Digital Control Systems

Lecture-1

Introduction to Digital Control Systems & Preliminary Concepts

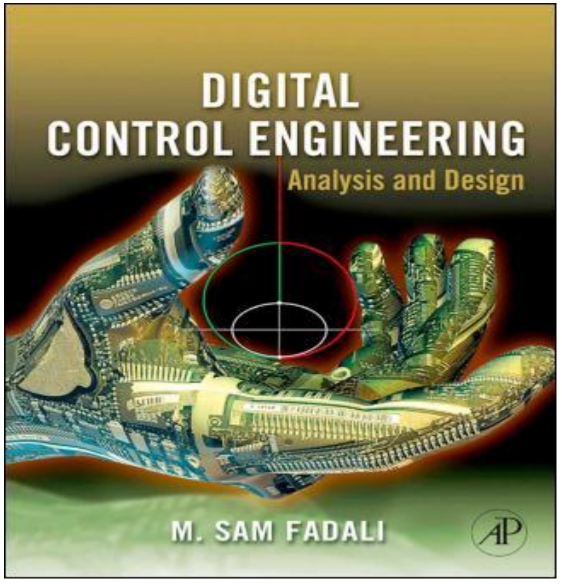
Note: The contents of this presentation are mostly taken from the book "Digital Control Engineering, Analysis and Design", by M S Fadali.

Lecture Outline

- Introduction
- Difference Equations
- Review of Z-Transform
- Inverse Z-transform
- Relations between s-plane and z-plane
- Solution of difference Equations

Recommended Book

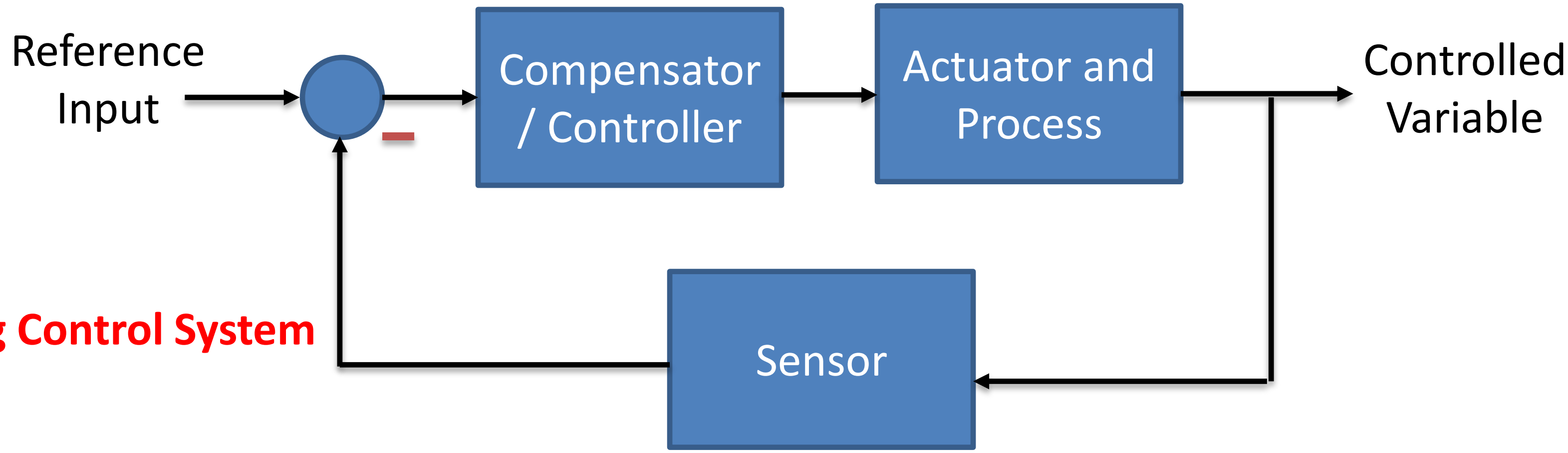
- M.S. Fadali, "Digital Control Engineering – Analysis and Design", Elsevier, 2009. ISBN: 13: 978-0-12-374498-2



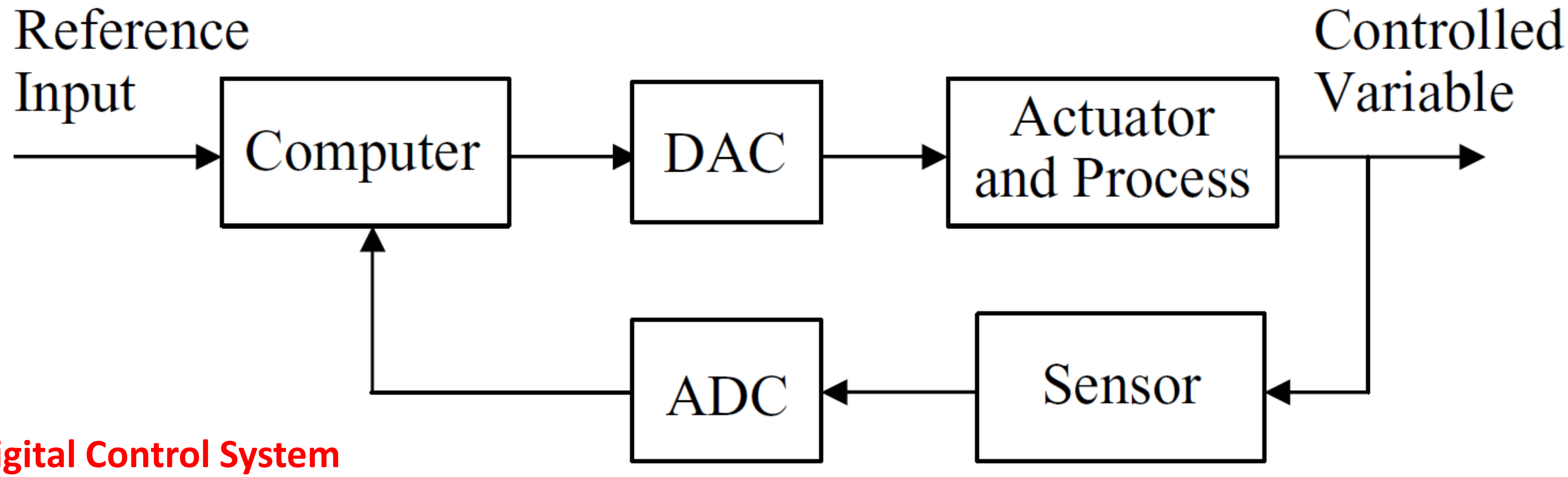
Introduction

- Digital control offers distinct advantages over analog control that explain its popularity.
- **Accuracy:** Digital signals are more accurate than their analogue counterparts.
- **Implementation Errors:** Implementation errors are negligible.
- **Flexibility:** Modification of a digital controller is possible without complete replacement.
- **Speed:** Digital computers may yield superior performance at very fast speeds
- **Cost:** Digital controllers are more economical than analogue controllers.

