Digital Control Systems (DCS)

Lecture-2
Modeling of Digital Control Systems

Lecture Outline

- Sampling Theorem
- ADC Model
- DAC Model
- Combined Models

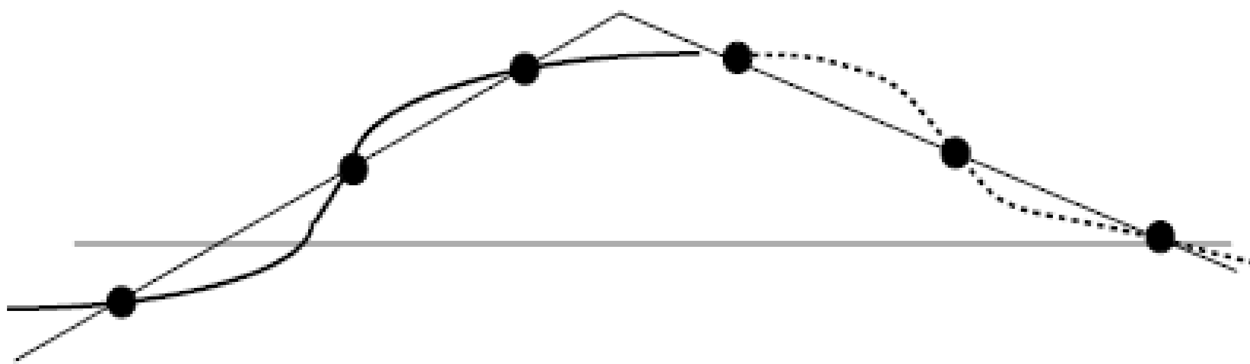
Sampling Theorem

- Sampling is necessary for the processing of analog data using digital elements.

- Successful digital data processing requires that the samples reflect the nature of the analog signal and that analog signals be recoverable from a sequence of samples.

Sampling Theorem

- Following figure shows two distinct waveforms with identical samples.



- Obviously, faster sampling of the two waveforms would produce distinguishable sequences.

Sampling Theorem

- Thus, it is obvious that sufficiently fast sampling is a prerequisite for successful digital data processing.

- The sampling theorem gives a lower bound on the sampling rate necessary for a given **band-limited** signal (i.e., a signal with a known finite bandwidth)

Sampling Theorem

- The band limited signal with

$$f(t) \overset{\mathscr{F}}{\rightarrow} F(j\omega), \qquad F(j\omega) \neq 0, \qquad -\omega_m \leq \omega \leq \omega_m$$
$$, \qquad F(j\omega) = 0, \qquad Elsewhere$$

- can be reconstructed from the discrete-time waveform

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - kT)$$

- if and only if the sampling angular frequency $\omega_s = 2\pi/T$ satisfies the condition

$$\omega_s > 2\omega_m$$

Selection of Sampling Frequency

- A given signal often has a finite "effective bandwidth" beyond which its spectral components are negligible.
- This allows us to treat physical signals as band limited and choose a suitable sampling rate for them based on the sampling theorem.
- In practice, the sampling rate chosen is often larger than the lower bound specified in the sampling theorem.
- A rule of thumb is to choose ω_s as

$$\omega_s = k\omega_m, \qquad 5 \le k \le 10$$

Selection of Sampling Frequency

$$\omega_s = k\omega_m, \qquad 5 \le k \le 10$$

- The choice of k depends on the application.

- In many applications, the upper bound on the sampling frequency is well below the capabilities of state-of-the-art hardware.

- A closed-loop control system cannot have a sampling period below the minimum time required for the output measurement; that is, the sampling frequency is upper-bounded by the **sensor delay**.

Selection of Sampling Frequency

- For example, oxygen sensors used in automotive air/fuel ratio control have a sensor delay of about 20 ms, which corresponds to a sampling frequency upper bound of 50 Hz.
- Another limitation is the computational time needed to update the control.
- This is becoming less restrictive with the availability of faster microprocessors but must be considered in sampling rate selection.

Selection of Sampling Frequency

- For a linear system, the output of the system has a spectrum given by the product of the frequency response and input spectrum.

- Because the input is not known a priori, we must base our choice of sampling frequency on the frequency response.

Selection of Sampling Frequency (1st Order Systems)

- The frequency response of first order system is

$$H(j\omega) = \frac{K}{j\omega/\omega_b + 1}$$

- where K is the DC gain and ω_b is the system bandwidth.

