# Chapter Two

# Vector Calculus

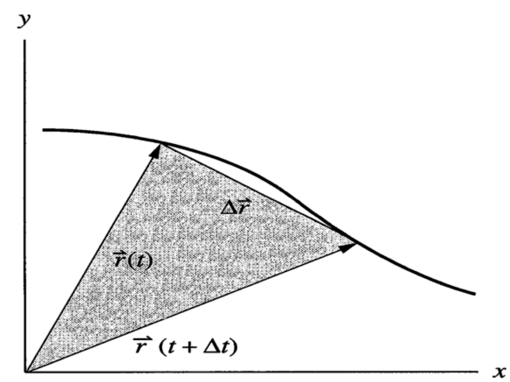
## TIME DERIVATIVES OF VECTORS

Suppose a vector A is changing with time; that is, its magnitude and/or direction is varying with time. If we choose a fixed coordinate system with basis vectors i and j , then

$$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j}.$$

The time rate of change of A is given by the derivative

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + A_x\frac{d\hat{i}}{dt} + \frac{dA_y}{dt}\hat{j} + A_y\frac{d\hat{j}}{dt}.$$




*2.1 The change Δr; in the position vector r(t) along a path in the x-y plane.*

But since we chose a fixed coordinate system (this is not necessary, in general),

$$d\hat{i}/dt = d\hat{j}/dt = 0, \text{ so}$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j}.$$

Let $\vec{A}(t)$ be the position vector $\vec{r}(t)$ The velocity vector is then

$$\vec{v}(t) = \lim_{\Delta t \to 0}\left[\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}\right]$$

$$= \lim_{\Delta t \to 0}\frac{\Delta \vec{r}}{\Delta t}$$

$$= \frac{d\vec{r}}{dt}.$$

## FLUID KINEMATICS

- A fluid is either a liquid or a gas.
- Many of the following examples and derivations will involve liquids, but most of the results will apply equally well to gases.
- One difference between liquids and gases is that it is very difficult to compress a liquid, but not so difficult to compress a gas.
- We will assume that the fluids are compressible. Since a fluid is made up of particles (atoms or molecules) with mass, we could apply Newton's laws and study the dynamics of each particle.
- However, because the number of particles is so large, this is impractical.
- Instead, we will concentrate on a "small" volume fixed in space and specify the density ρ and the velocity v of the fluid as it passes through this fixed volume in a given time.
- In order to make the analysis of a fluid feasible, we will make the following approximations:

1. **Steady flow**-at any given point (small volume), the velocity of the fluid,v,is constant in time
2. **Irrotational flow**-the fluid at a point has no net angular velocity. This will be made more quantitative.
3. **Non viscous fluid**-viscosity is a measure of the friction exerted on a fluid flowing past a surface. We will neglect any effects due to viscosity.

***For now we consider steady, irrotational, nonviscous flow.***

- If the fluid flow is steady, then at each point in space we can assign a unique velocity v of the fluid.
- If we draw a curve representing the path followed by a particle of the fluid, then every particle that reaches a point will follow the same curve.
- The curve describing the path taken by any particle passing through a given point is called a streamline. Notice that no two streamlines can cross.

- The velocity is tangent to the curve at any point on the curve.
- Shown in Figure 2.3 is a set of streamlines that form a tubular bundle, called a tube of flow.
- The velocity is always tangent to the streamlines; no fluid will flow through the side of the tube of flow.
- All of the fluid that enters at one end of the tube must exit at the other end.
- If we consider a thin tube, the velocity can be taken as constant over the area of the ends. However, the velocities at the two ends may differ.
- We will now derive a relationship between the flow into and out of a tube of flow.

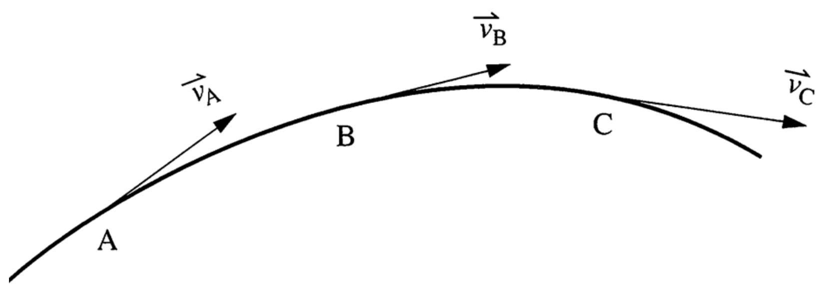



*Fig 2.2 A streamline in a fluid along which a tangent vector represents the velocity of the fluid at a given point.*