Structure of a Digital Control System



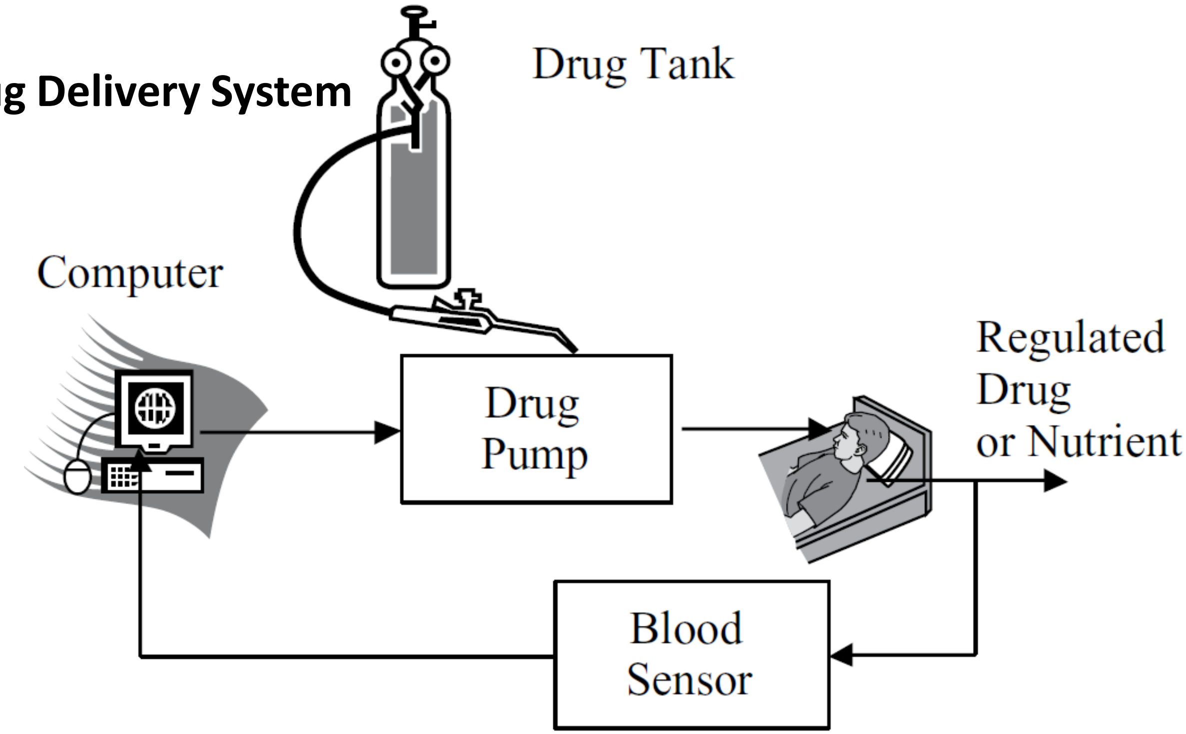
Analog Control System

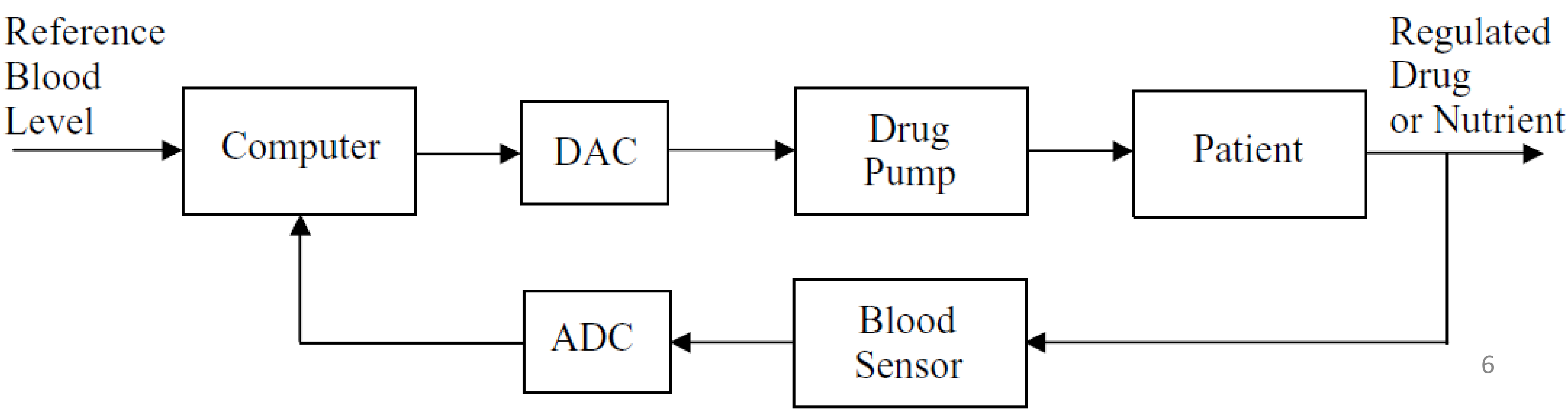


Digital Control System

Examples of Digital control Systems

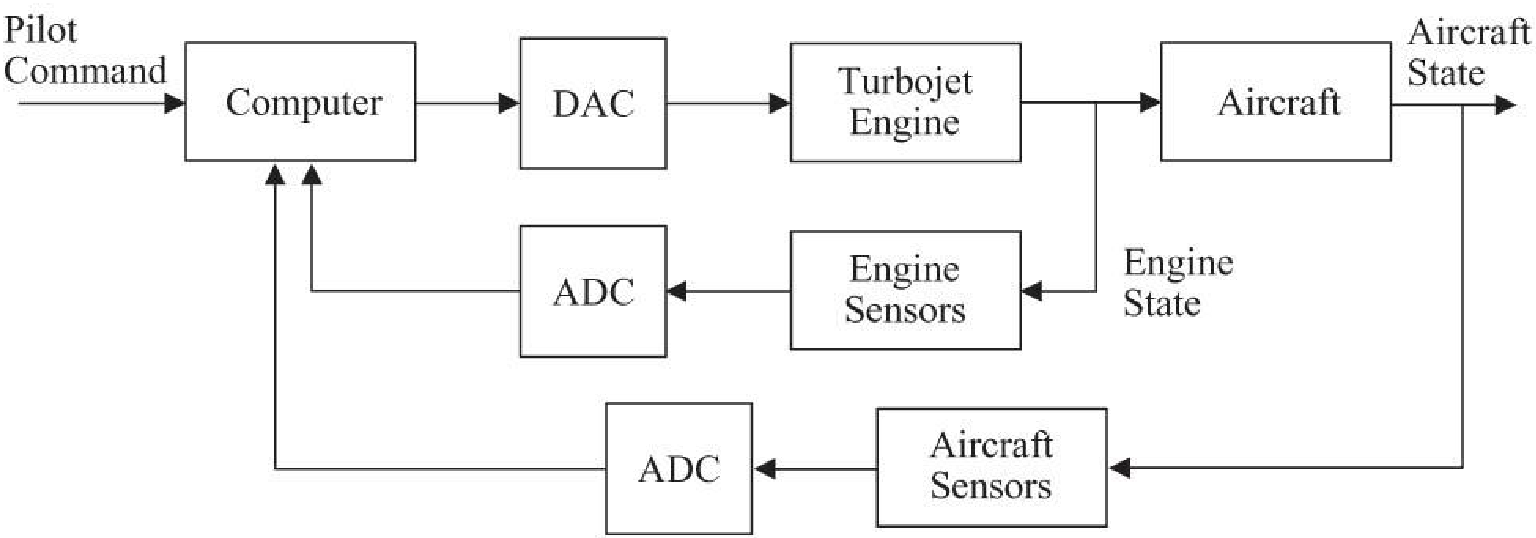
Closed-Loop Drug Delivery System





Examples of Digital control Systems

Aircraft Turbojet Engine



Difference Equation vs Differential Equation

- A difference equation expresses the change in some variable as a result of a *finite* change in another variable.

- A differential equation expresses the change in some variable as a result of an *infinitesimal* change in another variable.

Difference Equations

- Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values.

$$
\begin{aligned}
y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\
= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)
\end{aligned}
$$

- Where coefficients a_{n-1}, a_{n-2},… and b_n, b_{n-1},… are constant.
- $u(k)$ is forcing function

Difference Equations

- **Example-1:** For each of the following difference equations, determine the (a) order of the equation. Is the equation (b) linear, (c) time invariant, or (d) homogeneous?

1. $y(k+2) + 0.8y(k+1) + 0.07y(k) = u(k)$
2. $y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$
3. $y(k+1) = -0.1y^2(k)$

Difference Equations

- **Example-1:** For each of the following difference equations, determine the (a) order of the equation. Is the equation (b) linear, (c) time invariant, or (d) homogeneous?

1. $y(k+2) + 0.8y(k+1) + 0.07y(k) = u(k)$

Solution:

a) The equation is second order.
b) All terms enter the equation linearly
c) All the terms if the equation have constant coefficients. Therefore the equation is therefore LTI.
d) A forcing function appears in the equation, so it is nonhomogeneous.

Difference Equations

- **Example-1:** For each of the following difference equations, determine the (a) order of the equation. Is the equation (b) linear, (c) time invariant, or (d) homogeneous?

2. $y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$

Solution:

a) The equation is 4^{th} order.
b) All terms are linear
c) The second coefficient is time dependent
d) There is no forcing function therefore the equation is homogeneous.

Difference Equations

- **Example-1:** For each of the following difference equations, determine the (a) order of the equation. Is the equation (b) linear, (c) time invariant, or (d) homogeneous?

3. $y(k+1) = -0.1y^2(k)$

Solution:

a) The equation is 1^{st} order.
b) Nonlinear
c) Time invariant
d) Homogeneous

Z-Transform

- Difference equations can be solved using classical methods analogous to those available for differential equations.
- Alternatively, *z*-transforms provide a convenient approach for solving LTI equations.
- It simplifies the solution of discrete-time problems by converting LTI difference equations to algebraic equations and convolution to multiplication.

Z-Transform

- Given the causal sequence $\{u_0, u_1, u_2, \ldots, u_k\}$, its *z*-transform is defined as

$$U(z) = u_o + u_1 z^{-1} + u_2 z^{-2} + \ldots + u_k\, z^{-k}$$

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k}$$

- The variable z^{-1} in the above equation can be regarded as a time delay operator.

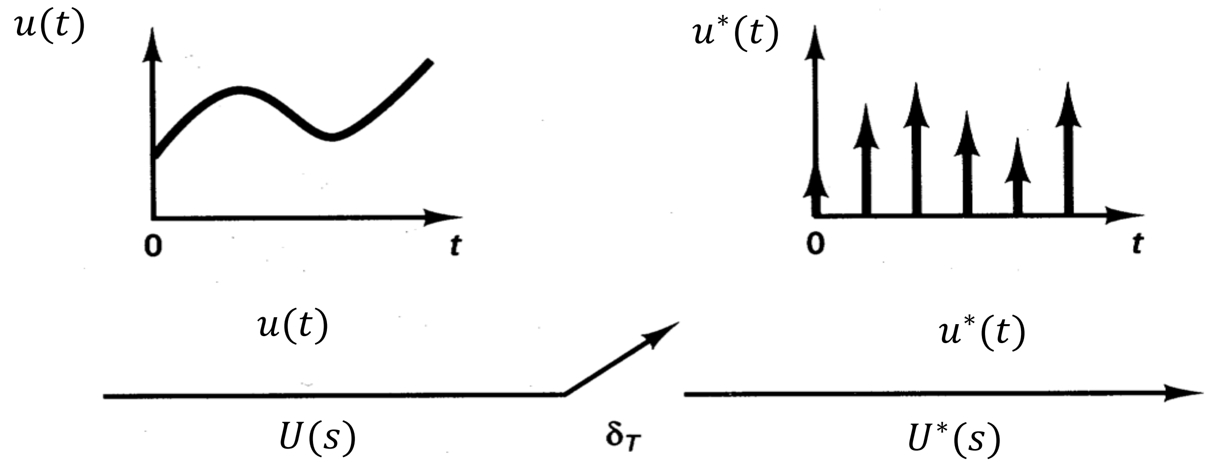
Z-Transform

- **Example-2**: Obtain the *z*-transform of the sequence

$$\{u_k\}_{k=0}^{\infty} = \{1, 1, 3, 2, 0, 4, 0, 0, 0, \ldots\}$$

Relation between Laplace Transform and Z-Transform

- Given the impulse train representation of a discrete-time signal



$$u^*(t) = u_o\delta(t) + u_1\delta(t-T) + u_2\delta(t-2T) + \cdots + u_k\delta(t-kT)$$

$$u^*(t) = \sum_{k=0}^{\infty} u_k\delta(t-kT)$$

Relation between Laplace Transform and Z-Transform

$$u^*(t) = u_o\delta(t) + u_1\delta(t-T) + u_2\delta(t-2T) + \cdots + u_k\delta(t-kT)$$