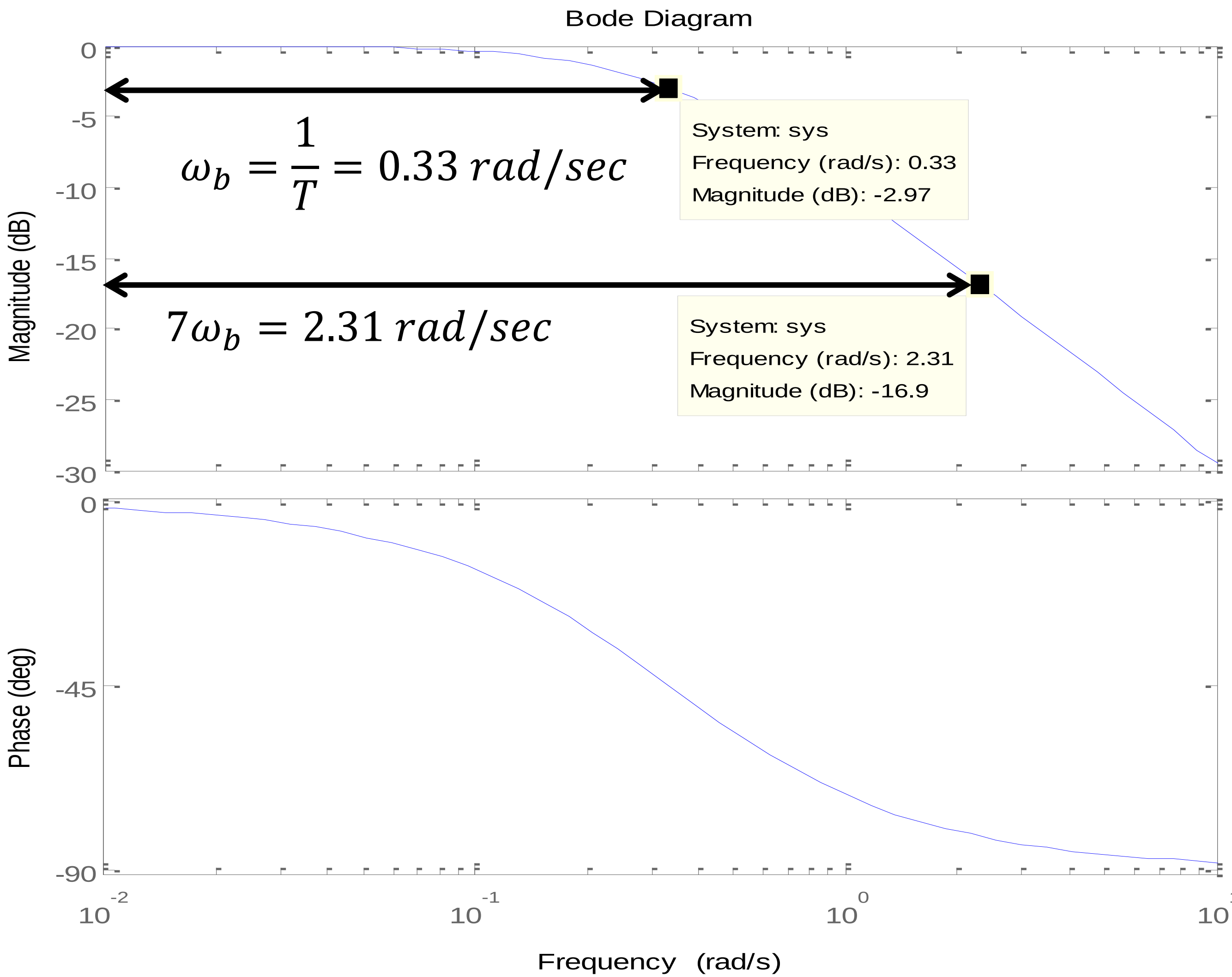
- Time constant and 3db bandwidth relationship

$$\omega_b = \frac{1}{T}$$

$$f_{3db} = \frac{1}{2\pi T}$$

Selection of Sampling Frequency (1st Order Systems)

$$G(s) = \frac{1}{3s+1}$$



Selection of Sampling Frequency (1st Order Systems)

- The frequency response amplitude drops below the DC level by a factor of about 10 at the frequency $7\omega_b$.

- If we consider $\omega_m = 7\omega_b$, the sampling frequency is chosen as

$$\omega_s = 7k\omega_b, \qquad 5 \leq k \leq 10$$

- OR

$$\omega_s = k\omega_b, \qquad 35 \leq k \leq 70$$

Selection of Sampling Frequency (2nd Order Systems)

- The frequency response of second order system is

$$H(j\omega) = \frac{K}{j2\zeta\omega/\omega_n + 1 - (\omega/\omega_n)^2}$$

- The bandwidth of the system is approximated by the damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

- Using a frequency of $7\omega_d$ as the maximum significant frequency, we choose the sampling frequency as

$$\omega_s = k\omega_d, \qquad 35 \le k \le 70$$

Example-1

- Given a first-order system of bandwidth 10 rad/s, select a suitable sampling frequency and find the corresponding sampling period.

Solution

$$\omega_b = 10\, rad/sec$$

- We know

$$\omega_s = k\omega_b, \qquad 35 \leq k \leq 70$$

- Choosing k=60

$$\omega_s = 60\omega_b = 600\, rad/sec$$

Example-1

- Corresponding sapling period is calculated as

$$T = \frac{2\pi}{\omega_s} = \frac{2 \times 3141}{600} = 0.01\, sec$$

Home work

- Fort he following first-order system select a suitable sampling frequency and find the corresponding sampling period.

$$G(s) = \frac{10}{s+1}$$

Home work

- Consider the following second order transfer function. Select a suitable sampling period for the system.

$$G(s) = \frac{16}{s^2 + 8s + 16}$$

Example-4

- A closed-loop control system must be designed for a damping ratio of about 0.7, and an undamped natural frequency of 10 rad/s. Select a suitable sampling period for the system if the system has a sensor delay of 0.02 sec.

Solution

- Let the sampling frequency be

$$\omega_s \geq 35\omega_d$$

$$\omega_s \geq 35\omega_n\sqrt{1-\zeta^2}$$

$$\omega_s \geq 35\times 10\sqrt{1-0.7^2}$$

Example-4

$$\omega_s \geq 249.95 \quad rad/s$$

- The corresponding sampling period is

$$T \leq \frac{2 \times 3.141}{249.95}$$

$$T \leq 0.025 \, sec$$
$$T \leq 25 \, ms$$

- A suitable choice is *T* = 20 ms because this is equal to the sensor delay.