Figure 2.3 shows a tube of flow whose ends have areas $A_1$ and A2 and velocities $V_1$ and $V_2$, respectively. In a time Δt, an element of fluid travels

**a distance $v\Delta t$. Then the mass crossing area $A_1$ in time $\Delta t$ is $\Delta m_1 = \rho_1 V_1$, where $V_1 = A_1 v_1 \Delta t$ is the volume of the cylinder with area $A_1$ and length $v_1 \Delta t$**

So we define the mass flux (or flow) at end 1 as

$$\frac{\Delta m_1}{\Delta t} = \rho_1 A_1 v_1 .$$ At end 2 the mass flux is $\frac{\Delta m_2}{\Delta t} = \rho_2 A_2 v_2 .$

No fluid can leave through the walls; thus, if we assume there is no creation or destruction of fluid inside the tube, the mass flux in must equal the mass flux out, or $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$. We can then say that

$$\rho A v = \text{constant.}$$

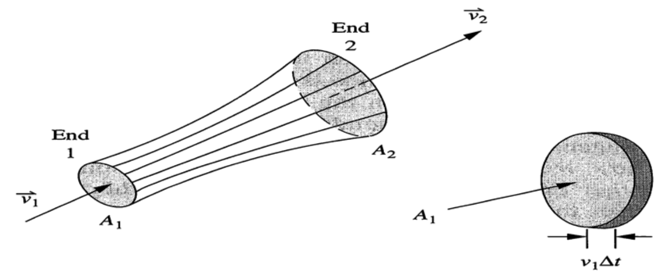



*Fig 2.3 A typical tube of flow. Also shown is a cylinder of fluid whose area is A, and whose length is $v_1$ Δt dt.*

If we further restrict our analysis to an incompressible fluid, then ρ = constant, so that

$$Av \equiv \text{flow rate} = \text{constant.}$$

## FLUID DYNAMICS

We can describe the motion of a fluid under the influence of pressure differences by using the concept of conservation of energy. Then the work done by the resultant force is equal to the change in the total mechanical energy of the system. Figure 2.4 shows a pipe through which a fluid is flowing subject to different pressures at the two ends, which are at different heights. The work done by external pressure P in changing the volume by ΔV is W = PΔV .

For small displacements Δx of the fluid, ΔV = AΔx, where A = area. Then W = PAΔx. For the situation shown in Figure 2.4, the total work is given by $W = P_1A_1\Delta x_1 - P_2A_2\Delta x_2$. The total mechanical energy is the sum of kinetic and gravitational potential energy, so for a small cylinder of fluid of mass m that moves from end 1 to end 2, the change in energy is

$$\begin{aligned}\Delta E &= \Delta E_k + \Delta U \\ &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 + mgy_2 - mgy_1.\end{aligned}$$

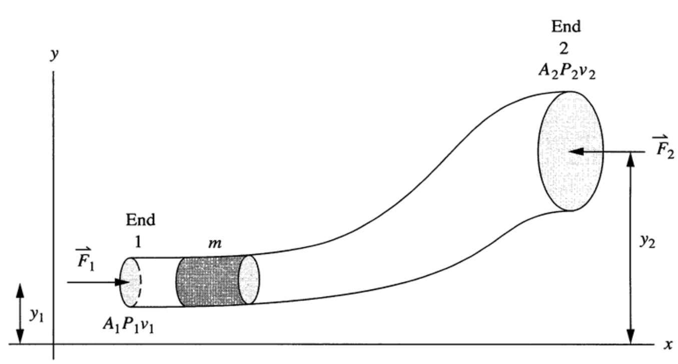



*Fig 2.4 A pipe through which fluid flows. The four quantities, pressure P, area A, velocity v, and height y, can be different at ends 1 and 2.*

The work-energy theorem states that

$$P_1A_1\Delta x_1 - P_2A_2\Delta x_2 = \frac{1}{2}m\left(v_2^2 - v_1^2\right) + mg(y_2 - y_1).$$

For an incompressible fluid, the volume occupied by a constant mass $m$ is constant. So $V_1 = A_1\Delta x_1 = V_2 = A_2\Delta x_2 = m/\rho$. Then

$$(P_1 - P_2)\frac{m}{\rho} = \frac{1}{2}m\left(v_2^2 - v_1^2\right) + mg(y_2 - y_1),$$

which can be rearranged to yield

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2.$$

This is equivalent to

$$P_1 + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}.$$

This Equation is called Bernoulli's equation.

