUM OF A PYRAMID **VII** FRUSTUM OF A CONE

B FINDING THE VOLUME OF A

I CYLINDER **II** PRISM **III** CONE **IV** PYRAMID

V SPHERE **VI** FRUSTUM OF A PYRAMID **VII** FRUSTUM OF A CONE

2 IF THE DIAMETER OF A SPHERE IS HALVED, WHAT EFFECT DOES THIS HAVE ON ITS V ITS SURFACE AREA?

3 WHAT IS THE RATIO OF THE VOLUME OF A SPHERE WHOSE RADIUS IS r UNITS TO THE CONE HAVING EQUAL RADIUS AND HEIGHT 2r UNITS?

CONSIDER THE FOLLOWING EXAMPLES.

EXAMPLE 1 A CANDLE IS MADE IN THE FORM OF A CIRCULAR CYLINDER 4 CM AT THE RADIUS BOTTOM AND A RIGHT CIRCULAR CONE OF ALTITUDE 3 CM, AS SHOWN IN FIGURE 7.47. IF THE OVERALL HEIGHT IS 12 CM, FIND THE TOTAL SURFACE AREA THE VOLUME OF THE CANDLE.

SOLUTION: SLANT HEIGHT OF THE CONE IS $\ell = \sqrt{3^2 + 4^2} = 5$ CM

THE TOTAL SURFACE AREA OF THE CANDLE IS THE SUM OF THE LATERAL SURFACE AREAS OF THE CONE, THE CYLINDER AND THE AREA OF THE BASE OF THE CYLINDER. THAT IS,

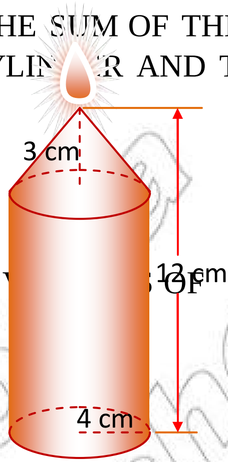


Figure 7.47

$A_T = \ r\ell + 2\ rh + \ r^2 = \ (4)\,5 + 2\ (4)\,9 + \ (4)^2$

$= 20 \ + 72 \ + 16 \ = 108 \ \text{CM}^2$

THE VOLUME OF THE CANDLE IS THE SUM OF THE V... OF THE CONE AND CYLINDER.

$V_T = V_{cone} + V_{cylinder} = \frac{1}{3}\ r^2 h_{co} + \ r^2 h_{cy}$

$= \frac{1}{3}\ (4)^2 \times 3 + \ (4)^2 \times 9 = 16 \ + 144 \ = 160 \ \text{CM}^3$

EXAMPLE 2 THROUGH A RIGHT CIRCULAR CYLINDER WHOSE BASE RADIUS IS 10 CM AND WHOSE HEIGHT IS ... DRILLED A TRIANGULAR PRISM HOLE WI... S EDGES 3 CM, 4 CM AND 5 CM AS SHOWN IN FIGURE 7.48. FIND THE TOTAL SURFACE AREA AN... THE REMAINING SOLID.

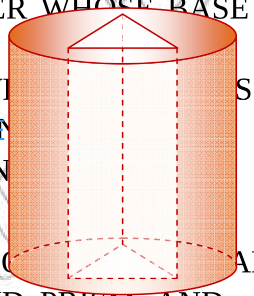


Figure 7.48

SOLUTION: THE TOTAL SURFACE AREA IS THE SUM O... AL SURFACE AREAS OF THE CYLINDER AND PRISM, AND THE BASE AREA OF THE CYLINDER, MINUS THE BASE AREA OF THE PRISM.

$A_T = 2\ rh + ph + 2\ r^2 - 2\left(\frac{1}{2}ab\right)$

$= 2\ (10)\,12 + (3 + 4 + 5)\,12 + 2\ (10)^2 - 2\left(\frac{1}{2}\times3\times4\right)$

$= 240 \ + 144 + 200 \ - 12 = (440 \ + 132)\ \text{CM}^2$

THE VOLUME OF THE RESULTING SOLID IS THE DIFFERENCE BETWEEN THE VOL... CYLINDER AND PRISM.

