Chapter 8

Stability in The frequency Domain

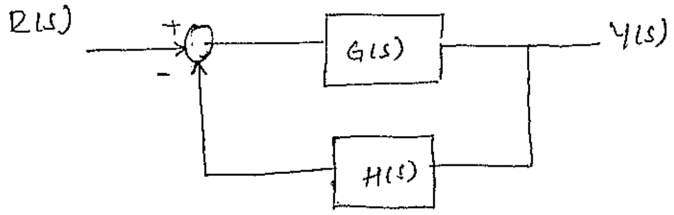
Introduction

For control system, it is necessary to determine whether the system is stable and also it's ~~the~~ relative stability. In this Chapter we will investigate the stability of a system ~~in the real~~ frequency responce based on the of the system.

A frequency domain stability criterion was developed by H. Nyquist (in 1932). The Nyquist stability criterion is based on a theorem on the theory of the function of complex variable due to Cauchy. Cauchy's theorem is concerned with mapping contours in the complex s-plane.

To determine the relative stability of a closed-loop system, we must investigate the characteristics eqn of the system.

$$F(s) = 1 + G(s)H(s) = 0$$



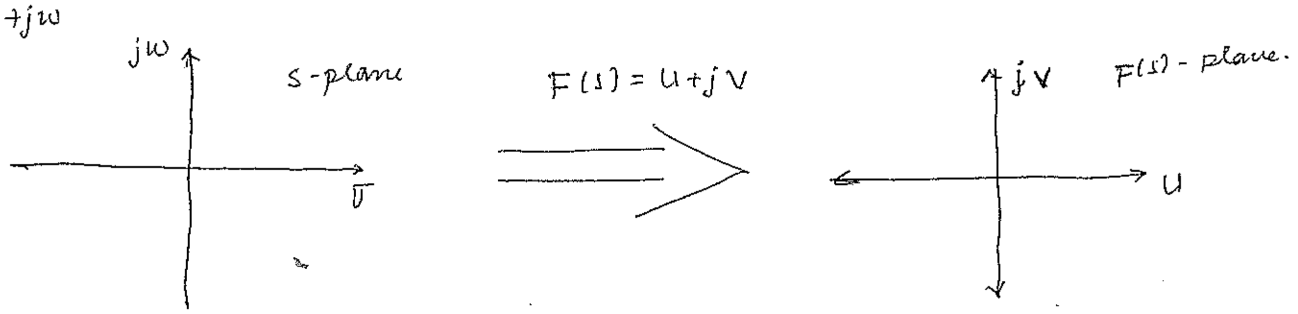
To ensure stability we must ascertain that the zeros of F(s) lie on the left hand ~~head~~ s-plane.

Nyquist thus proposed a mapping of the right-hand s-plane into the F(s) plane. Therefore to utilize and ~~understand~~ understand the Nyquist's criterion we shall first consider briefly the mapping contours on the complex plane.

Mapping contours in the s-plane

Mapping of contours in the s-plane by a function of F(s)

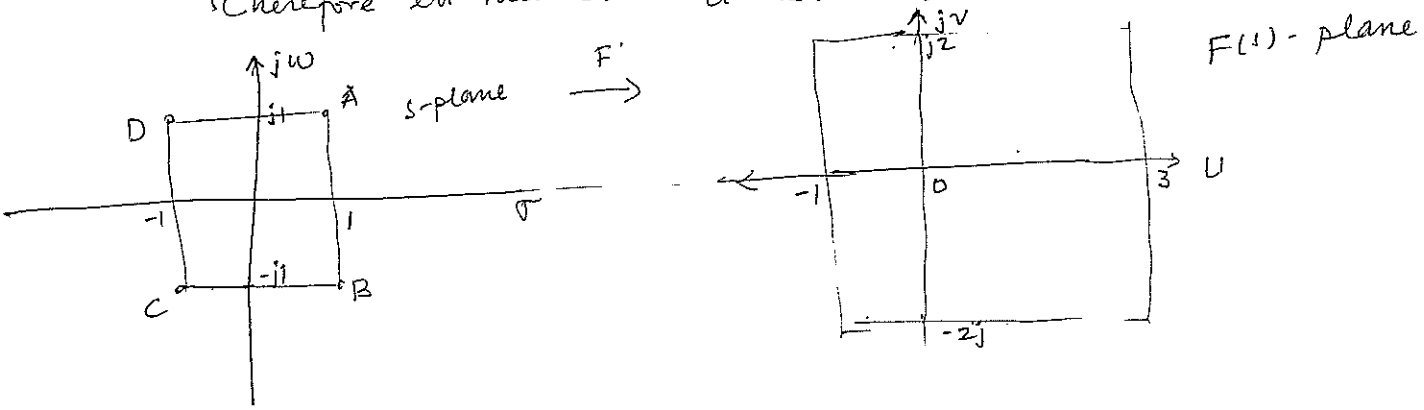
$s = \sigma + j\omega$



eg. Consider mapping of a s-plane unit square contour to the F(s) plane through the relation $F(s) = 2s+1$

$$u + jv = F(s) = 2s+1 = 2(\sigma + j\omega) + 1$$

Therefore in this case $u = 2\sigma + 1$ and $v = 2\omega$



The countour mapped by F(s) into a countour of an identical form, a square with the center shifted by one unit and the magnitude of the side ~~are~~ multiplied by 2.

This kind of mapping which retain the angle of s-plane on the F(s) plane are called conformal mapping.

