Debre Markos University

College of Natural and Computational Science

Department of Mathematics

Linear algebra II Worksheet II

1. Determine whether the mapping $\langle\ ,\ \rangle$ is an inner product or not on the indicated vector space V.
   a) $V=\mathbb{R}^2, \langle X\ ,Y\rangle = \mathrm{ax_1y_1+bx_2y_2}$, where $\mathrm{X=(x_1,x_2), Y=(y_1,y_2)}$, a > 0 and b > 0.
   b) $V=\mathbb{R}^2, \langle X\ ,Y\rangle = \mathrm{x_1y_2+x_2y_1}$, where $\mathrm{X=(x_1,x_2), Y=(y_1,y_2)}$.
   c) $V=P_2(\mathbb{R})$ = vector space of polynomials with dgree ≤ 2 over the field $\mathbb{R}$, $\langle f\ ,g\rangle = \mathrm{f(0)g(0)+f(1)g(1)+f(2)g(2)}$
   d) $V=M_2(\mathbb{R})$, for $A,B\in M_2(\mathbb{R})$, define $\langle A\ ,B\rangle = a_{11}\mathrm{b_{11}}+2\mathrm{a_{12}b_{21}}+3a_{21}\mathrm{b_{12}}+\mathrm{a_{22}b_{22}}$.
2. Find the inner product of u and v in the indicated space V if :
   a) $V=\mathbb{C}^3, u=(1+i,-3,4-3i)\, and\ v=(2-i,-i,2+i)\ with\ standard\ inner\ product\ in\ \mathbb{C}^3$.
   b) $V=C[0,1]=The\ vector\ space\ continuous\ functions\ on\ [0,1], u(t)=t^2+t+1\ and\ v(t)=t^3+2t^2+3t-1$ with an inner product $\langle f,g\rangle=\int_o^1 f(x)g(x)dx$.
3. Let $V=\mathbb{C}^3$ with the standard inner product. Let $x\ =(2,1+i,i)$ and $y\ =\ (2-i,2,l+\ 2i)$. Compute $\langle x,y\rangle$, $\|x\|, \|y\|$ and $\|x+y\|^2$. Then verify both Cauchy's inequality and the triangle inequality.
4. Suppose that $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are two inner products on a vector space V. Prove that $\langle\ ,\ \rangle = \langle\ ,\ \rangle_1 + \langle\ ,\ \rangle_2$ is another inner product on V.
5. Determine whether the following set of vectors is orthonormal or not.
   a) $V=\mathbb{C}^3$ with standard inner product, $S=\{(i,1{,}0),(0,i,1),(0{,}0,i)\}$
   b) $V=M_2$ with inner product $\langle A,B\rangle = trace(B^tA)$ $S=\left\{\begin{pmatrix}2&1\\-1&3\end{pmatrix},\begin{pmatrix}3&0\\0&2\end{pmatrix},\begin{pmatrix}2&1\\-1&-1\end{pmatrix}\right\}$
6. Let $V=R^2$.Let $u=(1{,}1,-2)$ and $v=(a,-1{,}2)$ for what values of a re u and v orthogonal?
7. Let V be an inner product space, and suppose that $T:V\to V$ is linear and that $||\mathrm{T(x)}|| \ =\ ||\mathrm{x}||$ for all x. Prove that T is one-to-one.
8. Let $W_1$and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1+\ W_2)^{\perp}={W_1}^{\perp}\cap {W_2}^{\perp}$ and $(W_1\cap\ W_2)^{\perp}={W_1}^{\perp}+{W_2}^{\perp}$.
9. Define an inner product on P_3 by $\langle f,g\rangle=\int_o^1 f(x)g(x)dx$. Let $f(x)=x\ and\ g(x)=x^2$. Find
   a) $Proj_{g}f$
   b) $Proj_{f}g$
10. Define an inner product on P_3 by $\langle f,g\rangle=\int_{-1}^1 f(x)g(x)dx$.

a) Use the standard basis $B = \{1, x, x^2, x^3\}$ to construct orthonormal basis for P_3.

b) Find $\langle x, 1\rangle, \langle x^2, x\rangle$ and $\langle x^2, 1\rangle$

11. Consider S = {(1, 1, -1), (-1, 2, 4)} which is a subspace of R^3.

a) Find the orthogonal complement of S

b) Find the dimension of the orthogonal complement of S

12. Let β be a basis for a finite-dimensional inner product space. Prove that if $\langle x, y\rangle = 0$ for all $x \in \beta$, then y = 0.

13. For each of the following linear operators below, determine whether it is self-adjoint, isometry , normal or not

a) $T: R^2 \to R^2$ defined by $T(x, y) = (y, -x)$

b) $T: \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = (iz_1 + z_2, z_1 + z_2)$

c) $T: M_{2x2}(R) \to M_{2x2}(R)$ defined by $T(A) = (A + A^t)$ with $\langle A, B\rangle = trace(AB^t)$

d) $T: R^2 \to R^2$ defined by $T(x, y) = (3x - y, -x + 4y)$

e) $T: \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = (2z_1 + iz_2, z_1 + 2z_2)$

14. For each of the following inner product spaces V and linear operators T on V, evaluate T* at the given element of V.

a) $V = \mathbb{R}^2, T(a, b) = (2a + b, a - 3b), x = (3,5)$.

b) $V = \mathbb{C}^2, T(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1), x = (3 - i, 1 + 2i)$.

c) $V = P_2(\mathbb{R})$, with $\langle f, g\rangle = \int_0^1 f(x)g(x)dx, T(f) = f' + 3f, f(x) = 4 - x + 3x^2$

15. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all x in V $\|T(x) \pm ix\|^2 = \|T(x)\|^2$.

16. For each of the linear operators below, determine whether it is normal, self- adjoint, or neither.

a) $T: R^2 \to R^2$ defined by $T(a, b) = (2a - 2b, -2a + 5b)$.

b) ($T: \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(a, b) = \{2a + ib, a + 2b)$.

c) T: $P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f) = f'$ where $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$

17. Let T and U be self-adjoint operators on an inner product space. Prove that TU is self-adjoint ⇔ TU = UT.

18. Assume that T is a linear operator on a complex (not necessarily finite dimensional) inner product space V with an adjoint T*. Prove

a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.

b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then T = I = identity operator. Hint: Replace x by $x + y$ and then by $x + iy$ and expand the resulting inner products.

c) (c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T*$.