

## Chapter Two

### Mathematical Modeling of Physical Systems

#### Introduction.

To understand and control complex systems, one must obtain quantitative mathematical models of these systems. It is therefore necessary to analyze the relationship between the system variables and to obtain a mathematical model.

The derivation of this model is based upon the fact that the dynamic system can be completely described by known differential equations or by experimental test data (system identification) or a mix of the two.

A mathematical model ~~may assume~~ of a dynamic system is defined as

"A set of equations that represents the dynamics of the system accurately or, at least, fairly well"

In control system, the systems under consideration are dynamic in nature, the descriptive equations are usually differential equations. Furthermore, if these equations can be linearized, then the Laplace transform can be utilized to simplify the method of solution, describing the operation of the system. In practice, the complexity of systems and our ignorance of all the relevant factors necessitate the introduction of assumptions concerning the system operation.

#### Simplicity Versus Accuracy

When attempting to build a model, a compromise must be made between the simplicity and of the model and the accuracy of the results of the analysis.

## Mathematical Modeling procedure

2

The procedure for obtaining a mathematical model for a system can be summarized as follows:

- 1- Define the system and its components
  - draw a schematic diagram of the system
  - define variables
- 2- Formulate the mathematical model and list the necessary assumptions
- 3- Write the differential equations describing the model
  - using physical laws, write equations for each components, combine them according to the system diagram, and obtain a mathematical model.
- 4- Solve the equations for the desired output variables
- 5- Examine the solutions and the assumption.  
(The question of the validity of any mathematical model can be answered only by experiment.)
- 6- If necessary, reanalyze or redesign the system.

## Differential Equations of Physical Systems.

The components of a control system are diverse in nature and may include Electrical, Mechanical, Thermal and Hydraulic devices. The differential equations describing the dynamic performance of a physical control system are obtained by utilizing **The Physical Laws of the process**. This approach applies well to all above various types of control components (Electrical, mechanical ---).

The differential equations relating the input and output quantities for the control system components are obtained **using the basic laws of physics**.

$$\frac{\partial^n y}{\partial t^n} + a_{n-1} \frac{\partial^{n-1} y}{\partial t^{n-1}} + \dots + a_1 \frac{\partial y}{\partial t} + a_0 y = b_{n-1} \frac{\partial^{n-1} r}{\partial t^{n-1}} + b_{n-2} \frac{\partial^{n-2} r}{\partial t^{n-2}} + \dots + b_1 \frac{\partial r}{\partial t} + b_0 r$$

The basic laws of physics include balancing of Force, Energy, and mass

→ For most physical systems one may classify the variables as either "through" or "across" Variable, in the sense that

→ Through refers to a point

→ Across refers between two points

A list of analogous variables for different systems are given below

System	Through Variable	Across Variable
Electrical	current, $i$	Potential difference or Voltage, $V$
Mechanical (Translational)	Force, $F$	Relative Velocity, $v$
Mechanical (Rotational)	Torque, $T$	Relative angular Velocity, $\omega$
Thermal	Rate of flow of Heat energy, $q$	Difference in temperature, $\Delta T$
Fluid	Volumetric rate of fluid flow, $Q$	Difference in pressure, $\Delta P$

### Modeling of Mechanical system.

Mechanical systems are classified into two types

A- Translational Mechanical system

B. Rotational Mechanical system.

#### A) Mechanical Translational system.

Mechanical systems obey the basic law that the sum of the forces must equal to zero.

Newton's law

"The sum of the applied forces must be equal to the sum of the reactive forces"

OR  $F = ma$

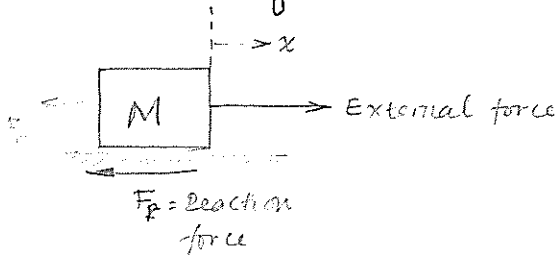


The three quantities characterizing elements in a mechanical translational system are:

- mass
- Elastic
- damping

1) The mass  $m$  is the inertial element.