Cauchy's theorem is concerned with mapping a function F(s) that has a finite number of poles and zeros within the countor.

$$F(s) = \frac{K \prod_{i=1}^{n} (s+s_i)}{\prod_{k=1}^{M} (s+s_k)}$$

s_i = zeros, s_k = poles. of F(s)

The Char. eqn

$$F(s) = 1 + L(s)$$
$$= 1 + G(s)H(s)$$

where $G(s)H(s) = \dfrac{N(s)}{D(s)}$

$$F(s) = 1 + G(s)H(s) = 1 + \frac{N(s)}{D(s)} = \frac{D(s)+N(s)}{D(s)} = \frac{K\prod_{i=1}^{n}(s+s_i)}{\prod_{k=1}^{M}(s+s_k)}$$

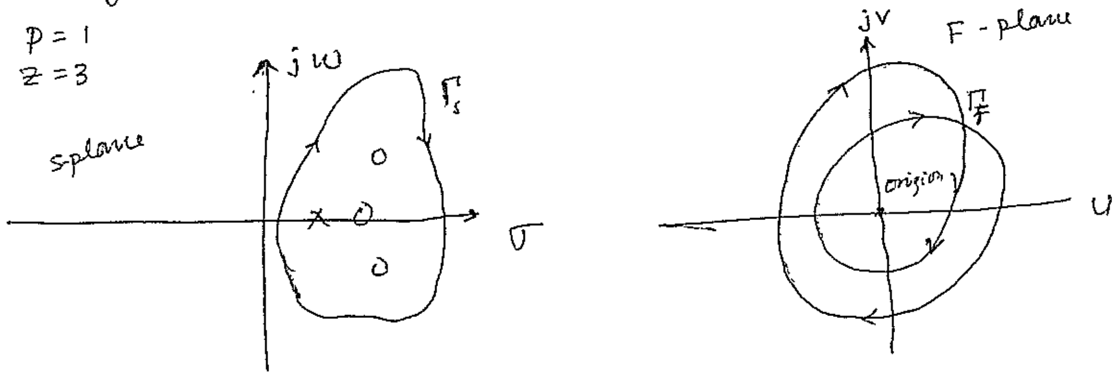
The poles of G(s) H(s) are the poles of F(s), ~~but~~ However, it is the zeros of F(s) that are the characteristics roots of the system and that indicates the system response

Cauchy's theorem (principle of the argument)

If a contour Γ_s in the s plane encircles Z zeros and P poles of F(s) and does not pass through any poles or zeros of F(s) and the traversal is in the clockwise direction along the contour, the corresponding countour Γ_F in the F(s) plane encircles the origion of the F(s) plane ($N = Z - P$) times in the clockwise direction.
$Z-P=Z$

An example of the use of Cauchy's theorem, Consider the pole-zero pattern shown in Fig with countour Γ_s to be considered.

$P = 1$
$Z = 3$



$$N = 3 - 1 = +2$$

Γ_F complets two clockwize encircledment of the origion in F(s) plane

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The Nyquist Criterion.

To investigate the stability of a control system, we consider the char. eqn which is $F(s) = 0$

i.e $$F(s) = 1 + G(s)H(s) = \frac{K \prod_{i=1}^{n} (s + s_i)}{\prod_{k=1}^{M} (s + s_k)}$$

The system to be stable, all the zeros of F(s) must lie in the left-hand s-plane. Thus we find that the roots of a stable system (zeros of F(s)) must lie to the left of the jw-axis. Therefore we choose a contour Γ_s in the s-plane that encloses the entire right-hand s-plane, and we determine whether any zeros of F(s) lie within in Γ_s by utilizing cauchy's theorem i.e we plot Γ_F in the F(s)-plane and determine the number of encircledment of the origion N. Then the number of zeros of F(s) within the Γ_s contour [and therefore unstable zeros of F(s)] is

$$Z = N + P$$

If $P = 0$
→ the number of unstable roots of the system is equal to N, the number of encircledment of the origion of the F(s)-plane.

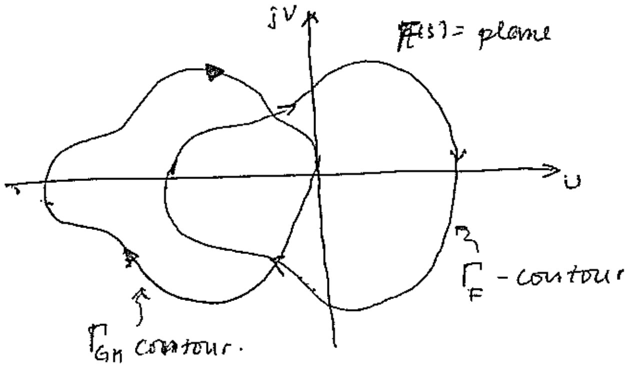
It is therefore follows that the contour Γ_{GH} of $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane is the same as to countour Γ_F of $1+G(s)H(s)$ drawn from the point $(-1+j0)$

Thus the encirclement of the origion by the contour ~~F~~ $\Gamma_{F=1+GH}$ is equivalent of the ~~origion~~ point $(-1+j0)$ by the countour Γ_{GH}.

Nyquist stability criterion stated as

If the countour Γ_{GH} of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist countour in the s-plane encircles the point $(-1+j0)$ in the counter-clockwise direction as many times as the number right half splane poles of $G(s)H(s)$; the closed-loop system is stable.

In the commonly occuring case of the open-loop stable system, the closed loop system is stable if the contour Γ_{GH} of $G(s)H(s)$ does not encircle $(-1+j0)$ point i.e the net encirclement is zero.



The mapping of the Nyquist contour into the contour Γ_{GH} is carried out as follows.