Home Work

- A closed-loop control system must be designed for a damping ratio of about 0.7, and an undamped natural frequency of 10 rad/s. Select a suitable sampling period for the system if the system has a sensor delay of 0.03 sec.

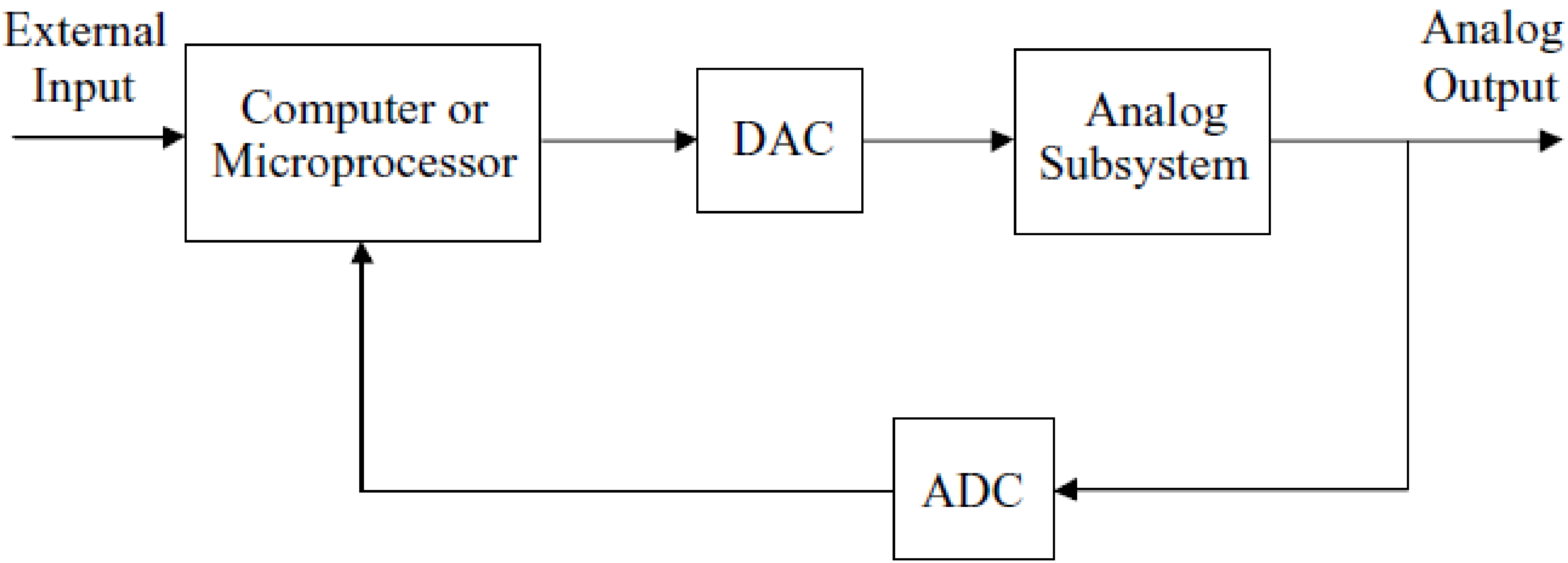
Home Work

- The following open-loop systems are to be digitally feedback-controlled. Select a suitable sampling period for each if the closed-loop system is to be designed for the given specifications.

1. $G(s) = \frac{1}{s+3}$, sensor delay=0.025s
2. $G(s) = \frac{1}{s^2+7s+25}$,sensor delay=0.03s

Digital Control Systems

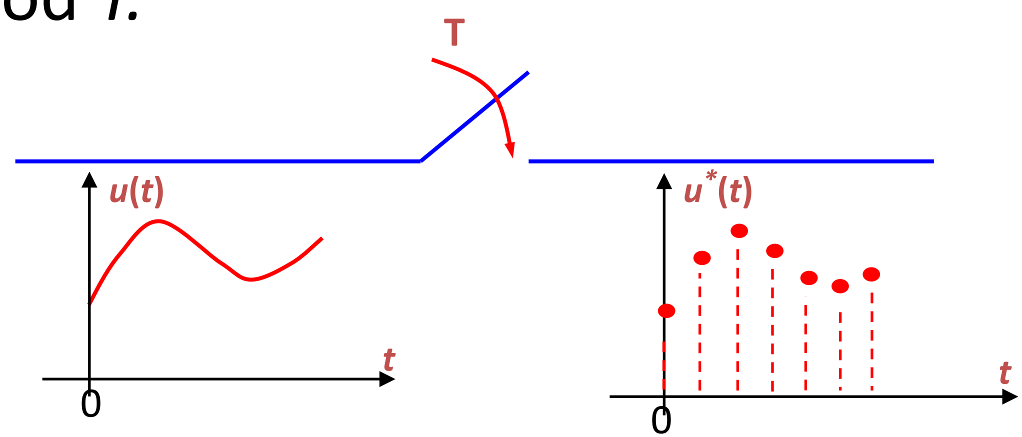
- A common configuration of digital control system is shown in following figure.