We can interpret this in terms of the energy density by realizing that P =pressure = work/volume, $\rho v^2/2$ = kinetic energy/volume, $\rho g y$ = gravitational potential energy/volume.

The above Equation states that the energy/volume is constant. This is a stronger statement than conservation of energy; conservation of energy density must be true at every point, whereas conservation of energy is a statement about the whole system.

**Example**

Suppose the fluid is at rest.

Then $v_1 = v_2 = 0$, and $P_1 + \rho g y_1 = P_2 + \rho g y_2$, Or

$$P_2 - P_1 = -\rho g(y_2 - y_1).$$

This is the result for pressure in a static, incompressible fluid.

## FIELDS AND THE GRADIENT

The potential energy of a force is defined as the negative of the work done by the force. For example, if we lift a mass m a height h above the ground, the change in potential energy due to gravity is the negative of the work done by gravity. Thus,

$$W = \int_0^h (m\vec{g})\cdot(d\vec{y}) = -mg\int_0^h dy = -mgh.$$

Then $\Delta U = -W$ or $\Delta U = mgh$ is the change in gravitational potential energy.

We can think of the potential energy as being a function of the position of the mass (in this case its height only) and we can assign a potential energy to each point in space.

A generalization of the previous example is the *scalar field*. This is a function that assigns a scalar to every point represented by a position vector $\vec{r}$. The symbol usually used is $\phi(\vec{r})$. This is interpreted to mean that for any position $\vec{r}$, there is a unique scalar quantity given by $\phi(\vec{r}) = \phi(x, y, z)$. If the $x$-$y$ plane of a rectangular coordinate axis lies on the surface of the earth and the $+z$-axis points upward from the surface, the gravitational potential energy (near the surface) can be represented by a scalar field that is written with respect to some origin in the $x$-$y$ plane as

$$\phi(\vec{r}) = \phi(x, y, z) = mgz.$$

In this case the function $\phi$ depends only on the $z$-component of $\vec{r}$.

Suppose we have a physical quantity that can be represented by a scalar field in two dimensions $\phi(\vec{r}) = \phi(x, y)$ and we want to find how it changes when we move an infinitesimal distance from one point to another. In moving from point 1 to point 2, x and y change by dx and dy, respectively. Then

$\phi$ changes by

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy.$$

We can represent the vector for displacement from point 1 to point 2 by ds = dxi + dyj.

Then dϕ can be written as the dot product of ds with another vector A defined by

$$\vec{A} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}.$$

So

$$\begin{aligned} d\phi &= \vec{A} \cdot d\vec{s} \\ &= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j}\right) \\ &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy. \end{aligned}$$

The vector $\vec{A}$ is called the *gradient* of the function $\phi$ and the symbol is $A \equiv \nabla\phi$.
In three dimensions,

$$\nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}.$$

It is usual to define the **vector operator** which is called "del" or "nabla"

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} \;+\; \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} \;+\; \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\;.$$

∇ called the gradient operator; that is,∇ operates on whatever scalar function appears to the right of it and creates a vector function.

$\phi(\vec{r}\,) = \phi(x, y, z)$ was called a scalar field because it assigns a scalar quantity to every point in space. If we calculate the gradient of the scalar field, $\nabla\phi$, we get another quantity, call it $\vec{A}(\vec{r}) = \vec{A}(x, y, z)$, that assigns a vector to every point in space. This is called a *vector field.*

**Example 1**

Suppose we have a scalar field in two dimensions given by $\phi(x, y) = x^2 - y^2$. This assigns a scalar to every point $(x, y)$ in the $x$-$y$ plane. The gradient of this function is

Solution

$$\vec{A} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{i}$$

$$= 2x\,\hat{i} - 2y\,\hat{j}\,.$$

Example 2 : Choose the gradient of a scalar fields for $f(x, y, z) = xy^2 - yz.$

(a) $\boldsymbol{i} + (2x - z)\boldsymbol{j} - y\boldsymbol{k}\,,$ (b) $2xy\boldsymbol{i} + 2xy\boldsymbol{j} + y\boldsymbol{k}\,,$

(c) $y^2\boldsymbol{i} - z\boldsymbol{j} - y\boldsymbol{k}\,,$ (d) $y^2\boldsymbol{i} + (2xy - z)\boldsymbol{j} - y\boldsymbol{k}\,.$

Example 3 : Find the gradient by using the following scalar fields.