$V_T = V_{cy} - V_p = \ r^2 h - \frac{1}{2}abh = \ (10)^2 \times 12 - \frac{1}{2} \times 3 \times 4 \times 12$

$= 1200 \ \text{CM}^3 - 72\ \text{CM}^3 = 24\,(50 \ - 3)\ \text{CM}^3.$

EXAMPLE 3 A CONE IS CONTAINED IN A CYLINDER ... THEIR BASE RADIUS AND HEIGHT ARE THE ... SHOWN IN FIGURE 7.49. CALCULATE THE VOLU... OF THE SPACE INSIDE THE CYLINDER BUT O... THE CONE.

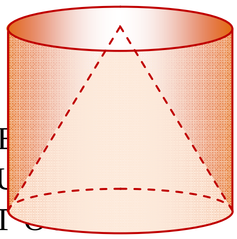


Figure 7.49

SOLUTION: THE REQUIRED VOLUME IS EQUAL TO THE DIFFERENCE BETWEEN THE VOLU
CYLINDER AND THE CONE. THAT IS,

$$V = V_{cy} - V_{co} = \ r^2h - \frac{1}{3}\ r^2h = \frac{2}{3}\ r^2h.$$

AS $r = h$, THEN $V = \frac{2}{3}\ r^3$.

Group Work 7.2

1 A CYLINDRICAL TIN 8 CM IN DIAMETER CONTAINS W... DEPTH OF 4 CM. IF A CYLINDRICAL WOODEN ROD 4 C... DIAMETER AND 6 CM LONG IS PLACED IN THE TIN IT FLOATS EXACTLY HALF SUBMERGED. WHAT IS THE NEW DEPTH OF WATER?

2 AN OPEN PENCIL CASE COMPRISES A CYLINDER OF LENGTH 20 CM AND RADIUS 2 CM AND A CONE OF HEIGHT 4 CM, AS SHOWN IN FIGURE 7.50. CALCULATE THE TOTAL SURFACE AREA AN THE VOLUME OF THE PENCIL CASE.

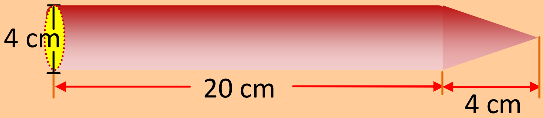


Figure 7.50

3 A BALL IS PLACED INSIDE A BOXINTO WHICH IT WI...TLY.

IF THE RADIUS OF THE BALL IS 8 CM, CALCULATE:

I THE VOLUME OF THE BALL

II THE VOLUME OF THE BOX

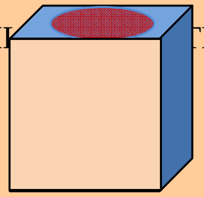


Figure 7.51

4 AN ICE-CREAM CONSISTS OF A HEMISPHERE AND A CONE.

CALCULATE ITS VOLUME AND TOTAL SURFAC...

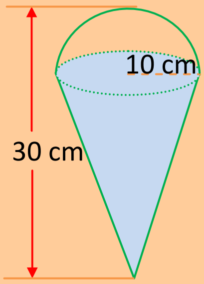


Figure 7.52

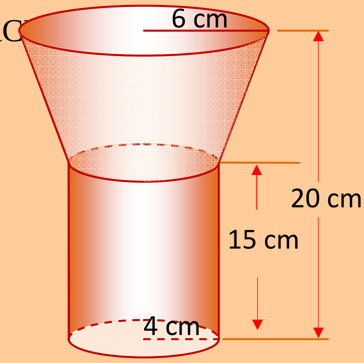


Figure 7.53

5 A TORCH 20 CM LONG IS IN THE FORM OF A RIGHT CIRCULAR CYLINDER OF HEIGHT 15 CM AND RADIUS 4 CM. JOINED TO IT IS A FRUSTUM OF A CONE OF RADIUS 6 CM. FIND THE VO THE TORCH.