- The Laplace Transform of above equation is

$$U^*(s) = u_o + u_1 e^{-sT} + u_2 e^{-2sT} + \cdots + u_k e^{-ksT}$$

$$U^*(s) = \sum_{k=0}^{\infty} u_k e^{-ksT} \qquad (A)$$

- And the Z-transform of $u^*(t)$ is given as

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k} \qquad (B)$$

- Comparing (A) and (B) yields

$$z = e^{sT}$$

Conformal Mapping between s-plane to z-plane

$$z = e^{sT}$$

- Where $s = \sigma + j\omega$.

$$z = e^{(\sigma + j\omega)T}$$

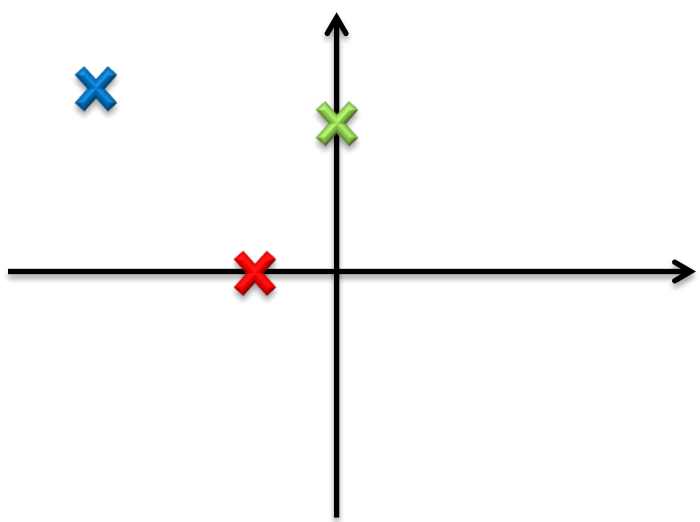
- Then z in polar coordinates is given by

$$z = e^{\sigma T} e^{j\omega T}$$

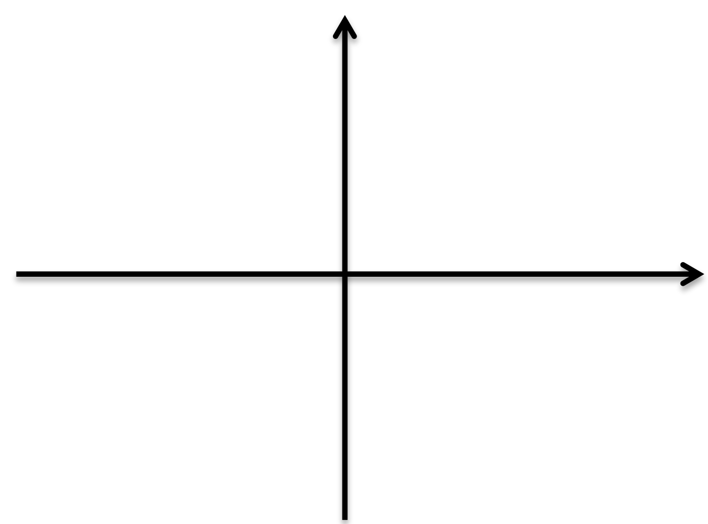
$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

Conformal Mapping between s-plane to z-plane

- We will discuss following cases to map given points on s-plane to z-plane.
  - **Case-1:** Real pole in s-plane ($s = \sigma$)
  - **Case-2:** Imaginary Pole in s-plane ($s = j\omega$)
  - **Case-3:** Complex Poles ($s = \sigma + j\omega$)



$s - plane$



$z - plane$

Conformal Mapping between s-plane to z-plane

- **Case-1:** Real pole in s-plane ($s = \sigma$)

- We know

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

- Therefore

$$|z| = e^{\sigma T} \qquad \angle z = 0$$

Conformal Mapping between s-plane to z-plane

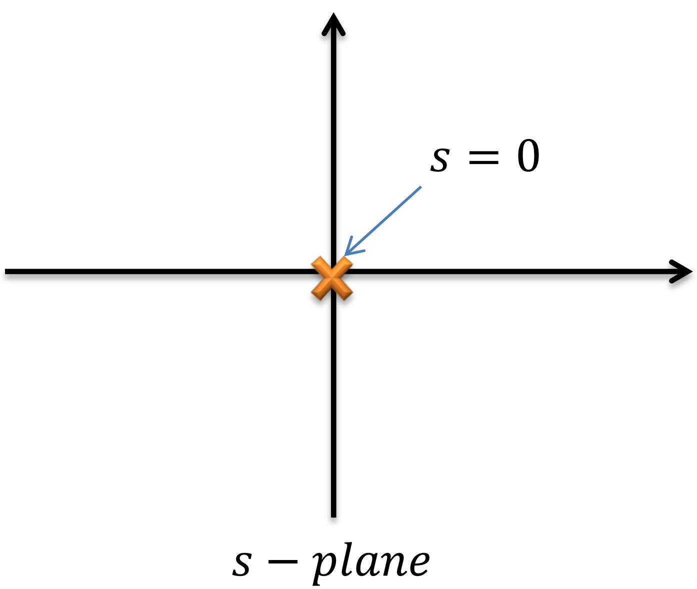
Case-1: Real pole in s-plane ($s = \sigma$)

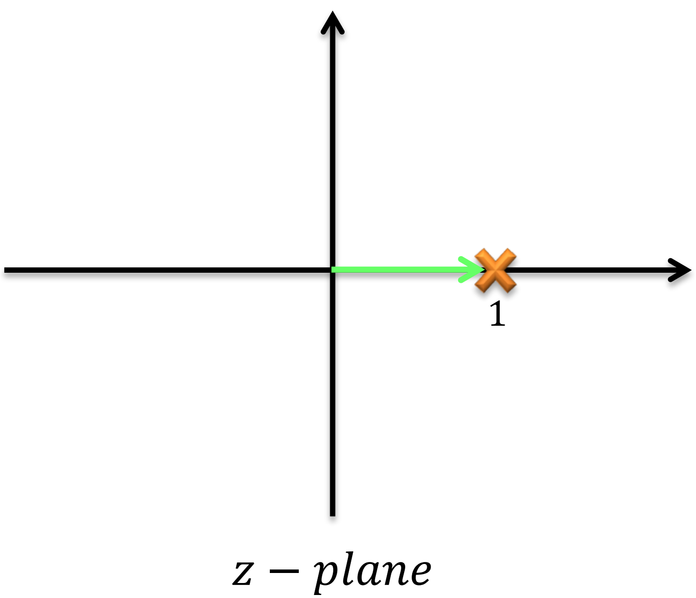
$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

When $s = 0$

$$|z| = e^{0T} = 1$$

$$\angle z = 0T = 0$$





Conformal Mapping between s-plane to z-plane

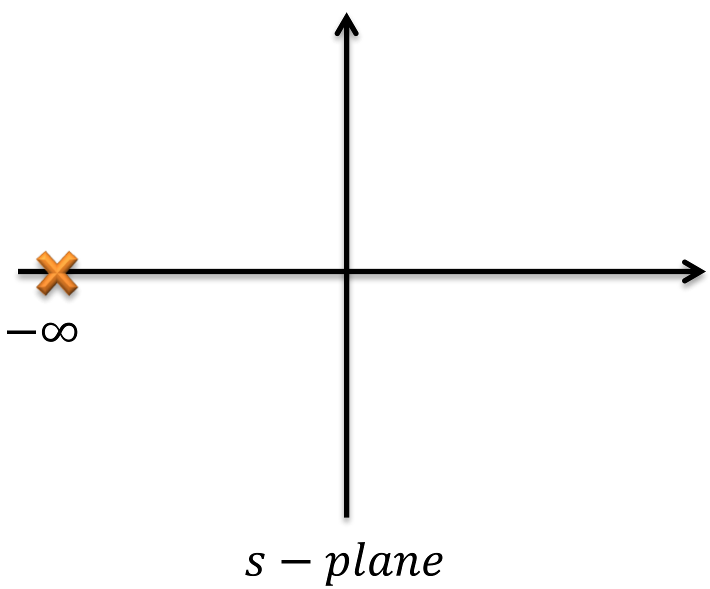
Case-1: Real pole in s-plane ($s = \sigma$)

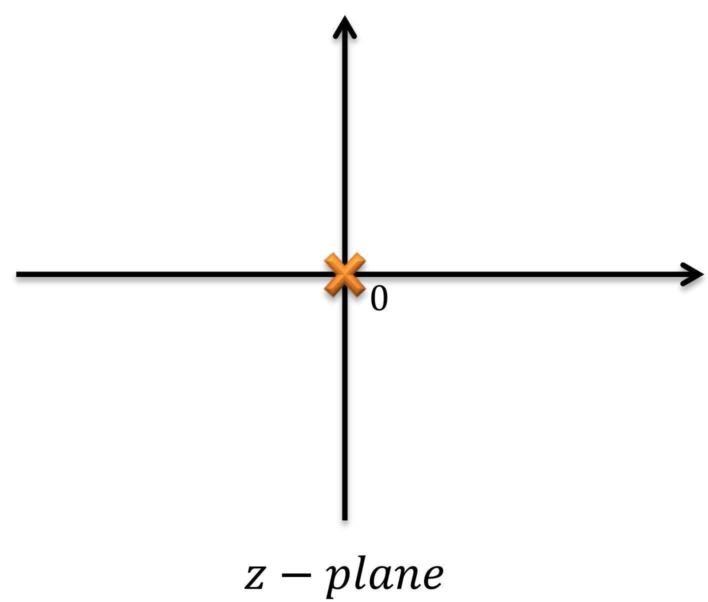
$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