The force applied to a mass produces an acceleration of the mass. The reaction force ( $F_r$ ) is equal to



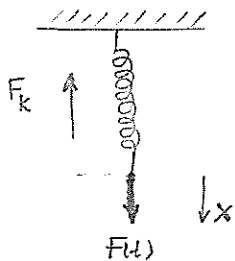
$$\begin{aligned} F_r &= Ma = M \frac{\partial v}{\partial t} \\ &= M \frac{\partial}{\partial t} (\vec{v}) \\ &= M \frac{\partial^2 x}{\partial t^2} \end{aligned}$$

where  $v = \frac{\partial x}{\partial t}$

The reaction force  $F_r$  is a function of time and acts "through"  $M$ .

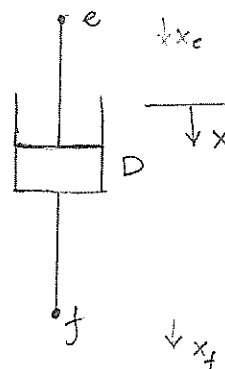
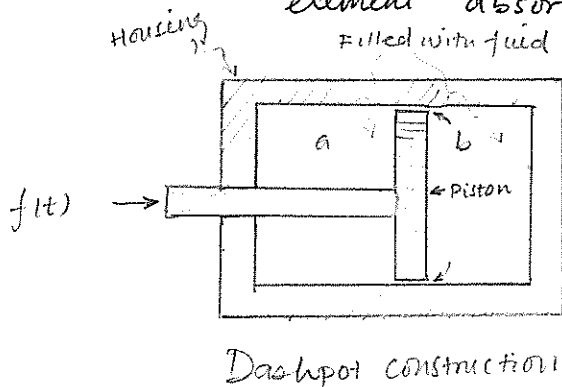
2) The Elastic (Elastance) or stiffness,  $K$  provides

Provides a restoring force as represented by a spring



$$\begin{aligned} F_k &= Kx \quad (\text{Hook's law}) \\ &= K \int v dt \end{aligned}$$

3) The damping or viscous friction,  $D$  characterizes the element absorbs energy.



The damping force is proportional to the difference in velocity of two bodies, and the assumption is made that the viscous friction is linear.

✓ [The reaction damping force  $F_D$  is equal to the product of damping  $D$  and the relative velocity of the two ends of the dash pot.] The direction of this force depends on the relative magnitude and directions of the velocities  $V_e$  and  $V_f$ .

$$F_D = D(V_e - V_f) \\ = D\left(\frac{dx_e}{dt} - \frac{dx_f}{dt}\right)$$

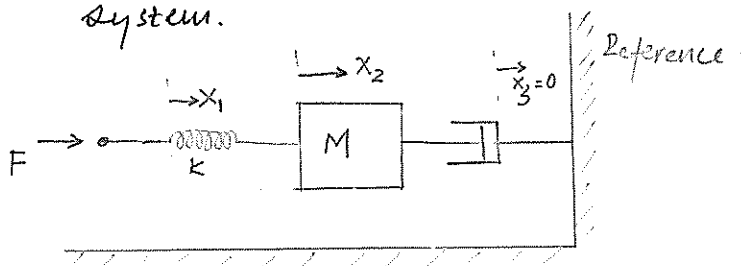
The basic operation

A dashpot in which the housing is filled with a fluid. If a force  $F$  is applied to the shaft, the piston presses against the fluid, increasing the pressure on side 'b' and decreasing the pressure on side 'a'. As a result, the fluid flows <sup>around</sup> the piston from side 'b' to side 'a'.

The damping  $D$  depends on the dimension and the fluid used.

Example.

Consider the simple mass-spring damper mechanical system as shown below. Obtain the differential eq<sup>n</sup> describing the dynamics of the system.

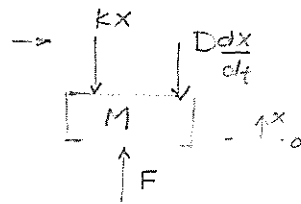
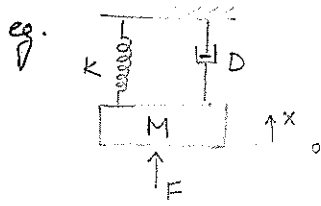


A Force  $F$  applied at the end of the spring must be balanced by a compression of the spring. The same force is also transmitted through the spring and acts at point  $x_2$ .