Exercise 7.4

1 FIND THE VOLUME OF EACH OF THE FOLLOWING.

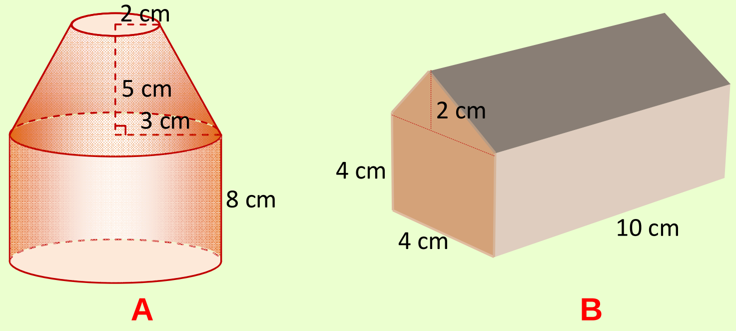


Figure 7.54

2 A STORAGE TANK IS IN THE FORM OF CYLINDER WITH ONE END HEMISPHERICAL, THE OTHER END BEING FLAT. THE DIAMETER OF THE CYLINDER IS 4 M AND THE OVERALL HEIGHT O 9 M. WHAT IS THE CAPACITY OF THE TANK?

3 AN IRON BALL 5 CM IN DIAMETER IS PLACED IN A CYLINDRICAL TIN OF DIAMETER 10 CM AND WATER IS POURED INTO THE TIN UNTIL ITS DEPTH IS 6 CM. IF THE BALL IS NOW HOW FAR DOES THE WATER LEVEL DROP?

4 FROM A HEMISPHERICAL SOLID OF RADIUS 8 CM, A CONICAL PART IS REMOVED AS SHOWN IN FIGURE 7.55. FIND THE VOLUME AND THE TOTAL SURFACE AREA OF THE RESULTING S

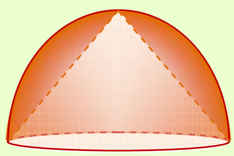


Figure 7.55

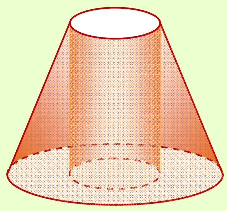


Figure 7.56

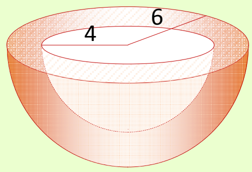


Figure 7.57

5 THE ALTITUDE OF A FRUSTUM OF A RIGHT CIRCULAR CONE IS 20 CM AND THE RADIUS OF ITS BASE IS 6 CM. A CYLINDRICAL HOLE OF DIAMETER 4 CM IS DRILLED THROUGH THE THE CENTRE OF THE DRILL FOLLOWING THE AXIS OF THE CONE, LEAVING A SOLID FIGURE 7.56. FIND THE VOLUME AND THE TOTAL SURFACE AREA OF THE RESULTING S

6 FIGURE 7.57 SHOWS A HEMISPHERICAL SHELL. FIND THE VOLUME AND THE TOTAL SURFAC OF THE SOLID.

7 A CYLINDRICAL PIECE OF WOOD OF RADIUS 8 CM CM HAS A CONE OF THE SAME RADIUS SCOOPED DEPTH OF 9 CM. FIND THE RATIO OF THE VOLUM SCOOPED OUT TO THE VOLUME OF WOOD WHIC FIGURE 7.58)

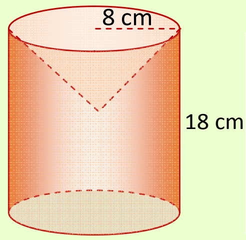


Figure 7.58

Key Terms

cone	lateral edge	regular pyramid
cross-section	lateral surface	slant height
cylinder	prism	sphere
frustum	pyramid	volume