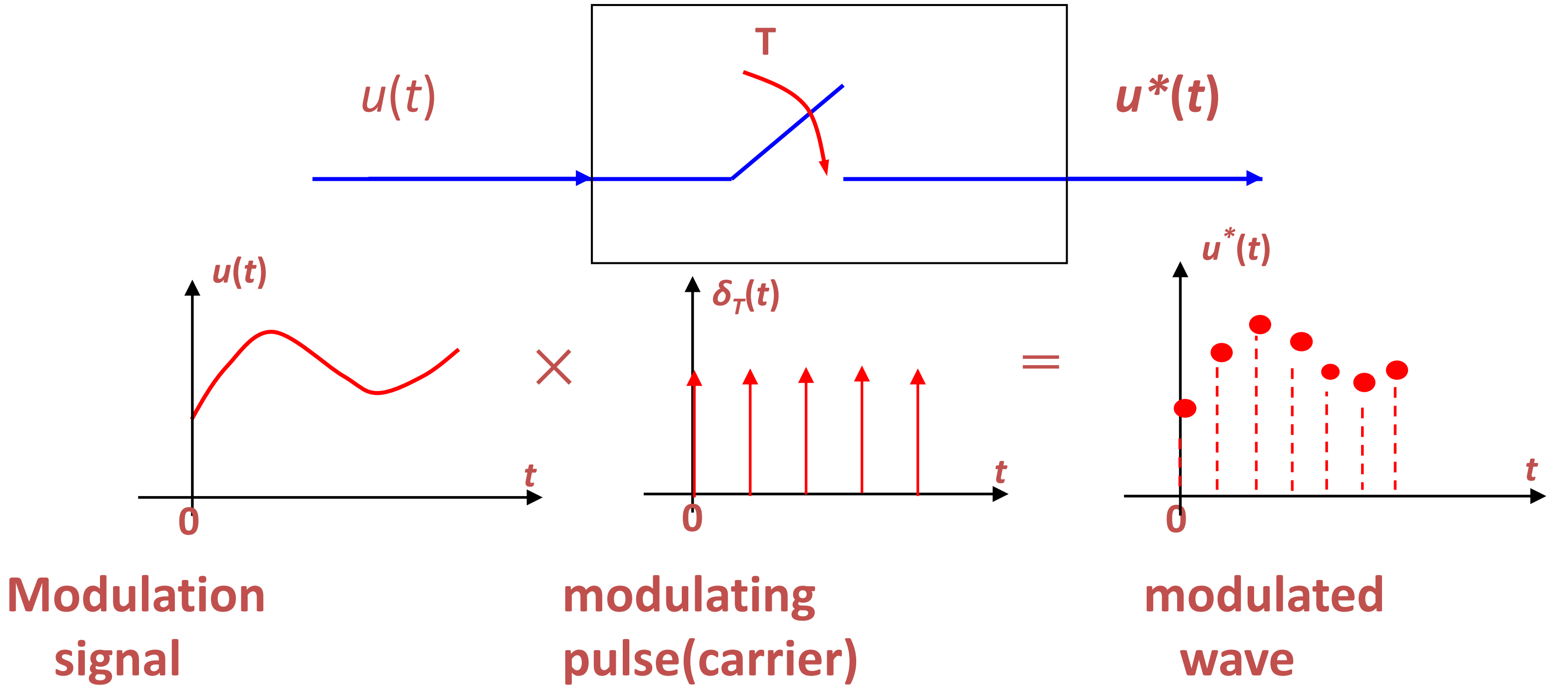
ADC Model

- Assume that
  - ADC outputs are exactly equal in magnitude to their inputs (i.e., quantization errors are negligible)
  - The ADC yields a digital output instantaneously
  - Sampling is perfectly uniform (i.e., occur at a fixed rate)

- Then the ADC can be modeled as an ideal sampler with sampling period *T.*



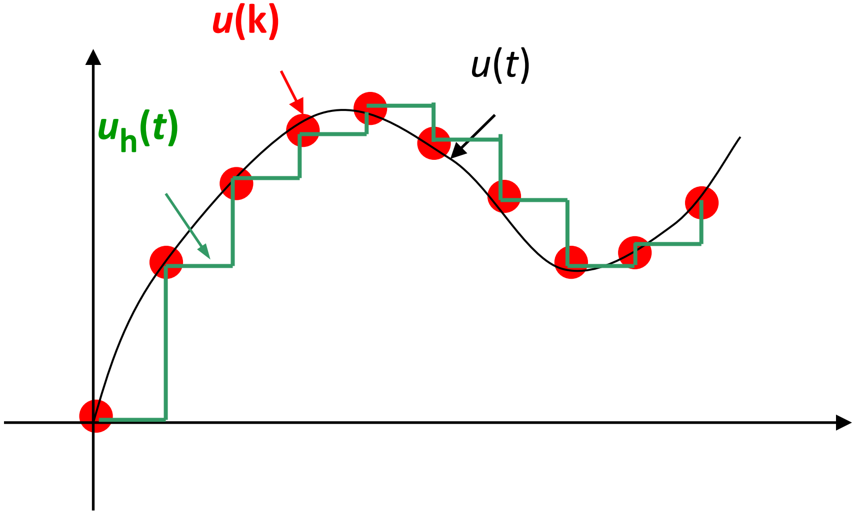
Sampling Process



$$u^*(t) = \sum_{k=0}^{\infty} u(t)\delta(t - kT)$$

DAC Model

- Assume that
  - DAC outputs are exactly equal in magnitude to their inputs.
  - The DAC yields an analog output instantaneously.
  - DAC outputs are constant over each sampling period.