1- The mapping of the imaginary axis is carried out by substitution of $s = j\omega$ in $G(s)H(s)$
resulting → $G(j\omega)H(j\omega)$ a frequency function

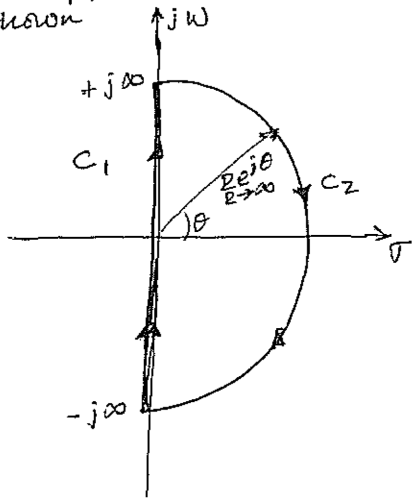
2. In physical system $(m \le n)$, $\lim_{\substack{s = Re^{j\theta} \\ R\to\infty}} G(s)H(s) =$ real constant (it is zero if $m < n$)

Thus the infinite ~~contour~~ arc of the Nyquist contour ~~mappa~~ maps ~~to~~ onto a point on the <u>real</u> axis

⇒ The complet contour Γ_{GH} is thus the polar plot of $G(j\omega)H(j\omega)$ with ω varying from $-\infty$ to ∞. This is usually known as Nyquist plot or Locous of $G(s)H(s)$

* Nyquist plot is symmetrical about the real axis since
$$G^*(j\omega)H^*(j\omega) = G(-j\omega)H(-j\omega)$$

the Nyquist contour that encloses the entire right-hand s-plane is shown



it is directed ~~cou~~ clockwise and comprises of infinite segment c_1 along jω axis and ~~C_2~~ an arc c_2 of infinite radius

Along c_1

$s = j\omega$ with s varying from $-j\infty$ to $+j\infty$

along c_2

$s = \underset{R \to \infty}{R e^{j\theta}}$ with θ varying from $+\pi/2$ to $-\pi/2$

The Nyquist contour so defined encloses all the right half s-plane. zeros and poles of $F(s) = 1 + G(s) H(s)$

Let there be Z zeros and P poles of F(s) in the right half s-plane, a closed contour Γ_F is traversed in the F(s) plane which encloses the origin

$N = P - Z$ (Remember $N = Z - P$ clockwise encircclement)

times in the counter-clockwise direction

In order for a system to be stable, there should be no zero of $F(s) = 1 + G(s) H(s)$ in the right half-s-plane i.e

$$Z = 0$$

This condition is meet if

$$N = P$$

i.e for closed-loop system to be stable, the number of counter-clockwise encircclement of the origion of F(s) plane by the countour Γ_F should equal to the number of the right half s-plane poles of F(s) which are the poles of the open-loop transfer function G(s) H(s).

In special cases (this is generally the case where in most single-loop practical system) ~~if~~ P=0 (i.e the open-loop stable system), the closed loop system is stable if

$$N = P = 0$$

which means that the net encircclements of the origion of the F(s) plane by ~~#~~ Γ_F countor should be zero. i.e easily observable

$$G(s) H(s) = \left[1 + G(s) H(s)\right] - 1$$

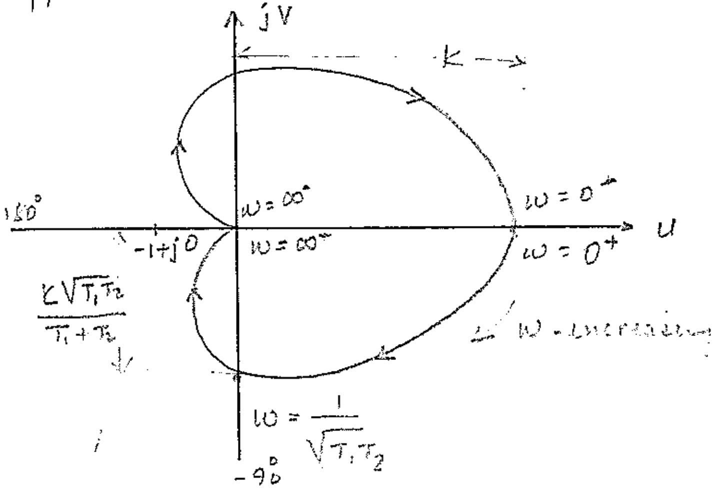
Example

Consider the feedback system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{(T_1 s+1)(T_2 s+1)}$$

sol^n $$G(j\omega)H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

- The infinite semi circular arc of the Nyquist contour mapps to origin
- The $j\omega$ axis is mapped to the solid lines.



Z = 0
P = 0 on R-HS-plane
N = P − Z
0 = 0 − Z
Z = 0
stable.

$G(j\omega)H(j\omega)$ does not encircle the point $-1+j0$, the system is stable for all positive values of K, T_1, T_2.

Example two

Consider now an open-loop unstable system with the transfer function

$$G(s)H(s) = \frac{s+2}{(s+1)(s-1)}$$

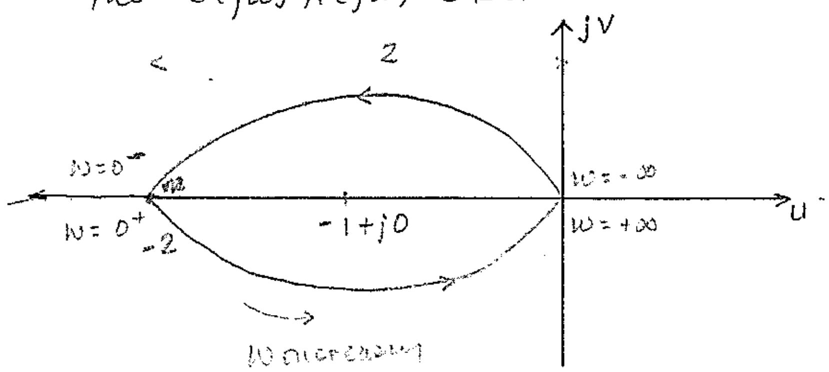
$\frac{s+2}{s^2-1} = \frac{2+j\omega}{-\omega^2-1}$

Let us consider determine whether the system is stable when the feedback path is closed.