Summary

Prism

$A_L = Ph$

$A_T = 2A_b + A_L$

$V = A_b\, h$

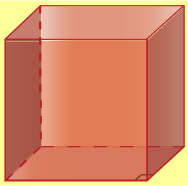


Figure 7.59

Right circular cylinder

$A_L = 2\quad rh$

$A_T = 2\quad r^2 + 2\quad rh = 2\quad r\,(r + h)$

$V = \quad r^2\, h$

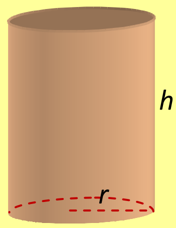


Figure 7.60

Regular pyramid

$A_L = \frac{1}{2} P\ell$

$A_T = A_b + \frac{1}{2} P\ell$

$V = \frac{1}{3} A_b\, h$

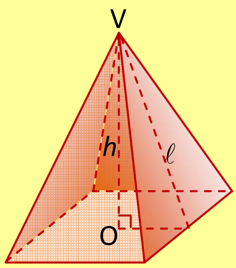


Figure 7.61

Right circular cone

$A_L = \quad r\ell$

$A_T = \quad r^2 + \quad r\ell = \quad r\,(r + \ell)$

$V = \frac{1}{3}\quad r^2 h$

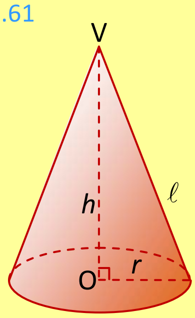


Figure 7.62

Sphere

$A = 4 \quad r^2$

$V = \frac{4}{3} \quad r^3$

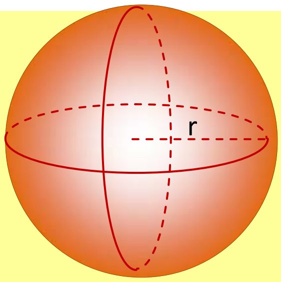


Figure 7.63

Frustum of a pyramid

$A_L = \frac{1}{2}\ell\,(P + P')$

$A_T = \frac{1}{2}\ell\,(P + P') + A_b + A'_b$

$V = \frac{1}{3}h'\left(A_b + A'_b + \sqrt{A_b A'_b}\right)$

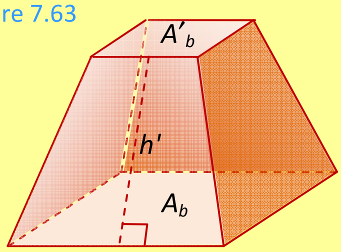


Figure 7.64

Frustum of a cone

$A_L = \frac{1}{2}\ell(2 \quad r + 2 \quad r') = \ell \quad (r + r')$

$A_T = \frac{1}{2}\ell(2 \quad r + 2 \quad r') + \quad r^2 + \quad (r')^2 = \ell \quad (r + r') + \quad (r^2 + r'^2)$