1. $U = x^2$

$$\Rightarrow \boldsymbol{\nabla} U = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} \;+\; \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} \;+\; \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) x^2 = 2x\hat{\boldsymbol{\imath}}\;.$$

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2. $U = r^2$

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \Rightarrow \boldsymbol{\nabla} U &= \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} \;+\; \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} \;+\; \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right)(x^2 + y^2 + z^2) \\ &= 2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}} \;=\; 2\,\mathbf{r}\;. \end{aligned}$$

## FLUID FLOW AND THE DIVERGENCE

Suppose we choose a small area A in a fluid over which the velocity v is approximately constant.

The flux is defined as:

$$\Phi = \rho\vec{v}\cdot\vec{A}$$

where $\vec{A} = \hat{n}A$ and $\hat{n}$ is a unit vector perpendicular to the area.

If v varies over the area A, then we define the flux by

$$\Phi = \int_A \rho\vec{v}\cdot d\vec{a},$$

where $d\vec{a} = \hat{n}\,da$ and $da$ is the infinitesimal surface area element.

If $\hat{n}$ is parallel to $\vec{v}$, then $\vec{v}$ is perpendicular to the area and $\vec{v}\cdot\vec{A} = vA\cos(0) = vA$; that is, the flux is a maximum. If $\hat{n}$ is perpendicular to $\vec{v}$, then $\vec{v}\cdot A = 0$ and the flux through area $A$ is zero

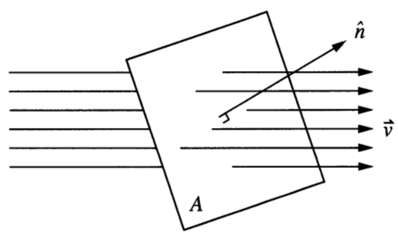



*Fig 2.5 An area A, the direction of which is given by ñ, with fluid flowing through it in a direction given by the direction of v.*

We defined the gradient operator by

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

If we treat $\nabla$ as an ordinary vector,

$$\left(\frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} + \frac{\partial \gamma_z}{\partial z}\right) = \nabla\cdot\vec{\gamma}.$$

This is called the *divergence* of the vector field $\vec{\gamma} = \rho\vec{v}$. The divergence is also written $div(\rho\vec{v})$. The order of $\nabla$ and $\rho\vec{v}$ in $\nabla\cdot(\rho\vec{v})$ is important. Since $\nabla$ is an operator, writing the divergence as $\rho\vec{v}\cdot\nabla$ leaves $\nabla$ without something on which to operate and in general gives a result different from $\nabla\cdot(\rho\vec{v})$.

The above two equations show a relationship between the divergence of the vector field and the flux:

$$\Phi_T = \oint_A \rho\vec{v} \cdot d\vec{A}$$
$$= \int_V \nabla \cdot (\rho\vec{v})\, dV.$$

This equation is called the **divergence theorem** and it holds for any vector field F:

$$\oint_A \vec{F} \cdot d\vec{A} = \int_V \nabla \cdot \vec{F}\, dV,$$

where A is the surface area enclosing the volume V.

**Examples 1:** Find the divergence of a vector field. If

$$\boldsymbol{F}(x, y) = 3x^2\boldsymbol{i} + 2y\boldsymbol{j}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{F}(x, y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2y) = 6x + 2\,. \end{aligned}$$

Quiz Select the divergence of $\boldsymbol{F}(x, y) = \frac{x}{y}\boldsymbol{i} + (2x - 3y)\boldsymbol{j}$.