When $s = -\infty$

$$|z| = e^{-\infty T} = 0$$

$$\angle z = 0$$





Conformal Mapping between s-plane to z-plane

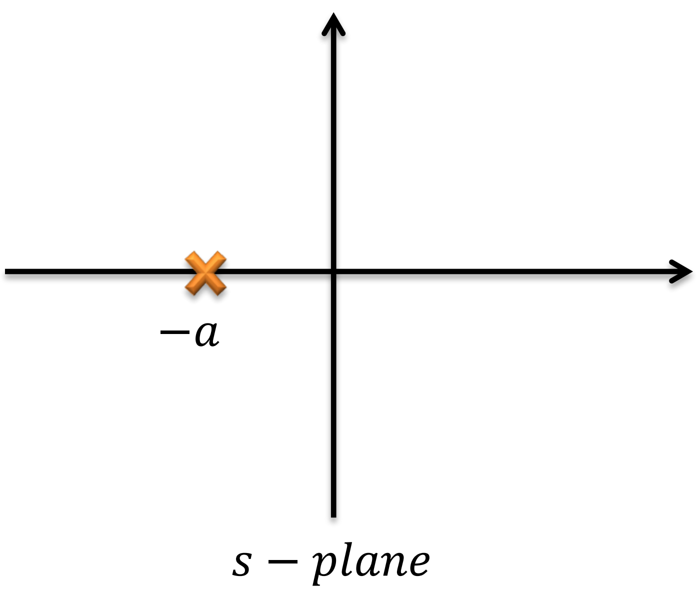
Case-1: Real pole in s-plane ($s = \sigma$)

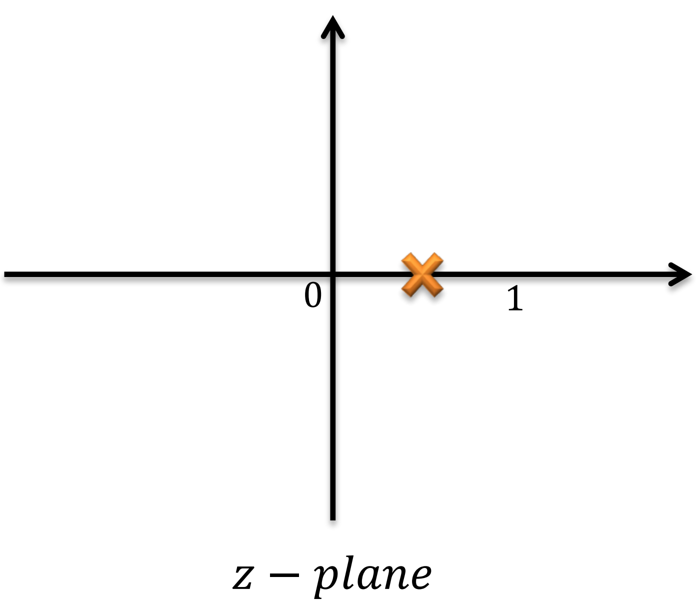
$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

Consider $s = -a$

$$|z| = e^{-aT}$$

$$\angle z = 0$$





Conformal Mapping between s-plane to z-plane

- **Case-2:** Imaginary pole in s-plane ($s = \pm j\omega$)

- We know

$$|z| = e^{\sigma T} \qquad\qquad \angle z = \omega T$$

- Therefore

$$|z| = 1 \qquad\qquad \angle z = \pm\omega T$$

Conformal Mapping between s-plane to z-plane

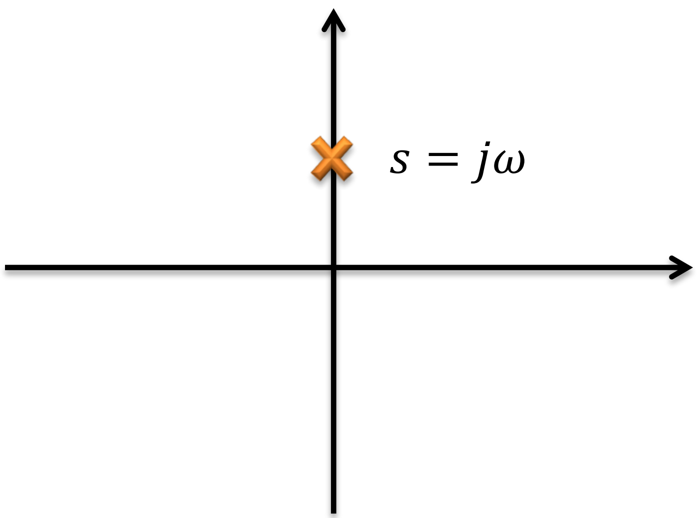
Case-2: Imaginary pole in s-plane ($s = \pm j\omega$)

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

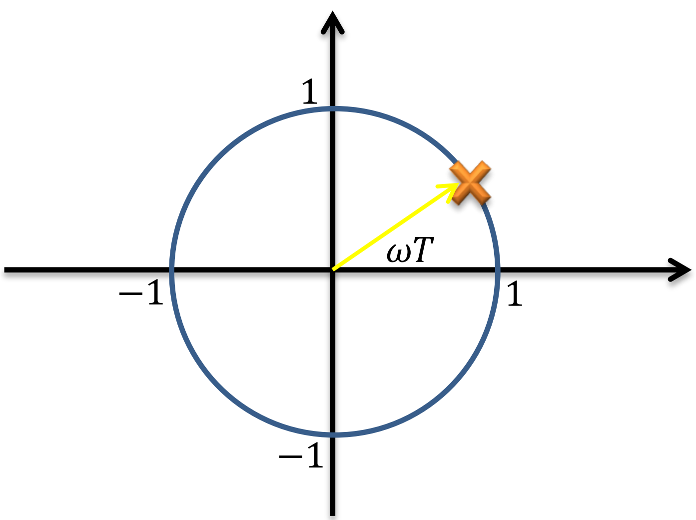
Consider $s = j\omega$

$$|z| = e^{0T} = 1$$

$$\angle z = \omega T$$



$s - plane$



$z - plane$

Conformal Mapping between s-plane to z-plane

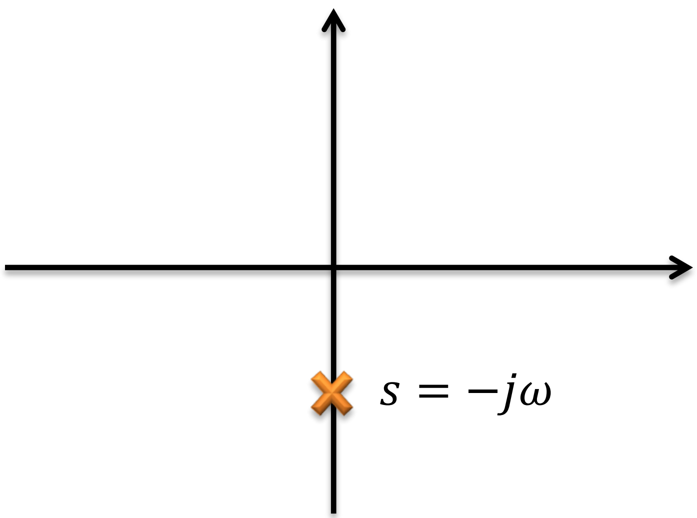
Case-2: Imaginary pole in s-plane ($s = \pm j\omega$)

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

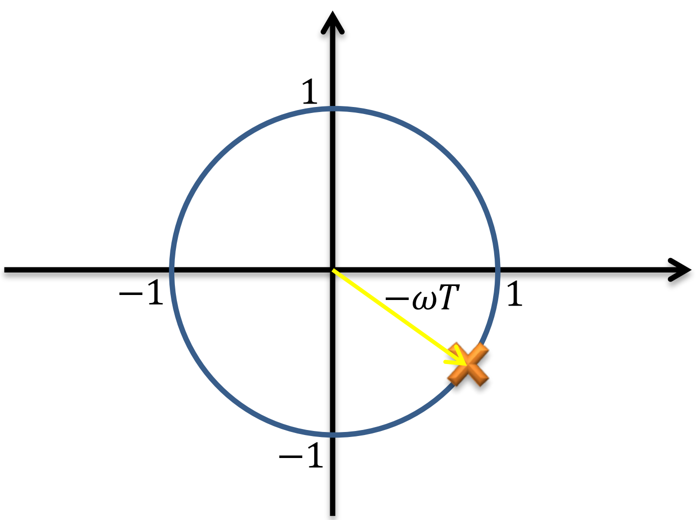
When $s = -j\omega$

$$|z| = e^{0T} = 1$$

$$\angle z = -\omega T$$



$s - plane$



$z - plane$

Conformal Mapping between s-plane to z-plane

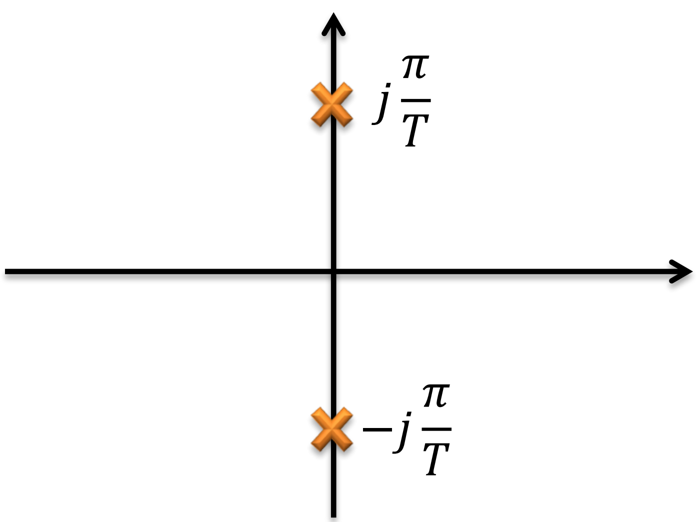
Case-2: Imaginary pole in s-plane ($s = \pm j\omega$)