$V = \frac{1}{3}h' \quad (r^2 + (r')^2 + rr')$

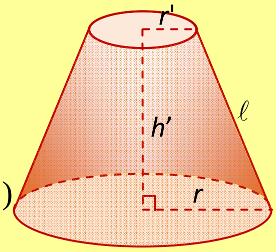


Figure 7.65

Review Exercises on Unit 7

1 FIND THE LATERAL SURFACE AREA AND VOLUME OF EACH OF THE FOLLOWING FIG

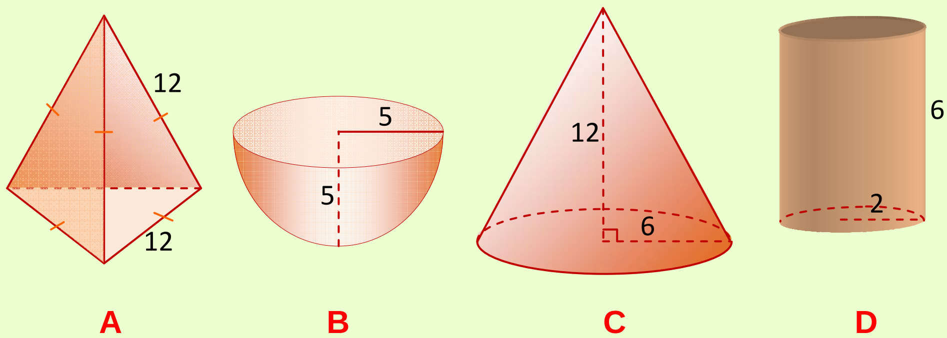


Figure 7.66

2 A LATERAL EDGE OF A RIGHT PRISM IS 6 CM AND THE PERIMETER OF ITS BASE IS 36 CM. FIND THE AREA OF ITS LATERAL SURFACE.

3 THE HEIGHT OF A CIRCULAR CYLINDER IS EQUAL TO THE RADIUS OF ITS BASE. FIND ITS TOTAL SURFACE AREA AND ITS VOLUME, GIVING YOUR ANSWER IN TERMS OF ITS RADIUS r

4 What is the volume of a stone in an Egyptian pyramid with a square base of side 100 m and a slant height of $50\sqrt{2}$ m for each of the triangular faces.

5 Find the total surface area of a regular hexagonal pyramid, given that the edge of the base is 8 cm and the altitude is 12 cm.

6 Find the area of the lateral surface of a right circular cone whose altitude is 8 cm and base radius 6 cm.

7 Find the total surface area of a right circular cone whose altitude is h and base radius is r. (Give the answer in terms of r and h)

8 When a lump of stone is submerged in a rectangular water tank whose base is 25 cm by 50 cm, the level of the water rises by 1 cm. What is the volume of the stone?

9 A frustum whose upper and lower bases are circular regions of radii 8 cm and 6 cm respectively, is 25 cm deep (*See* Figure 7.67). Find its volume.

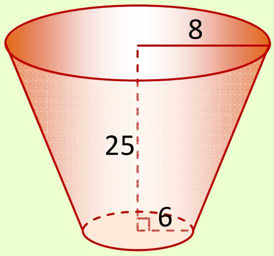


Figure 7.67

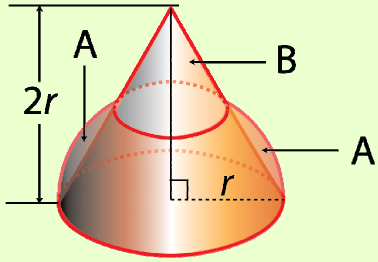


Figure 7.68

10 A cylindrical metal pipe of outer diameter 10 cm is 2 cm thick. What is the diameter of the hole? Find the volume of the metal if the pipe is 30 cm long.

11 A drinking cup in the shape of frustum of a cone with bottom diameter 4 cm and top diameter 6 cm, can contain a maximum of 80 cm^3 coffee. Find the height of the cup.

12 The slant height of a cone is 16 cm and the radius of its base is 4 cm. Find the area of the lateral surface of the cone and its volume.

13 The radius of the base of a cone is 12 cm and its volume is 720 cm^3. Find its height, slant height, and lateral surface area.

14 If the radius of a sphere is doubled, what effect does this have on its volume and on its surface area?

15 In Figure 7.68, a cone of base radius r and altitude $2r$ and a hemisphere of radius r whose base coincides with that of the cone are shown. A is the part of the hemisphere which lies outside the cone and B is the part of the cone lying outside the hemisphere. Prove that the volume of A is equal to the volume of B.

Table of Trigonometric Functions

	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	.....	1.000	.....	90°
1°	0.0175	0.9998	0.0175	57.29	1.000	57.30	89°
2°	0.0349	0.9994	0.0349	28.64	1.001	28.65	88°
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	87°
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	86°
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	85°
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	82°
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	81°
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	80°
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	79°
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	77°
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	75°
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	74°
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	73°
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	71°
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	70°
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	68°
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	67°
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	65°
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	64°
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	63°
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	61°
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	59°
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	58°
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	57°
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	56°
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	53°
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	52°
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	51°
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	50°
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	49°
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	47°
44°	0.6947	0.7193	0.9667	1.036	1.390	1.440	46°
45°	0.7071	0.7071	1 .0000	1.000	1.414	1.414	45°
	cos	sin	cot	tan	csc	sec	