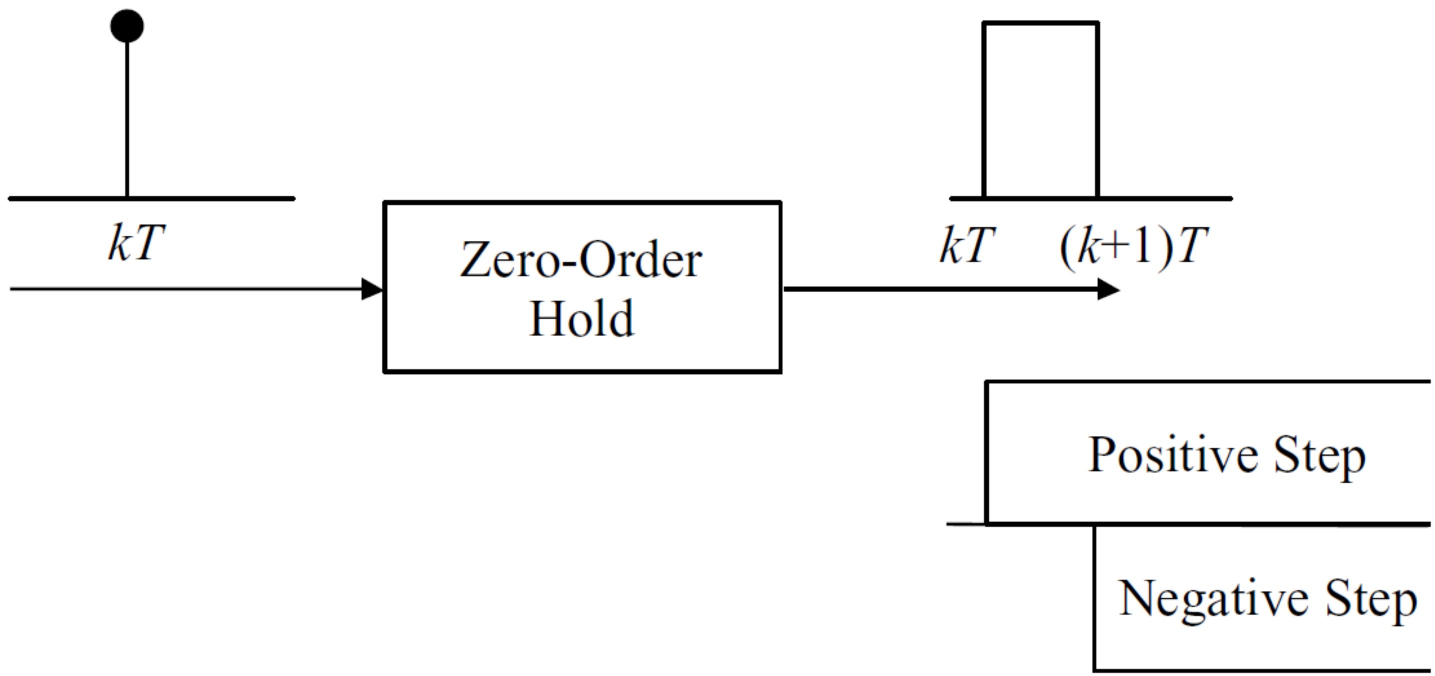


- Then the input-output relationship of the DAC is given by

$$u(k) \xrightarrow{ZOH} u_h(t) = u(k), \qquad kT \leq t \leq (k+1)T$$

DAC Model

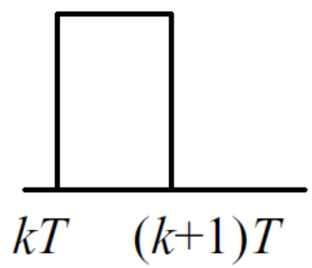
- Unit impulse response of ZOH



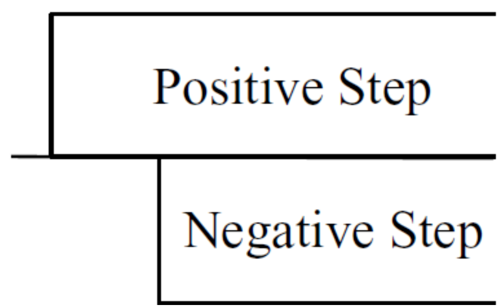
- The transfer function can then be obtained by Laplace transformation of the impulse response.

DAC Model

- As shown in figure the impulse response is a unit pulse of width T.



- A pulse can be represented as a positive step at time zero followed by a negative step at time *T*.



- Using the Laplace transform of a unit step and the time delay theorem for Laplace transforms,

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \qquad \mathcal{L}\{-u(t-T)\} = -\frac{e^{-Ts}}{s}$$