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

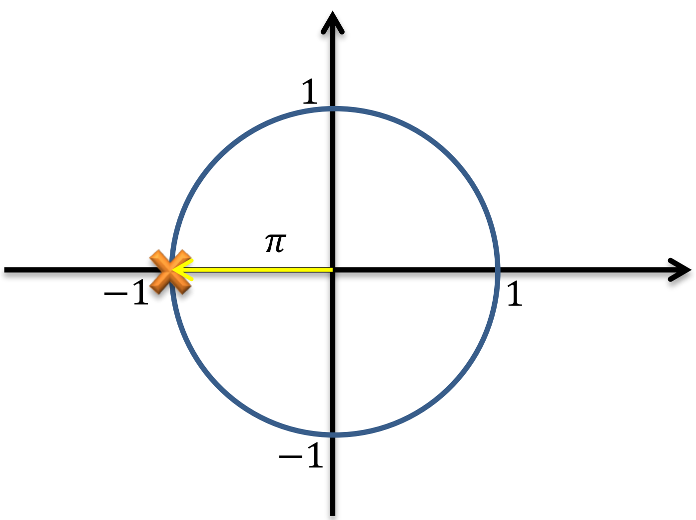
When $s = \pm j\frac{\pi}{T}$

$$|z| = e^{0T} = 1$$

$$\angle z = \pm\pi$$



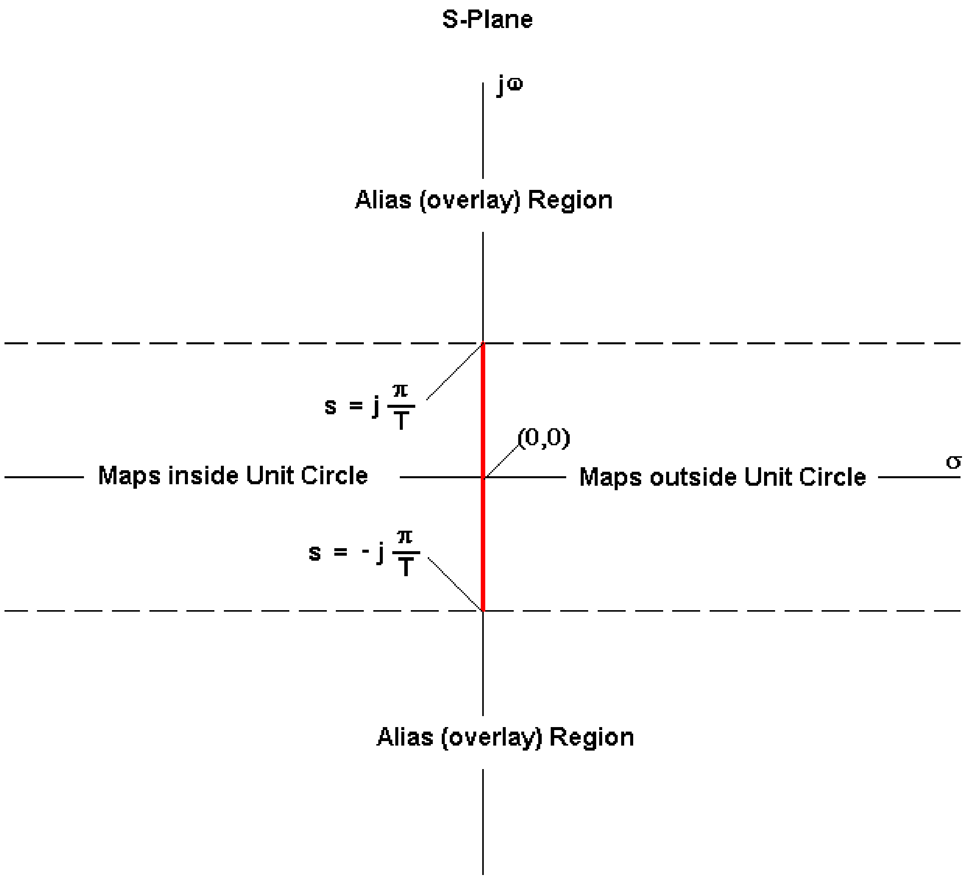
$s - plane$

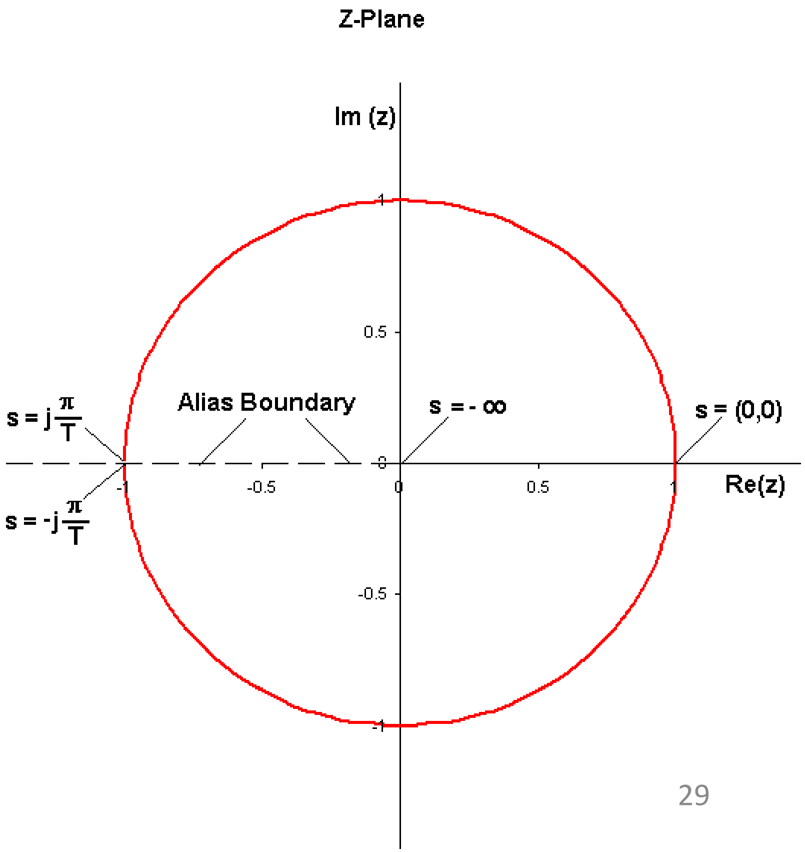


$z - plane$

Conformal Mapping between s-plane to z-plane

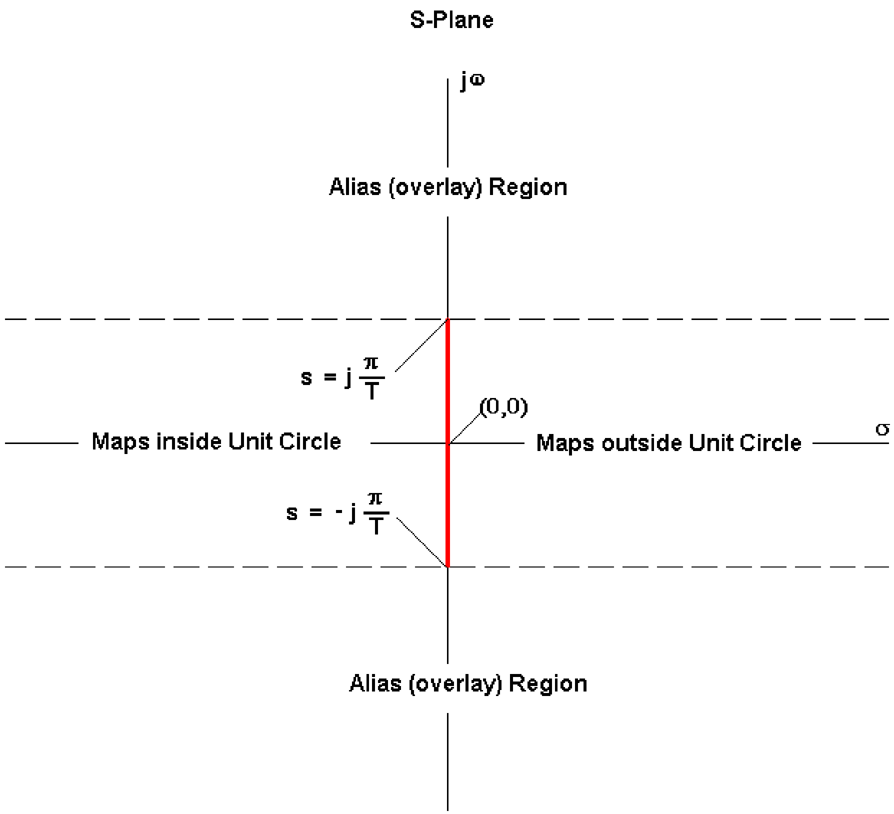
- Anything in the Alias/Overlay region in the S-Plane will be overlaid on the Z-Plane along with the contents of the strip between $\pm j\,\frac{\pi}{T}$.