Table of Common Logarithms

N	0	1	2	3	4		5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170		0.0212	0.0253	0.0294	0.0334	0.0374
1.1	0.0414	0.0453	0.0492	0.0531	0.0569		0.0607	0.0645	0.0682	0.0719	0.0755
1.2	0.0792	0.0828	0.0864	0.0899	0.0934		0.0969	0.1004	0.1038	0.1072	0.1106
1.3	0.1139	0.1173	0.1206	0.1239	0.1271		0.1303	0.1335	0.1367	0.1399	0.1430
1.4	0.1461	0.1492	0.1523	0.1553	0.1584		0.1614	0.1644	0.1673	0.1703	0.1732
1.5	0.1761	0.1790	0.1818	0.1847	0.1875		0.1903	0.1931	0.1959	0.1987	0.2014
1.6	0.2041	0.2068	0.2095	0.2122	0.2148		0.2175	0.2201	0.2227	0.2253	0.2279
1.7	0.2304	0.2330	0.2355	0.2380	0.2405		0.2430	0.2455	0.2480	0.2504	0.2529
1.8	0.2553	0.2577	0.2601	0.2625	0.2648		0.2672	0.2695	0.2718	0.2742	0.2765
1.9	0.2788	0.2810	0.2833	0.2856	0.2878		0.2900	0.2923	0.2945	0.2967	0.2989
2.0	0.3010	0.3032	0.3054	0.3075	0.3096		0.3118	0.3139	0.3160	0.3181	0.3201
2.1	0.3222	0.3243	0.3263	0.3284	0.3304		0.3324	0.3345	0.3365	0.3385	0.3404
2.2	0.3424	0.3444	0.3464	0.3483	0.3502		0.3522	0.3541	0.3560	0.3579	0.3598
2.3	0.3617	0.3636	0.3655	0.3674	0.3692		0.3711	0.3729	0.3747	0.3766	0.3784
2.4	0.3802	0.3820	0.3838	0.3856	0.3874		0.3892	0.3909	0.3927	0.3945	0.3962
2.5	0.3979	0.3997	0.4014	0.4031	0.4048		0.4065	0.4082	0.4099	0.4116	0.4133
2.6	0.4150	0.4166	0.4183	0.4200	0.4216		0.4232	0.4249	0.4265	0.4281	0.4298
2.7	0.4314	0.4330	0.4346	0.4362	0.4378		0.4393	0.4409	0.4425	0.4440	0.4456
2.8	0.4472	0.4487	0.4502	0.4518	0.4533		0.4548	0.4564	0.4579	0.4594	0.4609
2.9	0.4624	0.4639	0.4654	0.4669	0.4683		0.4698	0.4713	0.4728	0.4742	0.4757
3.0	0.4771	0.4786	0.4800	0.4814	0.4829		0.4843	0.4857	0.4871	0.4886	0.4900
3.1	0.4914	0.4928	0.4942	0.4955	0.4969		0.4983	0.4997	0.5011	0.5024	0.5038
3.2	0.5051	0.5065	0.5079	0.5092	0.5105		0.5119	0.5132	0.5145	0.5159	0.5172
3.3	0.5185	0.5198	0.5211	0.5224	0.5237		0.5250	0.5263	0.5276	0.5289	0.5302
3.4	0.5315	0.5328	0.5340	0.5353	0.5366		0.5378	0.5391	0.5403	0.5416	0.5428
3.5	0.5441	0.5453	0.5465	0.5478	0.5490		0.5502	0.5514	0.5527	0.5539	0.5551
3.6	0.5563	0.5575	0.5587	0.5599	0.5611		0.5623	0.5635	0.5647	0.5658	0.5670
3.7	0.5682	0.5694	0.5705	0.5717	0.5729		0.5740	0.5752	0.5763	0.5775	0.5786
3.8	0.5798	0.5809	0.5821	0.5832	0.5843		0.5855	0.5866	0.5877	0.5888	0.5899
3.9	0.5911	0.5922	0.5933	0.5944	0.5955		0.5966	0.5977	0.5988	0.5999	0.6010
4.0	0.6021	0.6031	0.6042	0.6053	0.6064		0.6075	0.6085	0.6096	0.6107	0.6117
4.1	0.6128	0.6138	0.6149	0.6160	0.6170		0.6180	0.6191	0.6201	0.6212	0.6222
4.2	0.6232	0.6243	0.6253	0.6263	0.6274		0.6284	0.6294	0.6304	0.6314	0.6325
4.3	0.6335	0.6345	0.6355	0.6365	0.6375		0.6385	0.6395	0.6405	0.6415	0.6425
4.4	0.6435	0.6444	0.6454	0.6464	0.6474		0.6484	0.6493	0.6503	0.6513	0.6522
4.5	0.6532	0.6542	0.6551	0.6561	0.6571		0.6580	0.6590	0.6599	0.6609	0.6618
4.6	0.6628	0.6637	0.6646	0.6656	0.6665		0.6675	0.6684	0.6693	0.6702	0.6712
4.7	0.6721	0.6730	0.6739	0.6749	0.6758		0.6767	0.6776	0.6785	0.6794	0.6803
4.8	0.6812	0.6821	0.6830	0.6839	0.6848		0.6857	0.6866	0.6875	0.6884	0.6893
4.9	0.6902	0.6911	0.6920	0.6928	0.6937		0.6946	0.6955	0.6964	0.6972	0.6981
5.0	0.6990	0.6998	0.7007	0.7016	0.7024		0.7033	0.7042	0.7050	0.7059	0.7067
5.1	0.7076	0.7084	0.7093	0.7101	0.7110		0.7118	0.7126	0.7135	0.7143	0.7152
5.2	0.7160	0.7168	0.7177	0.7185	0.7193		0.7202	0.7210	0.7218	0.7226	0.7235
5.3	0.7243	0.7251	0.7259	0.7267	0.7275		0.7284	0.7292	0.7300	0.7308	0.7316
5.4	0.7324	0.7332	0.7340	0.7348	0.7356		0.7364	0.7372	0.7380	0.7388	0.7396