DAC Model

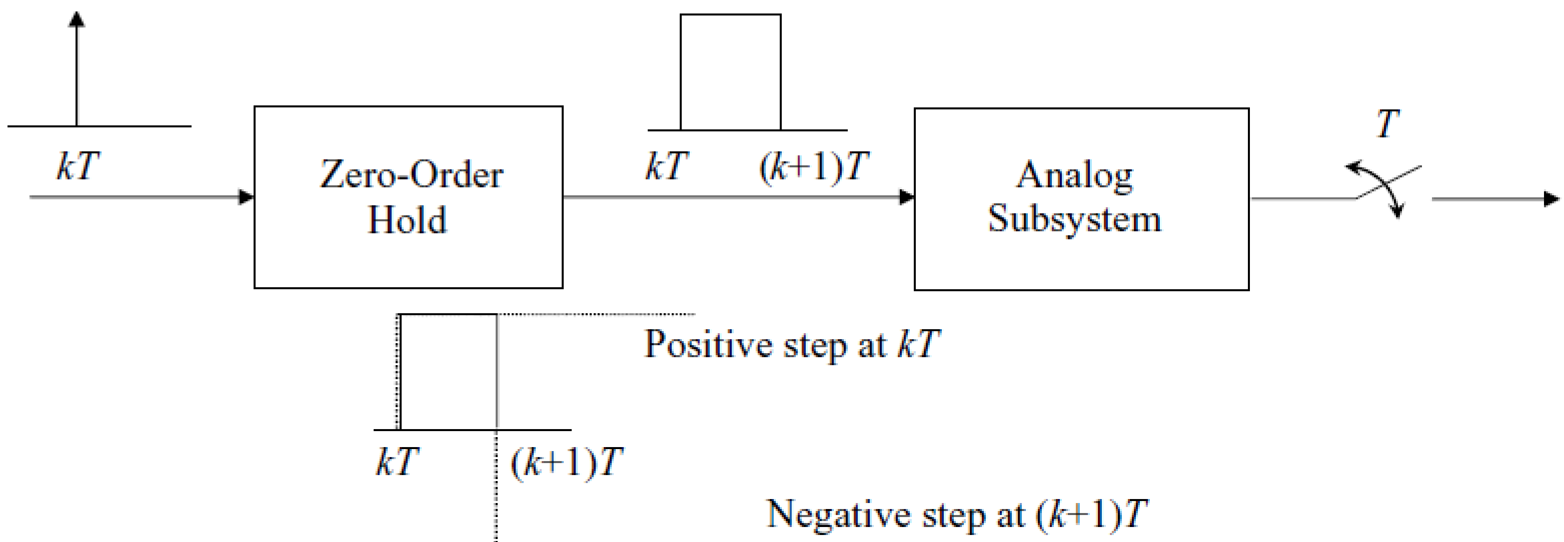
$$\mathcal{L}\{u(t)\} = \frac{1}{s} \qquad \mathcal{L}\{-u(t-T)\} = -\frac{e^{-Ts}}{s}$$

- Thus, the transfer function of the ZOH is

$$G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

DAC, Analog Subsystem, and ADC Combination Transfer Function

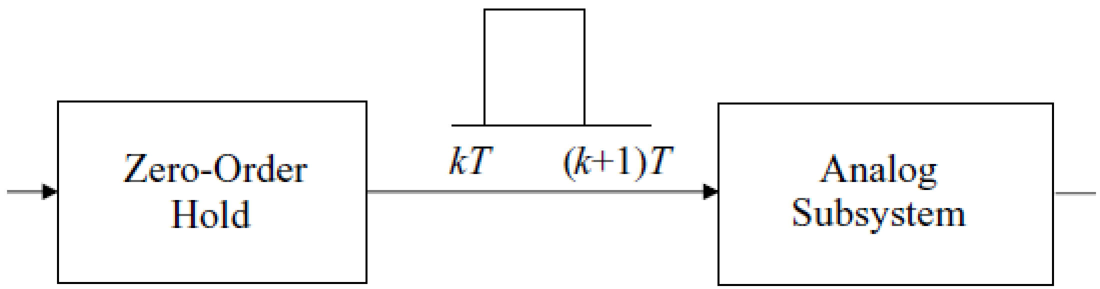
- The cascade of a DAC, analog subsystem, and ADC is shown in following figure.



- Because both the input and the output of the cascade are sampled, it is possible to obtain its *z*-domain transfer function in terms of the transfer functions of the individual subsystems.

DAC, Analog Subsystem, and ADC Combination Transfer Function

- Using the DAC model, and assuming that the transfer function of the analog subsystem is $G(s)$, the transfer function of the DAC and analog subsystem cascade is