(a) $\frac{1}{y} - 3\,,$ (b) $-\frac{x}{y^2} + 2\,,$ (c) $\frac{1}{y} - \frac{x}{y^2}\,,$ (d) $-2\,.$

The definition of the **divergence** may be directly extended to vector fields defined in three dimensions, $\boldsymbol{F}(x, y, z) = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\,.$$

## CIRCULATION AND THE CURL

The **curl** of a vector field, $\boldsymbol{F}(x, y, z)$, in three dimensions may be written curl $\boldsymbol{F}(x, y, z) = \boldsymbol{\nabla} \times \boldsymbol{F}(x, y, z)$, i.e.:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F}(x, y, z) &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}\,. \end{aligned}$$

It is obtained by taking the *vector product* of the vector operator $\boldsymbol{\nabla}$ applied to the vector field $\boldsymbol{F}(x, y, z)$. The second line is again a formal shorthand. The *curl* of a vector field is a *vector* field.

N.B. $\boldsymbol{\nabla} \times \boldsymbol{F}$ is sometimes called the *rotation* of $\boldsymbol{F}$ and written rot $\boldsymbol{F}$.

For example

$$(\nabla \times \vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.$$

Example 1

The curl of $\boldsymbol{F}(x, y, z) = 3x^2\boldsymbol{i} + 2z\boldsymbol{j} - x\boldsymbol{k}$ is:

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 2z & -x \end{vmatrix}$$

$$\begin{aligned} &= \left(\frac{\partial(-x)}{\partial y} - \frac{\partial(2z)}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(3x^2)}{\partial z}\right)\boldsymbol{j} \\ &\quad + \left(\frac{\partial(2z)}{\partial x} - \frac{\partial(3x^2)}{\partial y}\right)\boldsymbol{k} \\ &= (0-2)\boldsymbol{i} - (-1-0)\boldsymbol{j} + (0-0)\boldsymbol{k} \\ &= -2\boldsymbol{i} + \boldsymbol{j}\ . \end{aligned}$$

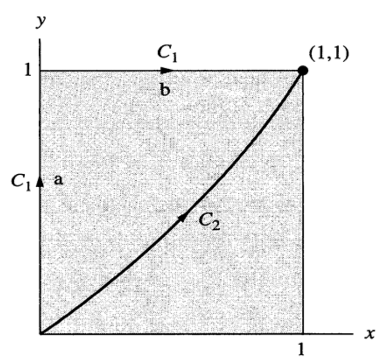
Let as see the curls in different coordinate systems. A line integral is defined as the integral of a function along a given path or curve. An example of a physical quantity that is defined in terms of a line integral is the work done by a force in moving an object along a particular path in space. This is given by

$$W = \int_C \vec{F} \cdot d\vec{r},$$

where $\int_C$ means the integral is along the curve $C$. So

$$W = \int_C (F_x\, dx + F_y\, dy + F_z\, dz).$$

Given $\vec{F} = xy\,\hat{i} - y^2\,\hat{j}$, find the work along paths $C_1$ and $C_2$ shown in Figure below.



*For curve* $C_1$: Along part a, $x = 0$ and $dx = 0$, so

$$W_a = \int_0^1 (-y^2)\,dy = -\frac{1}{3},$$

and along part b, $y = 1$ and $dy = 0$, so

$$W_b = \int_0^1 x\,dx = \frac{1}{2}.$$

Thus, $W_{C_1} = \frac{1}{6}$.

*For curve* $C_2$: $y = x^2$ (a parabola), so $x$ and $y$ are not independent. Replace $y$ by $x^2$ and $dy$ by $2xdx$:

$$W_{C_2} = \int_0^1 (x^3\,dx - 2x^5\,dx) = -\frac{1}{12}.$$

## CONSERVATIVE FORCES AND THE LAPLACIAN

Not all forces have a corresponding potential energy. For example, frictional forces cannot be represented by potential energies. Thus, we would like to have a method for determining whether a potential energy can be defined for a particular force.

A force that can be represented by a potential energy is called a conservative force. There are several equivalent definitions for conservative forces; three are given here.

1. For a conservative force, the work done by the force along a path from point A to point B is independent of the path.
2. The work done by a conservative force around a closed path is zero.
3. A force $\vec{F}$ is conservative if $\nabla \times \vec{F} = 0$.