GH = tf([1 2],[1 0 -1])
nyquist(H)

From the transfer function of the open-loop system
- Open-loop pole at in the right half s-plane ($s=1$)
  - ↳ $P=1$

The $G(j\omega)H(j\omega)$ sketch



$\omega \to 0^-$
$\frac{(-0.001+2)}{(1-0.001)(-0.001-1)}$

The plot indicates that the $(-1+j0)$ point is encircled by this locus once in the counter clockwise direction

$\therefore$ $N = 1 = P$ thus $Z = 0$ i.e there are no zeros of $1+G(s)H(s)$ in the right half s-plane

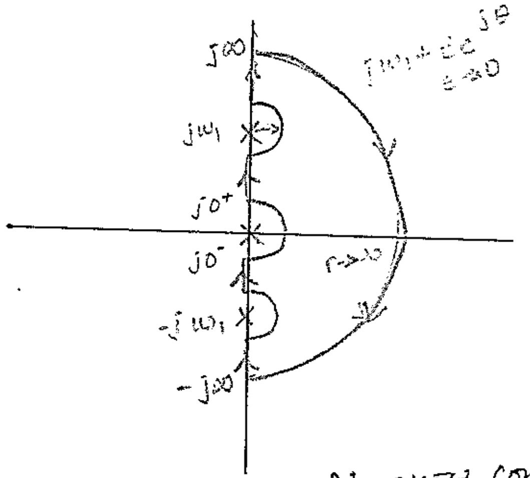
$\Rightarrow$ hence the closed loop system is stable.

$$N = P - Z$$
$$1 = 1 - Z$$
$$Z = 0 \quad \text{(there is no zero of } 1+GH\text{)}$$

Open-loop Poles On the jω-axis.

If $G(s)H(s)$ and therefore $1+G(s)H(s)$ has any poles on the jω-axis, the Nyquist contour defined earlier can not be used as such since the s-plane contour should not pass through a singularity of $1+G(s)H(s)$.

To study the stability in such a cases, the Nyquist contour must be modified so as to bypass any jω-axis pole. This is accomplished by indenting the Nyquist contour around the jω-axis poles along a semicircle of radius ϵ where $\epsilon \to 0$

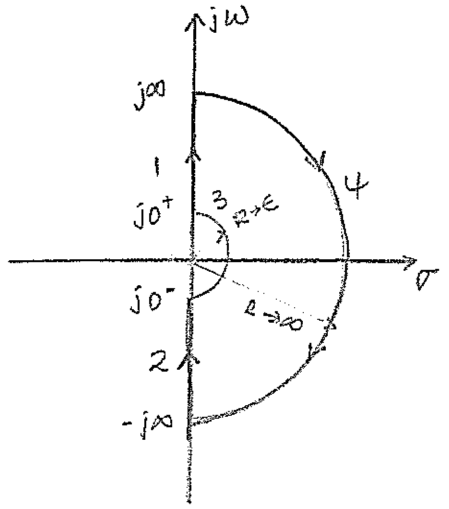


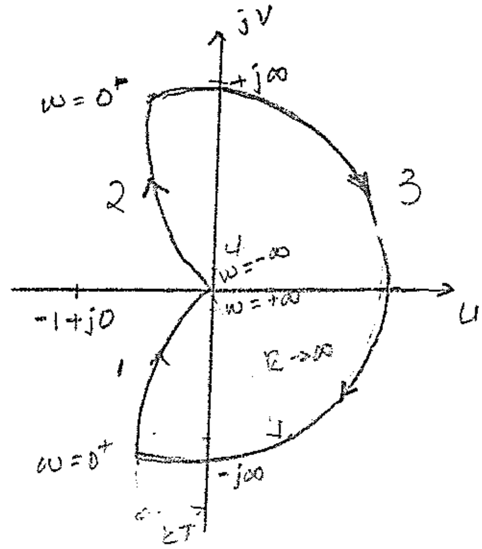
Indented Nyquist contour for jω-axis open-loop pole.

Example Consider a feedback system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{s(1+sT)}$$

- open loop system has a pole at origin
  - ↳ Therefore the Nyquist contour must therefore be indented to bypass the origin





The mapping of the Nyquist Contour

1- The semicircular endent around the pole at origion represented by $s = \underset{\epsilon \to 0}{\epsilon e^{j\theta}}$ (θ varying from -90° through 0° to $+90^\circ$) maps onto

$$\lim_{\epsilon \to 0} \frac{K}{\epsilon e^{j\theta}(1 + T\epsilon e^{j\theta})} = \lim_{\epsilon \to 0} \frac{K}{\epsilon e^{j\theta}} =$$

$$= \lim_{\epsilon \to 0} \frac{K}{\epsilon} e^{-j\theta}$$

The value $\frac{K}{\epsilon}$ approches infinity as ϵ approches zero and θ varies from $+90$ through 0° to -90° as s moves along the semi circle.