5.5	0.7404	0.7412	0.7419	0.7427	0.7435		0.7443	0.7451	0.7459	0.7466	0.7474
5.6	0.7482	0.7490	0.7497	0.7505	0.7513		0.7520	0.7528	0.7536	0.7543	0.7551
5.7	0.7559	0.7566	0.7574	0.7582	0.7589		0.7597	0.7604	0.7612	0.7619	0.7627
5.8	0.7634	0.7642	0.7649	0.7657	0.7664		0.7672	0.7679	0.7686	0.7694	0.7701
5.9	0.7709	0.7716	0.7723	0.7731	0.7738		0.7745	0.7752	0.7760	0.7767	0.7774
6.0	0.7782	0.7789	0.7796	0.7803	0.7810		0.7818	0.7825	0.7832	0.7839	0.7846
6.1	0.7853	0.7860	0.7868	0.7875	0.7882		0.7889	0.7896	0.7903	0.7910	0.7917
6.2	0.7924	0.7931	0.7938	0.7945	0.7952		0.7959	0.7966	0.7973	0.7980	0.7987
6.3	0.7993	0.8000	0.8007	0.8014	0.8021		0.8028	0.8035	0.8041	0.8048	0.8055
6.4	0.8062	0.8069	0.8075	0.8082	0.8089		0.8096	0.8102	0.8109	0.8116	0.8122
6.5	0.8129	0.8136	0.8142	0.8149	0.8156		0.8162	0.8169	0.8176	0.8182	0.8189
6.6	0.8195	0.8202	0.8209	0.8215	0.8222		0.8228	0.8235	0.8241	0.8248	0.8254
6.7	0.8261	0.8267	0.8274	0.8280	0.8287		0.8293	0.8299	0.8306	0.8312	0.8319
6.8	0.8325	0.8331	0.8338	0.8344	0.8351		0.8357	0.8363	0.8370	0.8376	0.8382
6.9	0.8388	0.8395	0.8401	0.8407	0.8414		0.8420	0.8426	0.8432	0.8439	0.8445
7.0	0.8451	0.8457	0.8463	0.8470	0.8476		0.8482	0.8488	0.8494	0.8500	0.8506
7.1	0.8513	0.8519	0.8525	0.8531	0.8537		0.8543	0.8549	0.8555	0.8561	0.8567
7.2	0.8573	0.8579	0.8585	0.8591	0.8597		0.8603	0.8609	0.8615	0.8621	0.8627
7.3	0.8633	0.8639	0.8645	0.8651	0.8657		0.8663	0.8669	0.8675	0.8681	0.8686
7.4	0.8692	0.8698	0.8704	0.8710	0.8716		0.8722	0.8727	0.8733	0.8739	0.8745
7.5	0.8751	0.8756	0.8762	0.8768	0.8774		0.8779	0.8785	0.8791	0.8797	0.8802
7.6	0.8808	0.8814	0.8820	0.8825	0.8831		0.8837	0.8842	0.8848	0.8854	0.8859
7.7	0.8865	0.8871	0.8876	0.8882	0.8887		0.8893	0.8899	0.8904	0.8910	0.8915