Definition 1 is equivalent to the statement that the potential energy is a function only of the end points of a path. This implies that the infinitesimal change in potential energy, $dU$, must be an exact differential. By definition, if $dU$ is an exact differential, we have

$$\int_A^B dU = U(B) - U(A).$$

Given the infinitesimal change

$$dU = \vec{F} \cdot d\vec{r} = F_x(x, y)dx + F_y(x, y)dy,$$

$dU$ will be an exact differential if

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}.$$

To show this, we use the fact that for a function $U(x, y)$, the total differential is

$$dU = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy$$

By comparing the above two equations,

$$F_x(x, y) = \frac{\partial U}{\partial x} \quad \text{and} \quad F_y(x, y) = \frac{\partial U}{\partial y}.$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial F_y}{\partial x}.$$

Definition 2 follows from definition 1 in a straightforward way by realizing that

$$\int_C \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r} + \int_B^A \vec{F} \cdot d\vec{r},$$

where $A$ and $B$ are two points on the closed curve $C$. Since, from definition 1,

$$\int_B^A \vec{F} \cdot d\vec{r} = -\int_A^B \vec{F} \cdot d\vec{r},$$

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

Definition 3 is a convenient method for determining whether a force is conservative.Which is,

$$\nabla \times \vec{F} = 0.$$

**Example**

The gravitational force between masses $m_1$ and $m_2$ separated by a distance $r = \sqrt{x^2 + y^2 + z^2}$ is given by Newton's law of gravity:

$$\vec{F} = G\frac{m_1 m_2}{r^3}\vec{r} \equiv K\frac{\vec{r}}{r^3}.$$

To show that this is a conservative force, we evaluate the curl of $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Kx/r^3 & Ky/r^3 & Kz/r^3 \end{vmatrix}$$

$$= -\frac{3}{r^5}\left[\hat{i}(zy - yz) + \hat{j}(zx - xz) + \hat{k}(xy - yx)\right]$$

$$= 0,$$

Quiz Which of the following is the curl of $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$?
(a) $2\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, (b) $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, (c) $0$, (d) $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

# Laplacian:

Recall that gradU of any scalar field U is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute div(gradU ), even if we don't know what it means yet.

We have just shown that $\vec{F}$ is a conservative field if

$$\nabla \times \vec{F} = 0.$$

$\nabla \times \nabla\phi = 0$ for any scalar field $\phi$. Thus, if

$\nabla \times \vec{F} = 0$, $\vec{F}$ can be written as the gradient of a scalar field $\phi$. This will be the potential energy field; more precisely,

( $\vec{F} = -\nabla U,$ (The scalar field U (r) is the Potential Function)

where $U$ is a scalar field and the minus sign is conventional.

If $\vec{F}$ is a conservative field, then

$$\nabla \cdot \vec{F} = -\nabla \cdot (\nabla U).$$

In rectangular coordinates,

$$\begin{aligned}\nabla \cdot (\nabla U) &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)\\ &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\\ &\equiv \nabla^2 U,\end{aligned}$$

where $\nabla^2$ is called the *Laplacian*. Then we have

$$\nabla^2 U = -\nabla \cdot \vec{F}.$$

If there are no sinks or sources of $\vec{F}$, then $\nabla \cdot \vec{F} = 0$,

$$\nabla^2 U = 0.$$

This is called *Laplace's equation*,

# ELECTRIC AND MAGNETIC FIELDS

Perhaps the most important application of vector calculus in physics is in electromagnetism. In this section we examine electric and magnetic fields and how they are described in terms of vector and scalar fields.

## ELECTRIC FIELDS

The electric field produced at the point r by a point charge q at the point r' is given in SI units by:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3}.$$

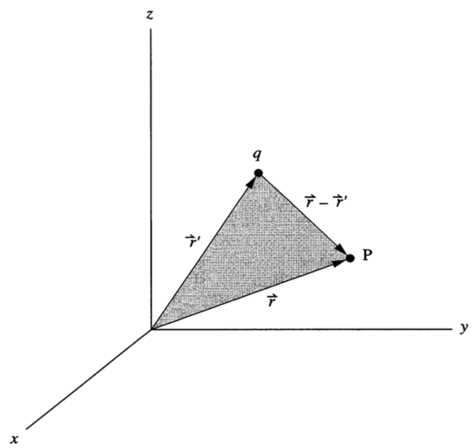



*Fig 2.6 Coordinate system for specifying the positions of the charge q and thpoint P.*

For a continuous distribution of charge represented by the charge density $\rho(\vec{r}\,')$,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}dV'.$$