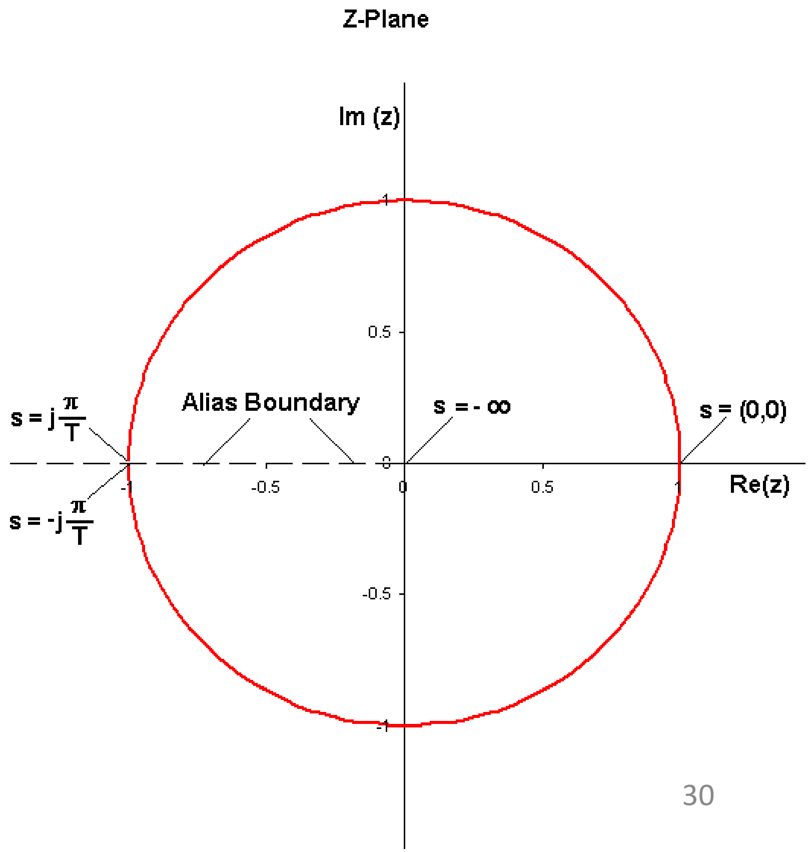




Conformal Mapping between s-plane to z-plane

- In order to avoid aliasing, there must be nothing in this region, i.e. there must be no signals present with radian frequencies higher than $\omega = \pi/T$, or cyclic frequencies higher than f = 1/2T.
- Stated another way, the sampling frequency must be at least twice the highest frequency present (Nyquist rate).





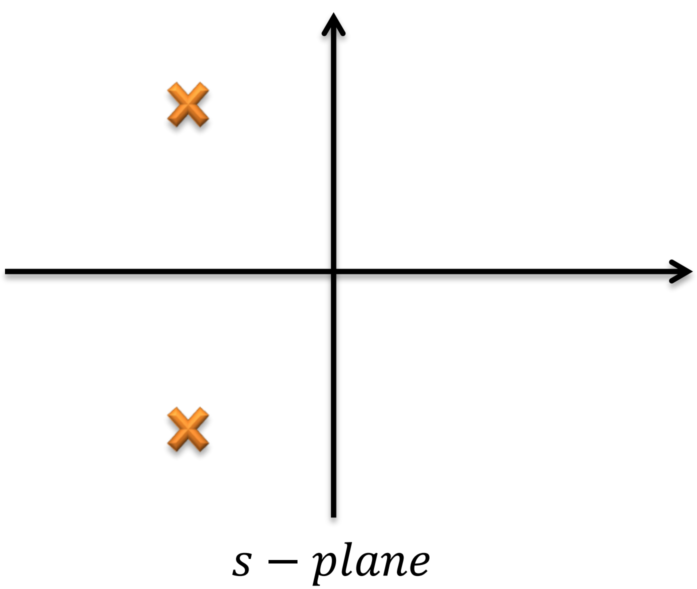
Conformal Mapping between s-plane to z-plane

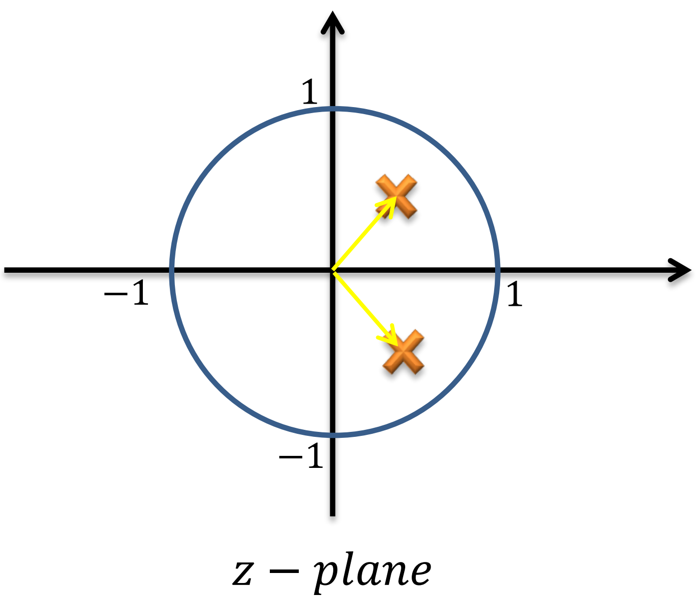
Case-3: Complex pole in s-plane ($s = \sigma \pm j\omega$)