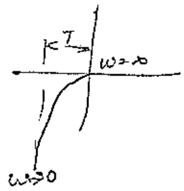
Thus the semicircleler endent around the origion maps into a semi circular arc of infinite radius in $G(s)H(s)$ -plane. (region 3)

2- The mapping of the positive emaginary axis ($\omega = 0^+$ to $+\infty$) is obtained by calculating

$\frac{K}{j\omega(1 + j\omega T)}$ at various values of ω and potting them

in the $G(s)H(s)$ plane. (part 1, 2)

. (part of locus of polar plot $\frac{1}{j\omega(1+j\omega T_1)}$



3. ~~Ae~~ The infinite semicircular arc of the Nyquist contour (4) represented by $s = \underset{R \to \infty}{R e^{j\theta}}$ (θ varying $+90$ through 0 to -90°) is mapped into

$$\lim_{R \to \infty} \frac{K}{Re^{j\theta}[TRe^{j\theta} + 1)]} = \lim_{R \to \infty} \frac{K}{TR^2(e^{j2\theta})} = 0 e^{-j2\theta}$$

i.e the origion of the $G(s)H(s)$ -plane.

The $G(s)H(s)$ locus thus turns at the origin with zero radius from $-180°$ through $0°$ to $+180°$

(part 2)

4 - The mapping of the negative imaginary axis is the mirror image of that for the positive imaginary axis.

In order to investigate the stability of the system, we first note that number of poles P of $G(s)H(s)$ in the right half's plane is zero. For all +ve values of K and T, the locous of $G(j\omega)H(j\omega)$ does not encircle $(-1+j0)$ point

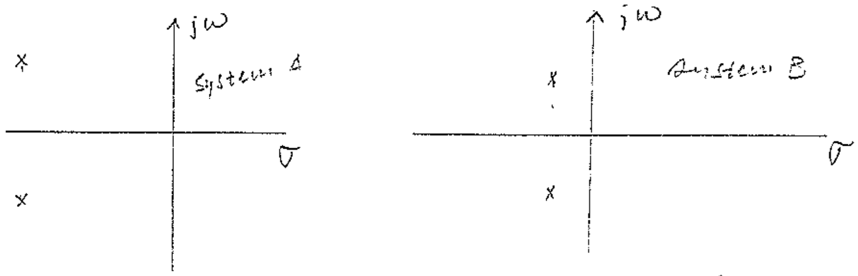
$\Rightarrow$ the system under consideration is always stable.

Assessment of Relative Stability Using Nyquist Criterion.

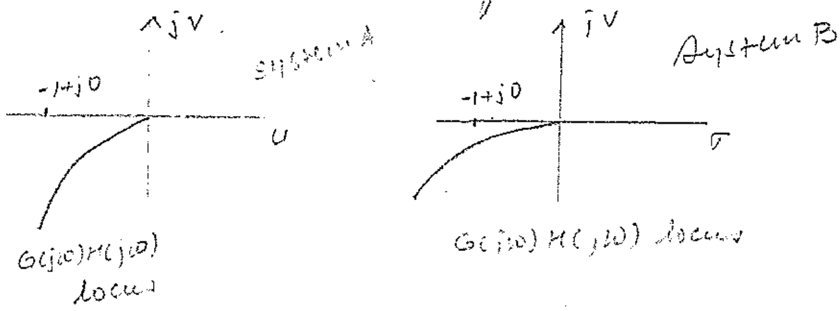
The Nyquist plot indicates not only whether a system is stable but also the degree of stability of a stable system. In addition the plot provides information as to how stability may be improved, if this is necessary.

The stability criterion of $G(s)H(s)$ (open-loop tran. fun) is merely the non-encirclement of $(-1+j0)$ point, it can be intuitively imagined that the polar plots gets closer to $(-1+j0)$ point, the system tends towards instability.

consider the closed loop poles of two systems A and B.



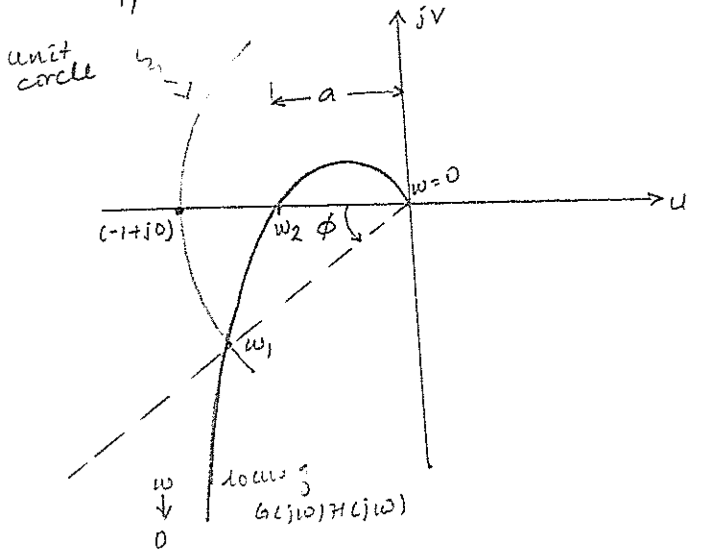
system A is more stable than B since the closed loop poles of A are further away from jω axis to the left than that of B's



The comparison of the closed loop-poles location of these two systems with their corresponding polar plots reveals that as polar plot moves closer to $(-1+j0)$ point, the system closed loop poles move closer to the jω axis hence the system becomes less stable and vice versa.

Consider

consider a typical $G(j\omega)H(j\omega)$ locus.