A systematic way to analyse the system is to draw a free-body diagram.

## Analogous Systems

Systems whose differential equations are of identical form are called Analogous Systems.

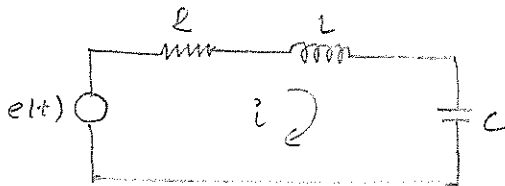


Newton eqn  $F = ma$

$$F - Kx - D \frac{dx}{dt} = M \frac{d^2x}{dt^2}$$

$$F = M \frac{d^2x}{dt^2} + D \frac{dx}{dt} + Kx \quad \dots (*)$$

AND



When from (\*)

$$e(t) = L \frac{d^2i}{dt^2} + \frac{1}{C} i + \dots \quad ***$$

Here from \*\* and \*\*\* the force F and Voltage  $e(t)$  are the Analogous Variables.

Analogous Quantities in Force (Torque) - Voltage / Force (torque) - current analogy.

Mechanical system  
(translational)

Mechanical system  
Rotational

Electrical system.

Force F

Torque

Voltage e / current i

Mass M

Moment of inertia J

Inductance L / capacitance

Viscous friction coefficient D

Viscous friction coefficient D

Resistance  $\frac{1}{R}$

Spring stiffness K

Torsional spring stiffness K

$\frac{1}{C}$  |  $\frac{1}{L}$

Displacement x

Angular displacement  $\theta$

Charge q / magnetic flux  $\phi$

Velocity  $\dot{x}$

Angular velocity  $\dot{\theta}$

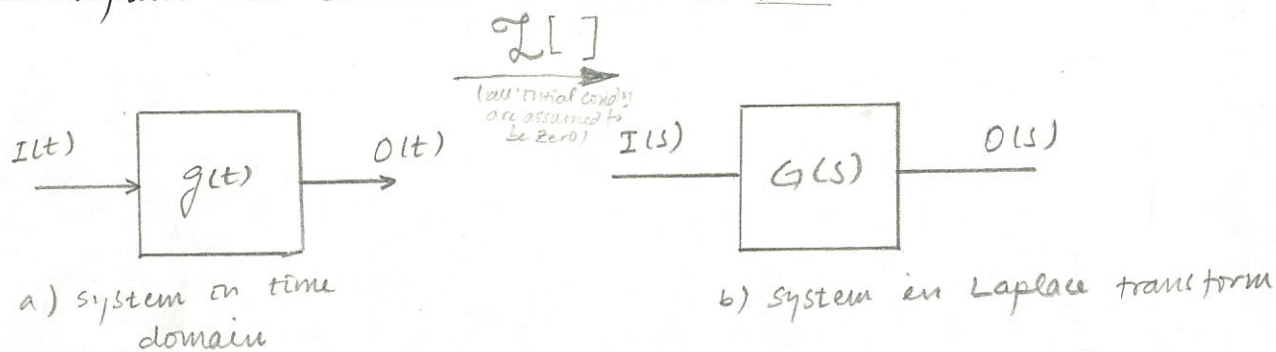
Current i / Voltage e

The concept of analogous systems is useful technique for the study of various systems like electrical, mechanical, thermal, liquid level etc. If the solution of one system is obtained, it can be extended to all other systems analogous to it. Generally it is convenient to study a non-electrical system in terms of its electrical analogue as electrical systems are more easily amenable to experimental study.

## Transfer function.

In control theory, Transfer functions are commonly used to characterize the Input to the Output relationship of components or systems that can be described by linear-time-invariant differential equation.

The Transfer Function of a linear-time-invariant system is defined as to be the ratio of the Laplace transform of the output Variable to the Laplace transform of the input Variable under the assumption that all initial conditions are Zero.



$$\text{Transfer function} = G(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} \bigg|_{\substack{\text{Initial} \\ \text{cond'n} = 0}} = \frac{\mathcal{L}[O(t)]}{\mathcal{L}[I(t)]} \bigg|_{\substack{\text{Initial} \\ \text{cond'n} = 0}}$$

The Transfer function of a system (or element) represents the relationship describing the dynamics of the system under consideration.