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

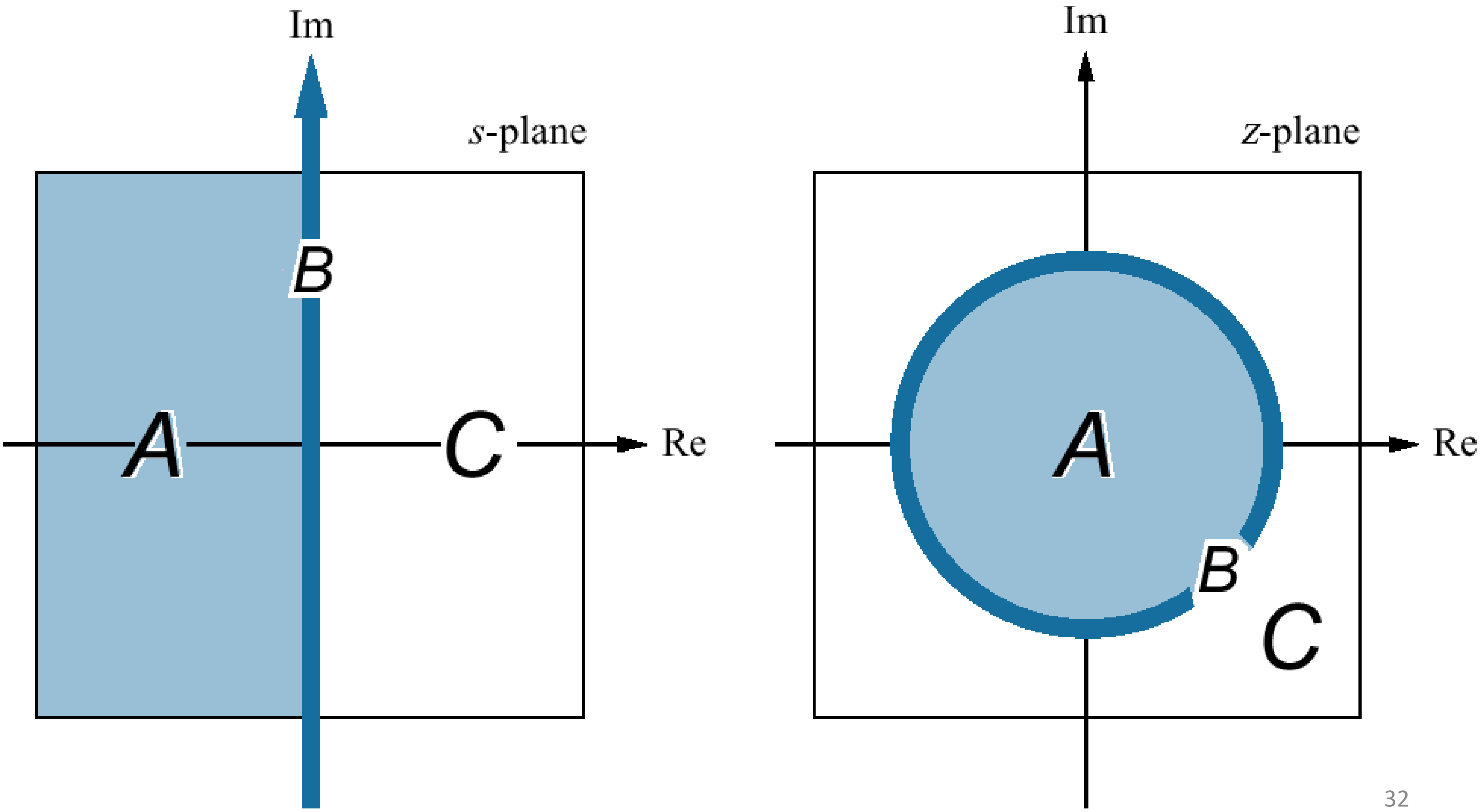
$$|z| = e^{\sigma T}$$

$$\angle z = \pm\omega T$$

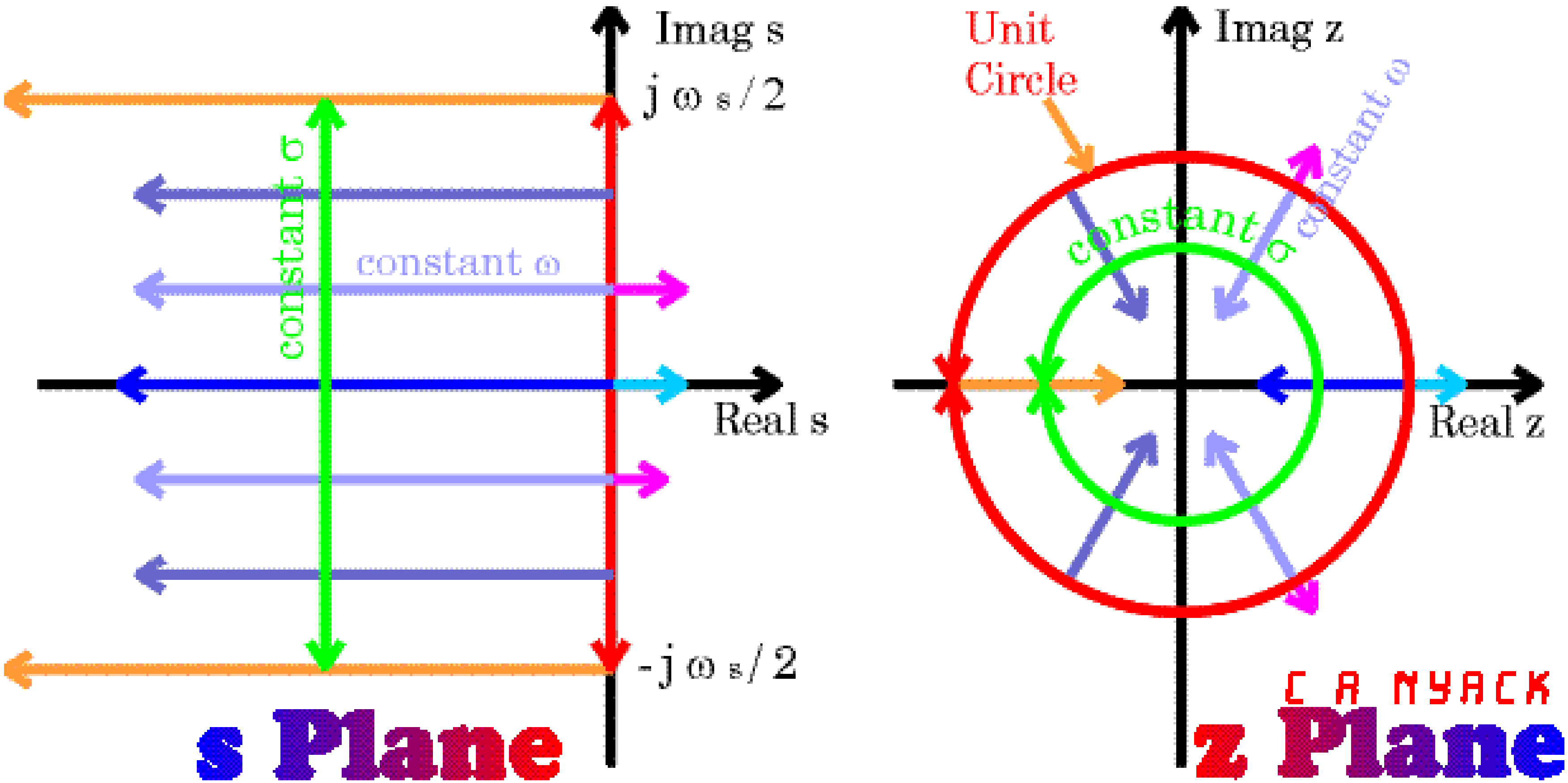




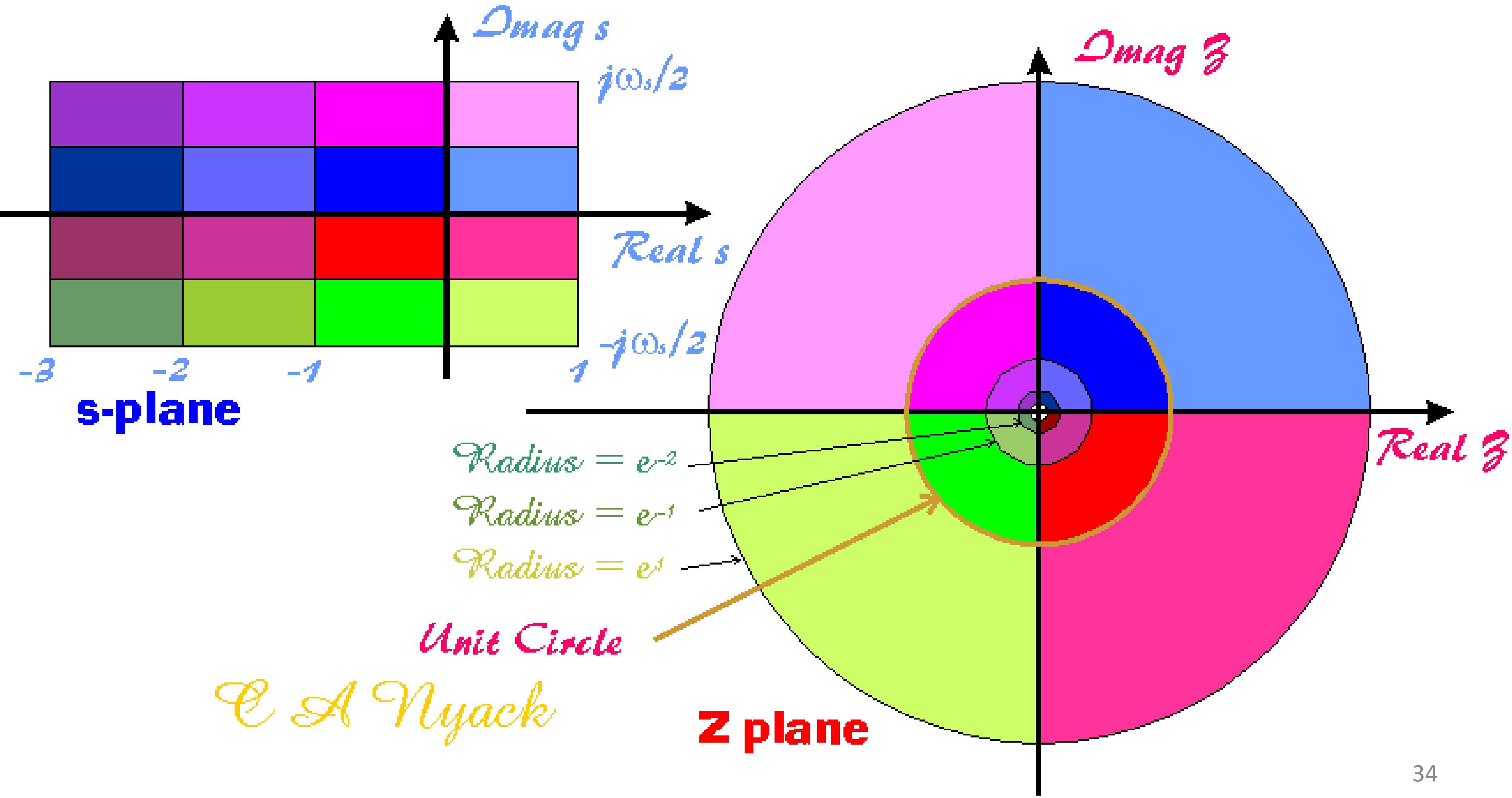
Mapping regions of the *s*-plane onto the *z*-plane



Mapping regions of the *s*-plane onto the *z*-plane



Mapping regions of the *s*-plane onto the *z*-plane



Example-3

- Map following s-plane poles onto z-plane assume (T=1). Also comment on the nature of step response in each case.

1. $s = -3$
2. $s = \pm 4j$
3. $s = \pm \pi j$
4. $s = \pm 2\pi j$
5. $s = -10 \pm 5j$

z-Transforms of Standard Discrete-Time Signals

- The following identities are used repeatedly to derive several important results.

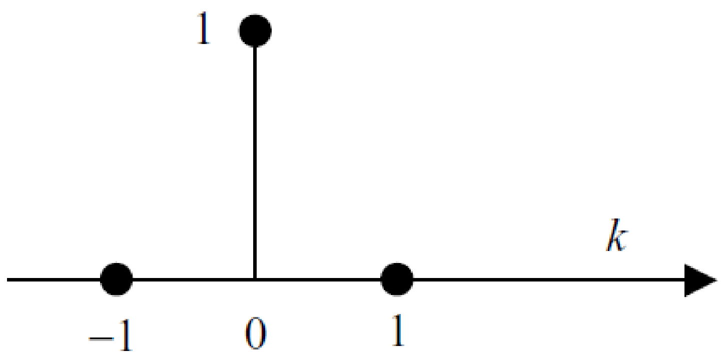
$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, \qquad a \neq 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a}, \qquad |a| \neq 1$$

z-Transforms of Standard Discrete-Time Signals

- Unit Impulse

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$



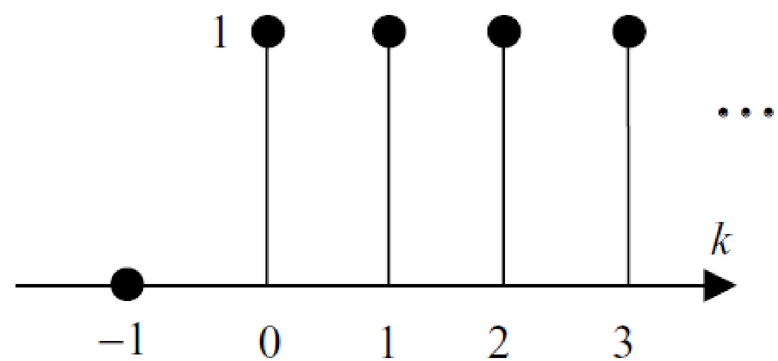
- Z-transform of the signal

$$\delta(z) = 1$$

z-Transforms of Standard Discrete-Time Signals

- Sampled Step

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



- or

$$u(k) = \{1, 1, 1, 1, \ldots\} \qquad k \geq 0$$

- Z-transform of the signal

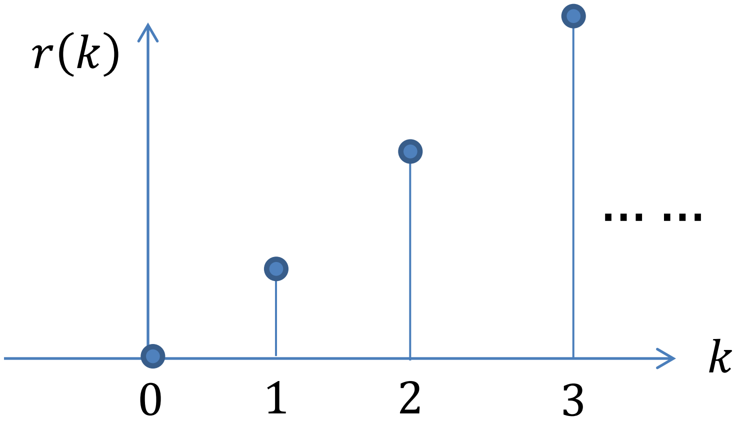
$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-k} = \sum_{k=0}^{n} z^{-k}$$