* The unit circle pass through the point (-1+j0) and intercept the $G(j\omega)H(j\omega)$ locus at a frequency ω_1.
* The $G(j\omega)H(j\omega)$ cross the the negative real axis at freq. ω_2 and with an intercept of a.

As $G(j\omega)H(j\omega)$ - locus approaches (-1+j0) point, the relative stability reduces. simultaneously, the value of 'a' approches unity and that of ϕ tends to zero.
The relative stability of could be measured interms of the intercept a or the phase angle ϕ. (interms of gain margin and phase margin)

Defn.
Gain Margin - it is the factor by which the system gain can be increased to drive it to the verge of instability.

In the above figure
at frequency $\omega = \omega_2$, the phase angle $\angle G(j\omega)H(j\omega) = 180°$
and $|G(j\omega)H(j\omega)| = a$
If the gain of the system is increased by a factor $\frac{1}{a}$
then $|G(j\omega)H(j\omega)|_{\omega=\omega_2}$ become $a\left(\frac{1}{a}\right) = 1$, hence the $G(j\omega)H(j\omega)$ plot pass through (-1+j0) point, driving the system to the verge of instability

=) Therefore the gain margin (GM) may be defined as the reciprocal of the gain at the frequency at which the phase angle becomes 180°.
The frequency at which the phase angle is 180° is called the phase cross over frequency. (ω_2)

$\therefore$ $GM = \frac{1}{a}$, where $a = |G(j\omega)H(j\omega)|_{\omega=\omega_2}$

The increase in gain for $G(j\omega)H(j\omega)$ plot to pass through (-1+j0) is given

$$GM = 20\log\frac{1}{a} = -20\log a \text{ dB}.$$

(since $a<1$, GM is +ve.)

The gain cross over frequency is the ω_1 frequency at which $|G(j\omega)H(j\omega)|$, the magnitude of the open loop transfer function, is unity.

Defn Phase margin : is the amount of <u>additional</u> phase lag at the gain cross over frequency required to bring the system to the verge of unstability.

The ~~phase margin~~ γ $= \angle \cancel{G(j\omega)H(j\omega)}|_{\omega=\omega_1} \cancel{+180^\circ}$

$= \cancel{\angle \phi + 180^\circ}$

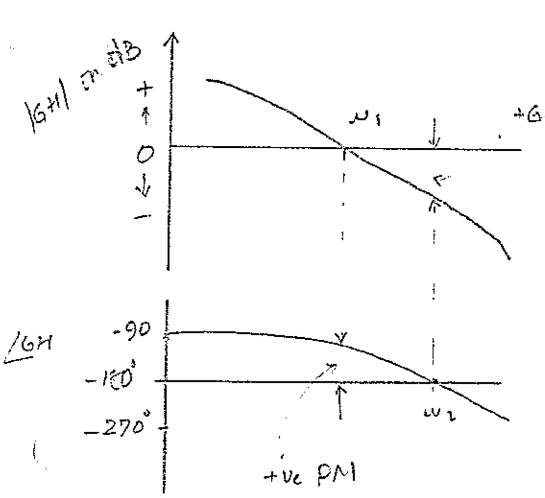
usually Practically GM about 6db or PM of 30-35° results in good degree of stability.

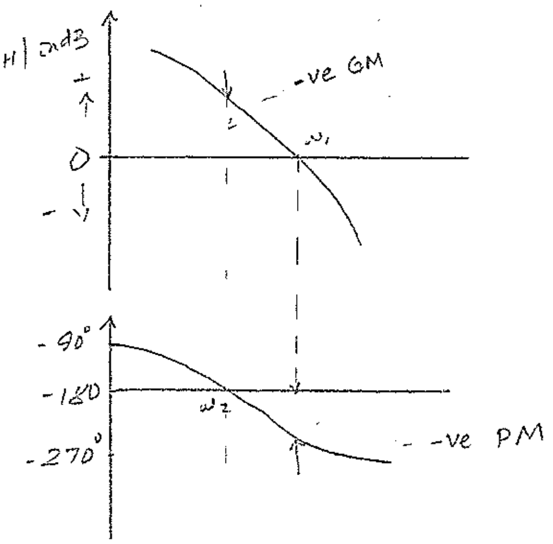
Phase margin $\phi = \angle G(j\omega)H(j\omega)|_{\omega=\omega_1} + 180^\circ$

where the angle at ω_1, the gain cross over frequency.

Ill.

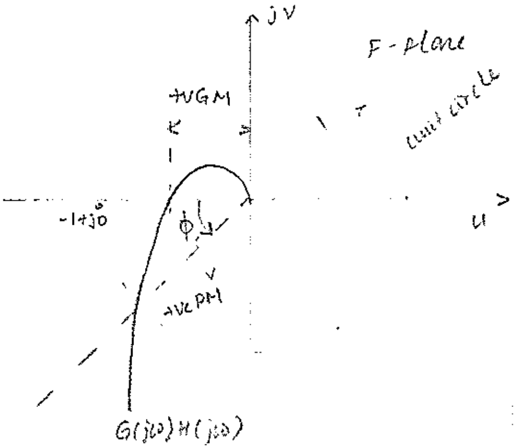
Illustration of phase & gain margin of both a stable and unstable system in Bode plot, ~~and~~ polar plot.

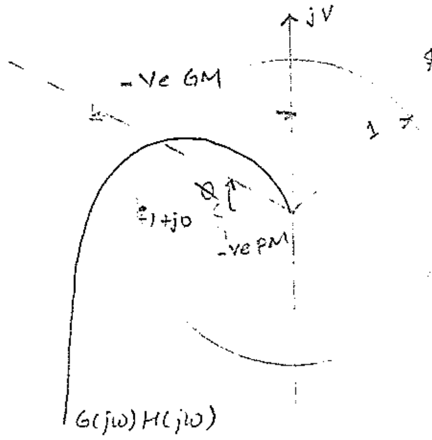




stable system

unstable system





The phase margin is measured positively in counter-clockwise direction from the negative real axis.