7.8	0.8921	0.8927	0.8932	0.8938	0.8943		0.8949	0.8954	0.8960	0.8965	0.8971
7.9	0.8976	0.8982	0.8987	0.8993	0.8998		0.9004	0.9009	0.9015	0.9020	0.9025
8.0	0.9031	0.9036	0.9042	0.9047	0.9053		0.9058	0.9063	0.9069	0.9074	0.9079
8.1	0.9085	0.9090	0.9096	0.9101	0.9106		0.9112	0.9117	0.9122	0.9128	0.9133
8.2	0.9138	0.9143	0.9149	0.9154	0.9159		0.9165	0.9170	0.9175	0.9180	0.9186
8.3	0.9191	0.9196	0.9201	0.9206	0.9212		0.9217	0.9222	0.9227	0.9232	0.9238
8.4	0.9243	0.9248	0.9253	0.9258	0.9263		0.9269	0.9274	0.9279	0.9284	0.9289
8.5	0.9294	0.9299	0.9304	0.9309	0.9315		0.9320	0.9325	0.9330	0.9335	0.9340
8.6	0.9345	0.9350	0.9355	0.9360	0.9365		0.9370	0.9375	0.9380	0.9385	0.9390
8.7	0.9395	0.9400	0.9405	0.9410	0.9415		0.9420	0.9425	0.9430	0.9435	0.9440
8.8	0.9445	0.9450	0.9455	0.9460	0.9465		0.9469	0.9474	0.9479	0.9484	0.9489
8.9	0.9494	0.9499	0.9504	0.9509	0.9513		0.9518	0.9523	0.9528	0.9533	0.9538
9.0	0.9542	0.9547	0.9552	0.9557	0.9562		0.9566	0.9571	0.9576	0.9581	0.9586
9.1	0.9590	0.9595	0.9600	0.9605	0.9609		0.9614	0.9619	0.9624	0.9628	0.9633
9.2	0.9638	0.9643	0.9647	0.9652	0.9657		0.9661	0.9666	0.9671	0.9675	0.9680
9.3	0.9685	0.9689	0.9694	0.9699	0.9703		0.9708	0.9713	0.9717	0.9722	0.9727
9.4	0.9731	0.9736	0.9741	0.9745	0.9750		0.9754	0.9759	0.9763	0.9768	0.9773
9.5	0.9777	0.9782	0.9786	0.9791	0.9795		0.9800	0.9805	0.9809	0.9814	0.9818
9.6	0.9823	0.9827	0.9832	0.9836	0.9841		0.9845	0.9850	0.9854	0.9859	0.9863
9.7	0.9868	0.9872	0.9877	0.9881	0.9886		0.9890	0.9894	0.9899	0.9903	0.9908
9.8	0.9912	0.9917	0.9921	0.9926	0.9930		0.9934	0.9939	0.9943	0.9948	0.9952
9.9	0.9956	0.9961	0.9965	0.9969	0.9974		0.9978	0.9983	0.9987	0.9991	0.9996

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