$$G_{ZA}(s) = G_{ZOH}(s)G(s)$$

$$G_{ZA}(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

DAC, Analog Subsystem, and ADC Combination Transfer Function

$$G_{ZA}(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

- The corresponding impulse response is

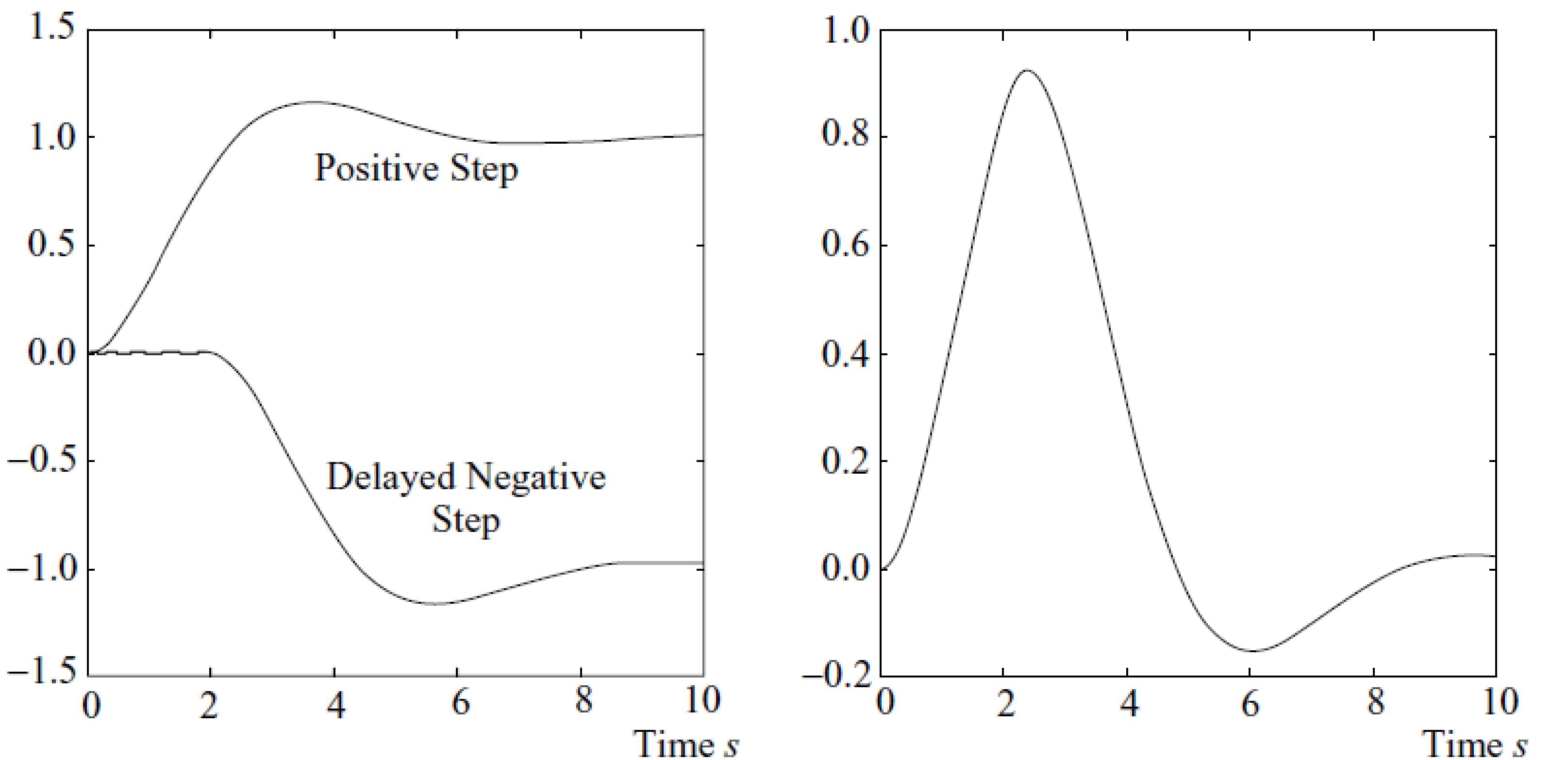
$$G_{ZA}(s) = \frac{G(s) - G(s)\, e^{-Ts}}{s}$$

$$G_{ZA}(s) = \frac{G(s)}{s} - \frac{G(s)\, e^{-Ts}}{s}$$

- The impulse response is the analog system step response minus a second step response delayed by one sampling period.

DAC, Analog Subsystem, and ADC Combination Transfer Function

$$G_{ZA}(s) = \frac{G(s)}{s} - \frac{G(s)\,e^{-Ts}}{s}$$



DAC, Analog Subsystem, and ADC Combination Transfer Function

$$G_{ZA}(s) = \frac{G(s)}{s} - \frac{G(s)\, e^{-Ts}}{s}$$

- Inverse Laplace yields

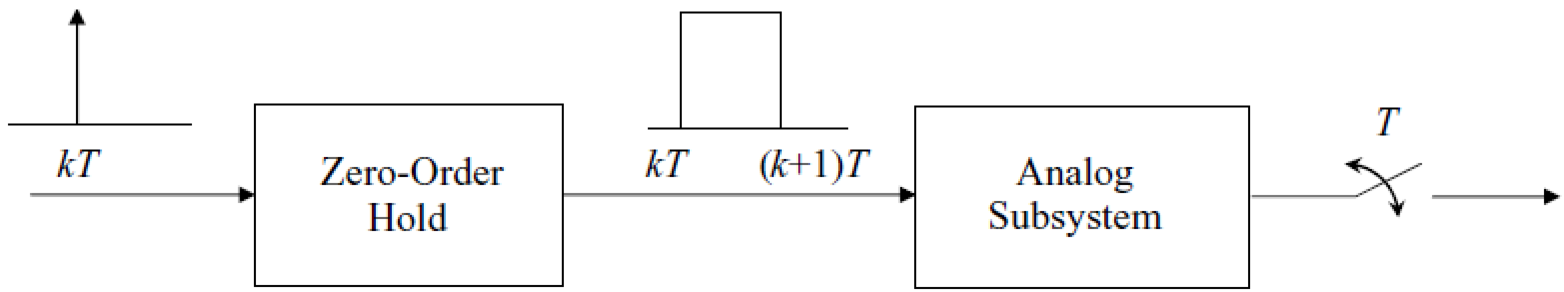
$$g_{ZA}(t) = g_s(t) - g_s(t - T)$$

- Where $g_s(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$

DAC, Analog Subsystem, and ADC Combination Transfer Function

$$g_{ZA}(t) = g_s(t) - g_s(t - T)$$

- The analog response is sampled to give the sampled impulse response



$$g_{ZA}(kT) = g_s(kT) - g_s(kT - T)$$

- By *z*-transforming, we can obtain the *z*-transfer function of the DAC (zero-order hold), analog subsystem, and ADC (ideal sampler) cascade.

DAC, Analog Subsystem, and ADC Combination Transfer Function

$$g_{ZA}(kT) = g_s(kT) - g_s(kT - T)$$

- Z-Transform is given as

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\{g_s{}^*(t)\}$$

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left[\mathscr{L}^{-1}\left\{\frac{G(s)}{s}\right\}^*\right]$$

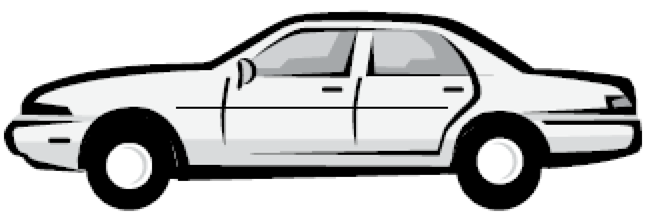
- The * in above equation is to emphasize that sampling of a time function is necessary before *z*-transformation.