Gain margin (GM) and phase margin (PM) are frequently used for frequency response specification by designers

- large GM or large PM indicates a very stable feedback system but usually a very sluggish one.
- A GM close to unity or PM close to zero corresponds to a high oscillatory system (i.e closed-loop poles close to $j\omega$ axis)

Computation of Gain Margin and phase margin.

GM and PM may be computed by the use of various plots:
- direct polar plot
- Bode plot

(inverse polar plot, log-magnitude Vs phase angle plot)

In relatively simple cases GM and PM may be computed directly.

Eg. Consider a unity feedback system having an open loop transfer function

$$G(j\omega) = \frac{K}{j\omega(j0.2\omega+1)(j0.05\omega+1)}$$

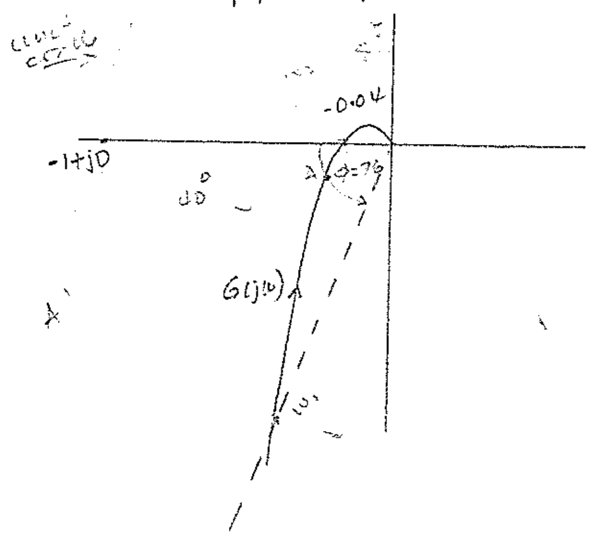
$$G(s) = \frac{K}{s(0.2s+1)(0.05s+1)}$$

Find GM and PM using Nyquist plot for $K=1$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+(0.2\omega)^2)}\sqrt{[1+(0.05\omega)^2]}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(0.2\omega) - \tan^{-1} 0.05\omega$$

The Nyquist plot.



$a = 0.04$

PM = 76°

$GM = 20\log\left(\frac{1}{a}\right)$

$= 20\log\left(\frac{1}{0.04}\right)$

$= 28\,dB$

Let us now use the Nyquist plot for adjustment of the system gain for specified GM or PM

Suppose it is desired to find the open-loop gain for

i) a GM of 20 dB

ii) a PM of 40°

Sol^n i) For a GM = 20 dB, the Nyquist plot should intersect the real axis at $-a$ where

$$20 \log \left(\frac{1}{a}\right) = 20$$

$$a = 0.1$$

This is achieved if the system gain is increased by a factor of $0.1/0.04 = 2.5$ Thus $k = 2.5$

$0.04 = 1 \ (k=1)$
$0.1 = k$
$k = 2.5$

ii) A PM $= 40^\circ$ is obtain if the system gain is increased such that a point A is shifted to location A'. This is achieved if the system gain is increased by a factor of $\frac{OA'}{OA} = \frac{1}{0.191} = 5.24$ Thus $k = 5.24$

The above can be done with only computation (for 1st and 2nd order system)

i) GM = 20 dB

GM $20 \log \frac{1}{a} = 20$

$a = 0.1$

The Nyquist plot intersects the real axis at a point where

$$G(j\omega) = \frac{k}{j\omega(j0.2\omega+1)(j0.05\omega+1)} = \frac{k}{\underbrace{-0.25\omega^2}_{a} + j\underbrace{\omega(1-0.01\omega^2)}_{b}}$$

$\frac{k}{a+jb} \Rightarrow \frac{k(a-jb)}{a^2-b^2}$

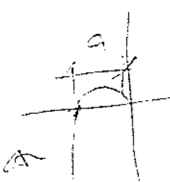
is real

setting the imaginary part equal to zero we have

$\frac{ka}{a^2-b^2} - \frac{jkb}{a^2-b^2}$

$\omega = \omega_2 = 10$ rad/sec

$$\text{Now } \left|G(j\omega)\right|_{\omega=\omega_2=10} = \frac{k}{0.25(10)^2} = a = 0.1$$



$-\frac{kb}{a^2-b^2} = 0$

$-k[\omega(1-0.01\omega^2)]$

$1-0.01\omega^2$, $\omega^2 = \frac{1}{0.01} = 100$

$\omega = 10$

which gives

$k = 2.5$

$$\frac{k}{-0.25(10)^2 + j(10)(1-0.01(10)^2)}$$

$= \frac{k}{\ldots}$

(ii) Let $\omega = \omega_1$ be the gain cross-over frequency. Then a PM = 40°

phase margin $\phi = \angle G(j\omega)H(j\omega) + 180$

$$-90^\circ - \tan^{-1}0.2\omega_1 - \tan^{-1}0.05\omega_1 + 180^\circ = 40^\circ$$

$$\tan^{-1}0.2\omega_1 + \tan^{-1}0.05\omega_1 = 50^\circ$$

$\tan^{-1}A + \tan^{-1}B = C$

$\tan C = \dfrac{A+B}{1-AB}$

$$\frac{0.2\omega_1 + 0.05\omega_1}{1 - 0.2\omega_1 \times 0.05\omega_1} = \frac{0.25\omega_1}{1 - 0.01\omega_1^2} = \tan 50^\circ = 1.2$$

$$0.012\omega_1^2 + 0.25\omega_1 - 1.2 = 0$$

solving for +ve values of ω_1

$\omega_1 = 4$ rad/sec

$$\Rightarrow \left| G(j\omega) \right|_{\omega=\omega_1} = \frac{k}{\omega_1\sqrt{(1+(0.2\omega_1)^2}\sqrt{(1+(0.05\omega_1)^2)}} = 1$$

$k = 5.2$

Example.