Transfer function may be defined only for a linear, stationary (constant parameter) system.

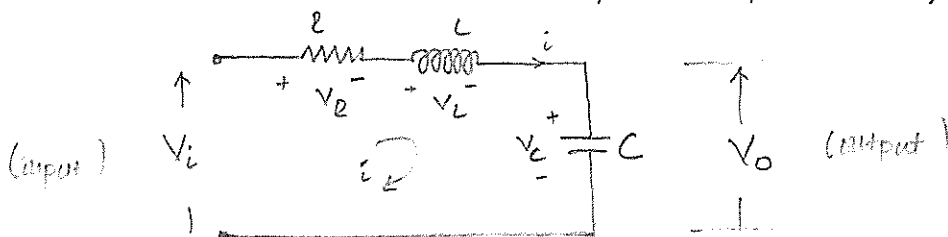
A transfer function is an input-output description of the behavior of a system. Thus the transfer function description does not include any information concerning the internal structure of the system and its behavior.

~~Exercise~~

Determine the transfer function of Electrical and mechanical system in the previous modeling section.

Electrical system.

consider the series RLC ckt shown below and obtain the transfer function relating the input-output voltage



Applying KVL

$$V_R + V_L + V_C = V_i$$

$$iR + L \frac{di}{dt} + \frac{1}{C} \int i dt = V_i$$

Applying Laplace transfer on both sides

$$I(s)R + sLI(s) + \frac{1}{sC} I(s) = V_i(s)$$

$$\left[ R + sL + \frac{1}{sC} \right] I(s) = V_i(s)$$

$$I(s) = \frac{V_i(s)}{R + sL + \frac{1}{sC}} \quad \text{--- (1)}$$

The output of the system  $V_o$

$$V_o = \frac{1}{C} \int i dt$$

Applying Laplace Transform

$$V_o(s) = \frac{1}{sC} I(s) \quad \text{--- (2)}$$

Substituting eqn (1) in (2)

$$V_o(s) = \frac{1}{sC} \left[ \frac{V_i(s)}{R + sL + \frac{1}{sC}} \right] = \frac{1}{sC} \left[ \frac{1}{R + sL + \frac{1}{sC}} \right] V_i(s)$$

The transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{s^2 LC + sRC + 1} //$$

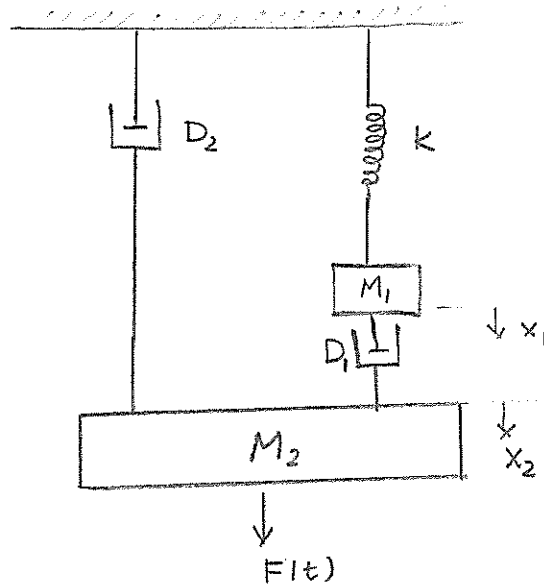


# Transfer Function of Mechanical system.

17  
10

Example

Determine the transfer function relating <sup>mass position</sup>  $x_2(s)$  to Input Force  $F(s)$  of the mass-spring mechanical system.