Even though nothing is "flowing," the electric field lines can be compared to the streamlines of fluid flow. In particular, we can define the electric field flux through an area in analogy with the mass flux in fluid flow. Thus, if E is constant over an area A, then the electric flux is given by

$$\Phi_E = \vec{E}\cdot\vec{A}.$$

If $\vec{E}$ is not constant over the total area, we generalize the definition to

$$\Phi_E = \int_A \vec{E}\cdot d\vec{A}.$$

From this definition, for a closed surface A, if the flux is nonzero, the number of lines going into the volume enclosed by the A is different from the number of lines coming out of the volume. Since electric field lines originate from positive charge and terminate on negative charge, a net flux through a closed surface is possible only if there is a net charge inside the volume. The precise relationship is given by Gauss' law:

$$\Phi_E = \oint_A \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} q_{\text{in}},$$

where $q_{\text{in}}$ is the net charge inside the volume enclosed by $A$.

We can use the divergence theorem

$$\int_V \nabla \cdot \vec{E}\, dV = \oint_A \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} q_{\text{in}},$$

and if we write

$$q_{\text{in}} = \int_V \rho(\vec{r})dV,$$

where V is the volume enclosed by A, we get

$$\int_V \nabla \cdot \vec{E}\, dV = \int_V \frac{1}{\epsilon_0} \rho(\vec{r})dV.$$

Since this equation is true for any volume V, the integrands must be equal, and

$\nabla \cdot \vec{E} = \rho(\vec{r})/\epsilon_0.$

From the maxwell's equation

$\nabla \times \vec{E} = 0.$ and we know that $\nabla \times \nabla V = 0$ for any scalar field $V$; then,

since $\nabla \times \vec{E} = 0$, we can write

$$\vec{E} = -\nabla V,$$

where the minus sign is conventional. $V$ is called the *electric scalar potential*.

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \nabla \cdot (-\nabla V),$ Or $\nabla^2 V = -\rho/\epsilon_0.$ **This equation is called Poisson's equation**.

If $\rho = 0$ in the region of interest, the above equation reduces to Laplace's equation for V,

$$\nabla^2 V = 0.$$

# MAGNETIC FIELDS

The magnetic field produced at a point $\vec{r}$ due to a current density is given by the *Biot-Savart law*:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dV',$$

where $\vec{J}(\vec{r}\,')$ is the current density, or current/area.

The magnetic flux is defined as

$$\Phi_B = \vec{B} \cdot \vec{A},$$

or in the case where $\vec{B}$ varies over $\vec{A}$,

$$\Phi_B = \int_A \vec{B} \cdot d\vec{A}.$$

Note that, $\nabla \cdot \vec{B} = 0$.
From the divergence theorem,

$$\oint_A \vec{B} \cdot d\vec{A} = \int_V \nabla \cdot \vec{B}\, dV = 0,$$

so the net flux through a closed surface A is zero.
Since $\nabla \cdot \vec{B} = 0$ and since $\nabla \cdot (\nabla \times \vec{F}) \equiv 0$ for any vector field $\vec{F}$ $\vec{B}$ can be written as the curl of a vector field, usually called $\vec{A}$:

$\vec{B} = \nabla \times \vec{A}$. $\vec{A}$ is the *magnetic vector potential*.

*Ampere's law* relates the line integral of $B$ around a closed loop $C$ to the current passing through the area $A$ bounded by the loop

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{thru}}.$$

Thus, the circulation of $\vec{B}$ around a closed loop equals $\mu_0$ times the current passing through the loop. This is useful for calculating $\vec{B}$ in highly symmetric situations, and in that sense is the magnetic analog of Gauss' law. We can apply Stokes' theorem to obtain a differential version

$$\oint_C \vec{B} \cdot d\vec{l} = \int_A (\nabla \times \vec{B}) \cdot d\vec{A}.$$

By combining the two equations:

$$\mu_0 I_{\text{thru}} = \mu_0 \int_A \vec{J} \cdot d\vec{A},$$

$$\int_A (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \int_A \vec{J} \cdot d\vec{A}.$$

Since this must be true for any area $A$, the integrands must be equal, and

$$\nabla \times \vec{B} = \mu_0 \vec{J}.$$

This equation states that the curl of the magnetic field at a point equals $\mu_0$ times the current density at that point.

We can summarize the four basic results derived for static $\vec{E}$ and $\vec{B}$ fields:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J}.$$

These are *Maxwell's equations* for static fields.