$$U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| < 1$$

z-Transforms of Standard Discrete-Time Signals

- Sampled Ramp

$$r(k) = \begin{cases} k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



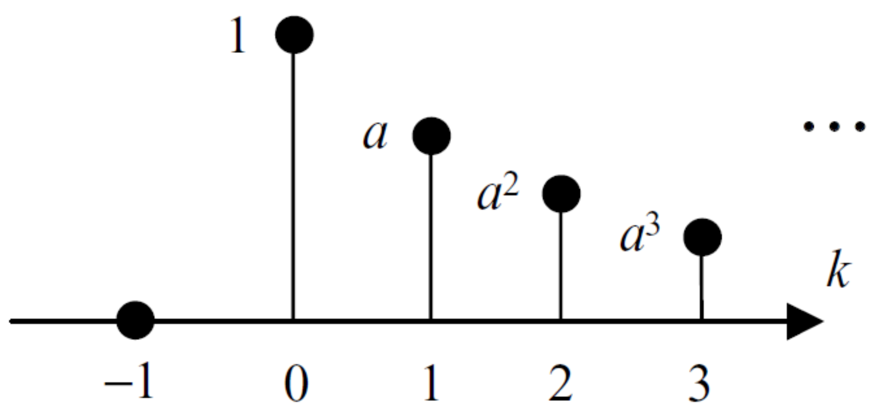
- Z-transform of the signal

$$U(z) = \frac{z}{(z-1)^2}$$

z-Transforms of Standard Discrete-Time Signals

- Sampled Parabolic Signal

$$u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



- Then

$$U(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots + a^kz^{-k} = \sum_{k=0}^{n}(az)^{-k}$$

$$U(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| < 1$$

Properties of Z-Transform

- Linearity Property

$$\mathcal{Z}\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$$

- Time delay Property

$$\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$$

- Time advance Property

$$\mathcal{Z}\{f(k+n)\} = z^{n}F(z) - z^{n}f(0) - z^{n-1}f(1) - \cdots - zf(n-1)$$

- Multiplication by exponential

$$\mathcal{Z}\{a^{-k}f(k)\} = F(az)$$

Exercise

- Find the z-transform of following causal sequences.

1. $f(k) = 2 \times 1(k) + 4 \times \delta(k), \qquad k = 0,1,2,\ldots$

2. $f(k) = \begin{cases} 4, & k = 2,3,\ldots \\ 0, & otherwise \end{cases}$

3. $f(k) = \{4, 8,16, 24, \ldots\}, \qquad k = 0,1,2,\ldots$

4. $f(k) = e^{-akT}, \qquad k = 0,1,2,\ldots$

Exercise

- Find the z-transform of following causal sequences.

1. $f(k) = 2 \times 1(k) + 4 \times \delta(k), \qquad k = 0,1,2,\ldots$

Solution: Using Linearity Property

$$F(z) = \mathcal{Z}\{2 \times 1(k) + 4 \times \delta(k)\}$$

$$F(z) = 2 \times \mathcal{Z}\{1(k)\} + 4 \times \mathcal{Z}\{\delta(k)\}$$

$$F(z) = 2 \times \frac{z}{z-1} + 4$$

$$F(z) = \frac{6z-4}{z-1}$$

Exercise

- Find the z-transform of following causal sequences.

$$2. \quad f(k) = \begin{cases} 4, & k = 2,3,\ldots \\ 0, & otherwise \end{cases}$$

Solution: The given sequence is a sampled step starting at k-2 rather than k=0 (i.e. it is delayed by two sampling periods). Using the delay property, we have

$$F(z) = \mathcal{Z}\{4 \times 1(k-2)\}$$

$$F(z) = 4z^{-2}\,\mathcal{Z}\{1(k-2)\}$$

$$F(z) = 4z^{-2}\frac{z}{z-1} = \frac{4}{z(z-1)}$$

Exercise

3. $f(k) = \{4, 8, 16, 24, \dots\}, \qquad k = 0,1,2, \dots$

- **Solution:** The sequence can be written as

$$f(k) = 2^{k+2} = g(k+2), \qquad k = 0, 1, 2, \dots$$

- where $g(k)$ is the exponential time function

$$g(k) = 2^k, \qquad k = 0, 1, 2, \dots$$

- Using the time advance property, we write the transform

$$F(z) = z^2 G(z) - z^2 g(0) - z g(1)$$

$$F(z) = z^2 \frac{z}{z-2} - z^2 - 2z = \frac{4z}{z-2}$$

Exercise

$$4.\quad f(k) = e^{-akT}, \qquad\qquad k = 0,1,2,\ldots$$

- observe that $f(k)$ can be rewritten as

$$f(k) = (e^{aT})^{-k} \times 1, \qquad\qquad k = 0,1,2,\ldots$$

- Then apply the multiplication by exponential property to obtain

$$\mathcal{Z}\{(e^{aT})^{-k} \times f(k)\} = \frac{e^{aT}z}{e^{aT}z - 1}$$

$$F(z) = \frac{z}{z - e^{-aT}}$$

Inverse Z-transform

1. **Long Division:** We first use long division to obtain as many terms as desired of the *z*-transform expansion.

2. **Partial Fraction:** This method is almost identical to that used in inverting Laplace transforms. However, because most *z*-functions have the term z in their numerator, it is often convenient to expand $F(z)/z$ rather than $F(z)$.

Inverse Z-transform

- **Example-4:** Obtain the inverse *z*-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.2z + 0.1}$$

- **Solution**
- *1. Long Division*

Inverse Z-transform

- *1. Long Division*

$$F(z) = \frac{z+1}{z^2 + 0.2z + 0.1}$$

$$\begin{array}{r l}
 & z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \ldots\ldots \\
z^2 + 0.2z + 0.1 \,\big)\! & z + 1 \\
 & \underline{z + 0.2 + 0.1z^{-1}} \\
 & 0.8 - 0.10z^{-1} \\
 & \underline{0.8 + 0.16z^{-1} + 0.08z^{-2}} \\
 & -0.26z^{-1} - \ldots
\end{array}$$

- *Thus*

$$F(z) = 0 + z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \cdots$$

- *Inverse z-transform*

$$f(k) = \{0, \quad 1, \quad 0.8, \quad -0.26, \quad \ldots\}$$

Inverse Z-transform

- **Example-5:** Obtain the inverse *z*-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

- **Solution**
- *2. Partial Fractions*

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.3z + 0.02)}$$

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.1z + 0.2z + 0.02)}$$

Inverse Z-transform

$$\frac{F(z)}{z} = \frac{z+1}{z(z+0.1)(z+0.2)}$$

$$\frac{F(z)}{z} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$

$$A = z\frac{F(z)}{z}\bigg|_{z=0} = F(0) = \frac{1}{0.1\times 0.2} = \frac{1}{0.02} = 50$$

$$B = (z+0.1)\frac{F(z)}{z}\bigg|_{z=-0.1} = (z+0.1)\frac{1}{z}\frac{z+1}{(z+0.1)(z+0.2)}\bigg|_{z=-0.1} = \frac{-0.1+1}{(-0.1)(-0.1+0.2)} = -90$$

$$C = (z+0.2)\frac{F(z)}{z}\bigg|_{z=-0.2} = (z+0.2)\frac{1}{z}\frac{z+1}{(z+0.1)(z+0.2)}\bigg|_{z=-0.2} = \frac{-0.2+1}{(-0.2)(-0.2+0.1)} = 40$$

Inverse Z-transform

$$\frac{F(z)}{z} = \frac{50}{z} - \frac{90}{z + 0.1} + \frac{40}{z + 0.2}$$

$$F(z) = 50 - \frac{90z}{z + 0.1} + \frac{40z}{z + 0.2}$$

- Taking inverse z-transform (using z-transform table)

$$f(k) = 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k$$

Home Work

- For each of the following equations, determine the order of the equation and then test it for (i) linearity, (ii) time invariance, (iii) homogeneity.

a) $y(k+2) = y(k+1)y(k) + u(k)$

b) $y(k+3) + 2y(k) = 0$

c) $y(k+4) + y(k-1) = u(k)$

d) $y(k+5) = y(k+4) + u(k+1) - u(k)$

e) $y(k+2) = y(k)u(k)$

Home Work

- Find the z-transforms of the following sequences

$a)\ \{0, 1, 2, 4, 0, 0, \dots\}$

$b)\ \{0, 0, 0, 1, 1, 1, 0, 0, 0, \dots\}$

$c)\ \{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, 0, \dots\}$

Home Work

- Find the inverse transforms of the following functions

a) $F(z) = 1 + 3z^{-1} + 4z^{-2}$

b) $F(z) = 5z^{-1} + 4z^{-5}$

c) $F(z) = \frac{z}{z^2+0.3z+0.02}$

d) $F(z) = \frac{z-0.1}{z^2+0.04z+0.25}$

e) $F(z) = \frac{z}{(z+0.1)(z+0.2)(z+0.3)}$