The differential equation describing the dynamics of the mechanical system are given by

$$M_2 \frac{d^2 x_2}{dt^2} + D_2 \frac{dx_2}{dt} + D_1 \left( \frac{dx_2}{dt} - \frac{dx_1}{dt} \right) = F \quad \dots (1)$$

$$M_1 \frac{d^2 x_1}{dt^2} + D_1 \left( \frac{dx_1}{dt} - \frac{dx_2}{dt} \right) + K x_1 = 0 \quad \dots (2)$$

Taking the Laplace transform of eqn (1) and (2)

$$\left[ s^2 M_2 + s D_2 + s D_1 \right] x_2(s) - s D_1 x_1(s) = F(s) \quad \dots (3)$$

$$\left[ s^2 M_1 + s D_1 + K \right] x_1(s) - s D_1 x_2(s) = 0 \quad \dots (4)$$

From eqn (4) we get

$$x_1(s) = \frac{s D_1}{s^2 M_1 + s D_1 + K} x_2(s) \quad \dots (5)$$

Substituting Eq (5) into eqn (3) we get

$$\left[ s^2 M_2 + s D_2 + s D_1 \right] x_2(s) - s D_1 \left[ \frac{s D_1}{s^2 M_1 + s D_1 + K} \right] x_2(s) = F(s)$$

$$\left[ s^2 M_2 + s D_2 + s D_1 - \frac{s^2 D_1^2}{s^2 M_1 + s D_1 + K} \right] x_2(s) = F(s)$$

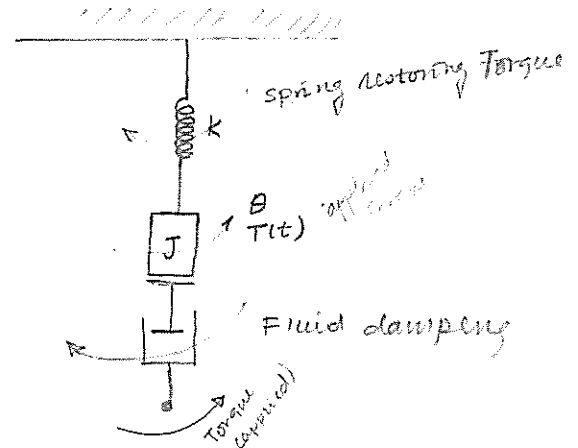
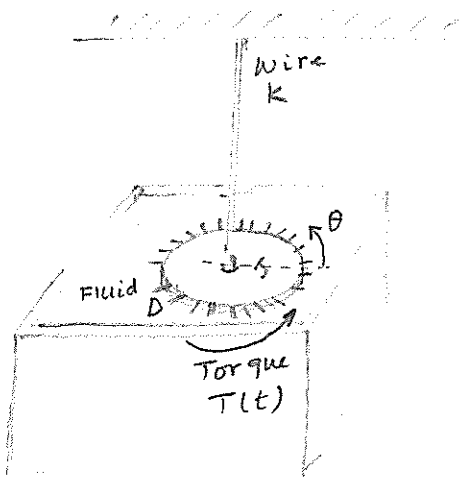
$$\left[ \frac{(s^2 M_2 + s D_2 + s D_1)(s^2 M_1 + s D_1 + k) - s^2 D_1^2}{s^2 M_1 + s D_1 + k} \right] X_2(s) = F(s) \quad 18$$

The transfer function  $\frac{X_2(s)}{F(s)}$  becomes

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2 M_1 + s D_1 + k}{(s^2 M_2 + s D_2 + s D_1)(s^2 M_1 + s D_1 + k) - s^2 D_1^2}$$

Example.

A system shown below has a mass with a moment of inertia  $J$  immersed in a fluid. A torque  $T$  is applied to the mass. The wire produces a reactive torque proportional to the stiffness  $k$  and the angle of twist. The fins moving through the fluid have a damping  $D$  which requires a torque proportional to the rate at which they are moving.



Free body diagram

$$\begin{aligned} T_{\text{spring}} &= k\theta \\ T_{\text{fluid}} &= D \frac{d\theta}{dt} \end{aligned}$$

$$T(t) - T_{\text{fluid}}(t) - T_{\text{spring}}(t) = J \frac{d^2\theta}{dt^2}$$

$$T(t) - D \frac{d\theta}{dt} - k\theta = J \frac{d^2\theta}{dt^2}$$

$$T(t) = J \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + k\theta$$

Transferring using Laplace transform

$$T(s) = [s^2 J + s D + k] \theta(s)$$

The transfer function

$$G(s) = \frac{\theta(s)}{T(s)} = \frac{1}{s^2 J + s D + k}$$