## VECTOR CALCULUS EXPRESSIONS AND IDENTITIES

This section presents expressions for the vector operators in three coordinate systems. In addition, several useful vector identities are listed.

## CARTESIAN COORDINATES

Line element: $d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Volume element $dV = dx\,dy\,dz$

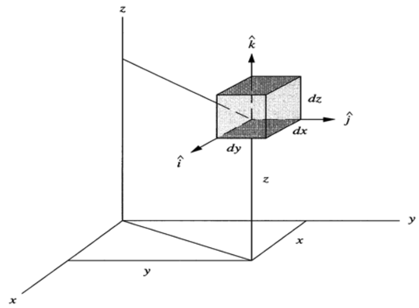



*Fig 2.7 Coordinates and infinitesimal volume element in the cartesian (rectangular)*

| | |
|---|---|
| Gradient: | $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ |
| Divergence: | $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ |
| Curl: | $\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$ |
| Laplacian: | $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ |

# CYLINDRICAL COORDINATES:

Relation to cartesian coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Line element: $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{z}$

Volume element $dV = r\,dr\,d\theta\,dz$

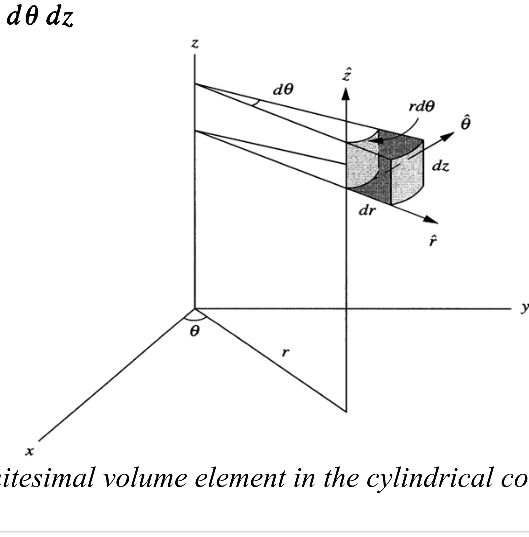



*Fig 2.8 Coordinates and infinitesimal volume element in the cylindrical coordinate system*

| | |
|---|---|
| Gradient: | $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$ |
| Divergence: | $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$ |
| Curl: | $\nabla \times \vec{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right]\hat{r} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_r}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{z}$ |
| Laplacian: | $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$ |

## SPHERICAL COORDINATES

Relation to cartesian coordinates:

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

Line element: $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

Volume element $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

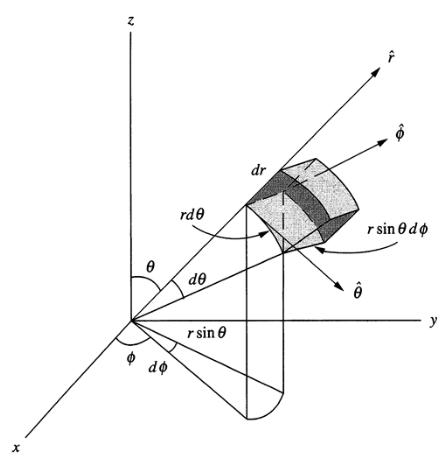



*Fig 2.9 Coordinates and infinitesimal volume element in the spherical coordinate system.*

| | |
|---|---|
| Gradient: | $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$ |
| Divergence: | $\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$ |
| Curl: | $\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$ |
| Laplacian: | $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$ |

## VECTOR IDENTITIES

1. $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$
2. $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$
3. $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$
4. $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f)$
5. $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$
6. $\nabla\cdot(\nabla\times\vec{A}) = 0$
7. $\nabla\times(\nabla f) = 0$
8. $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

**……………………..END……………..**

# Exercise

1. Calculate the divergence of the vector fields

(a) $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ , (b) $\boldsymbol{F} = y^3\boldsymbol{i} + xy\boldsymbol{j}$ ,

2. Calculate the curl of the following vector fields $\boldsymbol{F}(x, y, z)$

(a) $\boldsymbol{F} = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$ , (b) $\boldsymbol{F} = y^3\boldsymbol{i} + xy\boldsymbol{j} - z\boldsymbol{k}$ ,

(c) $\boldsymbol{F} = \dfrac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2 + y^2 + z^2}}$