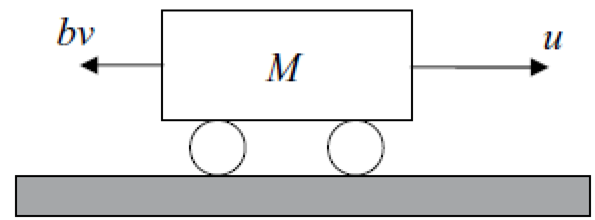
- Having made this point, the equation can be rewritten more concisely as

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left[\frac{G(s)}{s}\right]$$

Example-3

- Find $G_{ZAS}(z)$ for the cruise control system for the vehicle shown in figure, where u is the input force, v is the velocity of the car, and b is the viscous friction coefficient.





Solution

- The transfer function of system is given as

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{Ms + b}$$

- Re-writing transfer function in standard form

$$G(s) = \frac{K}{\tau s + 1} = \frac{K/\tau}{s + 1/\tau}$$

Example-3

$$G(s) = \frac{K/\tau}{s + 1/\tau}$$

- Where $K = 1/b$ and $\tau = M/b$
- Now we know

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left[\frac{G(s)}{s}\right]$$

- Therefore,

$$\frac{G(s)}{s} = \frac{K/\tau}{s(s + 1/\tau)}$$

- The corresponding partial fraction expansion is

$$\frac{G(s)}{s} = \left(\frac{K}{\tau}\right)\left[\frac{\tau}{s} - \frac{\tau}{s + 1/\tau}\right]$$

Example-3

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left[\left(\frac{K}{\tau}\right)\left\{\frac{\tau}{s} - \frac{\tau}{s + 1/\tau}\right\}\right]$$

- Using the *z*-transform table, the desired *z*-domain transfer function is

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left[K\left\{\frac{1}{s} - \frac{1}{s + 1/\tau}\right\}\right]$$

$$G_{ZAS}(z) = \frac{z - 1}{z}\left[K\left\{\frac{z}{z - 1} - \frac{z}{z - e^{-T/\tau}}\right\}\right]$$

$$G_{ZAS}(z) = \left[K\left\{1 - \frac{z - 1}{z - e^{-T/\tau}}\right\}\right]$$

Example-3

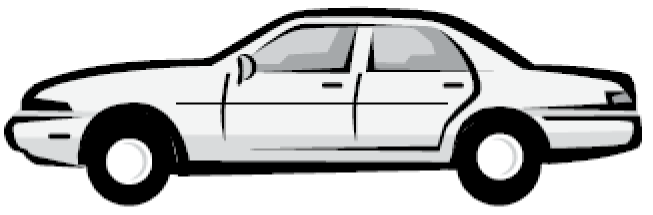
$$G_{ZAS}(z) = \left[K\left\{1 - \frac{z-1}{z-e^{-T/\tau}}\right\}\right]$$

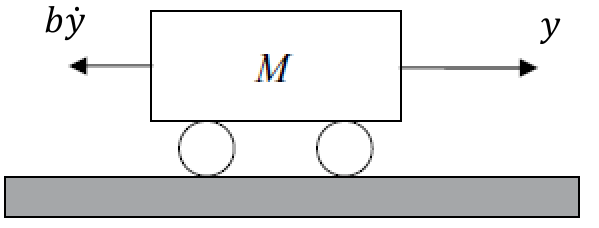
$$G_{ZAS}(z) = K\ \frac{z - e^{-\frac{T}{\tau}} - z + 1}{z - e^{-T/\tau}}$$

$$G_{ZAS}(z) = K\ \frac{1 - e^{-\frac{T}{\tau}}}{z - e^{-T/\tau}}$$

Example-4

- Find $G_{ZAS}(z)$ for the vehicle position control system, where *u* is the input force, *y* is the position of the car, and *b* is the viscous friction coefficient.





Solution

- The transfer function of system is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(Ms+b)}$$

- Re-writing transfer function in standard form

$$G(s) = \frac{K}{s(\tau s+1)} = \frac{K/\tau}{s(s+\frac{1}{\tau})}$$

Example-4

$$G(s) = \frac{K/\tau}{s(s+\frac{1}{\tau})}$$

- Where $K = 1/b$ and $\tau = M/b$
- Now we know

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]$$

- Therefore,

$$\frac{G(s)}{s} = \frac{K/\tau}{s^2(s+1/\tau)}$$

- The corresponding partial fraction expansion is

$$\frac{G(s)}{s} = K\left[\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s+1/\tau}\right]$$

Example-4

- The desired z-domain transfer function can be obtained as

$$G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}K\left[\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s + 1/\tau}\right]$$

$$G_{ZAS}(z) = \frac{z-1}{z}\ K\left[\frac{z}{(z-1)^2} - \frac{\tau z}{z-1} + \frac{\tau z}{z - e^{-T/\tau}}\right]$$

$$G_{ZAS}(z) = K\left[\frac{1}{z-1} - \tau + \frac{\tau(z-1)}{z - e^{-T/\tau}}\right]$$

$$G_{ZAS}(z) = K\left[\frac{\left(1 - \tau + \tau e^{-\frac{T}{\tau}}\right)z + \left[\tau - e^{-\frac{T}{\tau}}(\tau + 1)\right]}{(z-1)(z - e^{-T/\tau})}\right]$$

Home work

- Find $G_{ZAS}(z)$ for the series *R*-*L* circuit shown in Figure with the inductor voltage as output.