Determine the gain margin and phase margin of a unity feedback system having an open loop transfer function.

$$G(j\omega) = \frac{10}{j\omega(j0.1\omega+1)(j0.05\omega+1)}$$

by use of Bode plot.

The Bode plot is shown below

$$\tan^{-1}A + \tan^{-1}B = C$$

$$\tan C = \tan(\tan^{-1}A + \tan^{-1}B)$$

$$= \frac{\tan(\tan^{-1}A) + \tan(\tan^{-1}B)}{1 - \tan(\tan^{-1}A)\tan(\tan^{-1}B)}$$

$$= \frac{A+B}{1-AB}$$

Stability design via mapping positive $j\omega$-axis.

15th

eg. Find the range of gain for stability and instability, and the gain for marginally stability, for the unity feedback system. where

$$G(s) = \frac{K}{(s^2+2s+2)(s+2)}$$

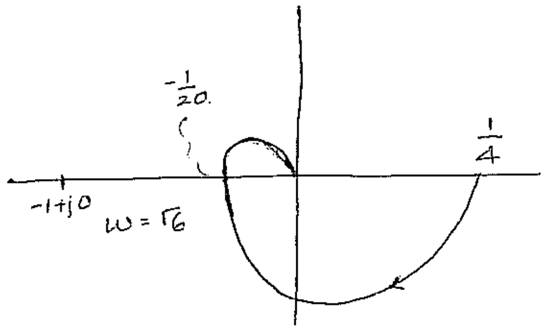
For marginal stability find the radian frequency of oscillation. Use the Nyquist criterion and the mapping of only +ve imaginary axis.

sol/n

since the open-loop poles are only in the left half-plane, the Nyquist criterion tells us that we want no encirclement of -1 for stability.

→ Hence a gain less than unity at ±180° is required

By setting K=1, draw the portion of the contour along the +ve imaginary axis.



Intersection with the imaginary axis:, setting the imaginary part equal to zero.

$$G(j\omega)H(j\omega) = \left.\frac{1}{(s^2+2s+2)(s+2)}\right|_{s \to j\omega}$$

$$= \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} \quad \cdots\, *$$

setting the imaginary part equal to zero

$$\frac{-j\omega(6-\omega^2)}{16(1-\omega^2)^2+\omega^2(6-\omega^2)^2} = 0$$

$$\Rightarrow \omega = \sqrt{6}$$

Substituting back this value of ω yeilds in (*) yeilds the real part

$$\frac{4(1-\omega^2)}{16(1-\omega^2)^2+\omega^2(6-\omega^2)^2} = \frac{4(1-6)}{16(1-6)^{\cancel{2}}} = \frac{4}{16(-5)} = \frac{-1}{20}$$

$$\Rightarrow \frac{-1}{20} \angle 180^\circ$$

This closed-loop system is stable if the magnitude of the frequency response is less than unity at 180°. 15+II

-) Hence, the system is stable for $k < 20$, unstable for $k > 20$, and marginally stable for $k = 20$, when the system is marginally stable, the radian frequency of oscillation is $\sqrt{6}$.

Eg. for $k = 6$

find the gain and phase margine

To find gain margine

$$G(j\omega)H(j\omega) = \left.\frac{6}{(s^2+2s+2)(s+2)}\right|_{s \to j\omega}$$

the Nyquist diagram cross the real axis at $\omega = \sqrt{6}$
=) The real part calculate for $k = 6$ =) $(~~-0.3~~)$ $\frac{-1}{20} * 6 = -3.33$
thus the gain can be increased by -3.33 before the real part becomes -1. Hence

$$GM = 20 \log 3.33 = 10.45 dB.$$

To find PM, find freq when $|G(j\omega)H(j\omega)| = 1$

where $G(j\omega)H(j\omega)$ has unity magnitude at $\omega = 1.253$ rad/sec

At this freq. the phase angle is $-112.3°$

=) the phase angle b/n this angle and $-180°$ is $67.7°$ which is the phase marigine.

PM = 67.7°

Correlation between phase margin and damping factor.

Consider a unity feedback second-order system with an open loop trans. function

$$G(s)H(s) = \frac{K}{s(s\tau+1)} = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

where $\omega_n = \sqrt{K/\tau}$ and $2\xi\omega_n = \frac{1}{\tau}$

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)}$$

At the gain cross over frequency $\omega = \omega_1$

the magnitude $|G(j\omega)H(j\omega)| = 1$

$$\Rightarrow \frac{\omega_n^2}{\omega_1\sqrt{(\omega_1^2+4\xi^2\omega_n^2)}} = 1$$

$$(\omega_1^2)^2 + 4\xi^2\omega_n^2(\omega_1)^2 - 4\omega_n^4 = 0$$

$$\Rightarrow \left(\frac{\omega_1}{\omega_n}\right)^2 = \sqrt{(4\xi^4+1)} - 2\xi^2$$

The phase margin of this system is given

$$\phi = \angle G(j\omega_1)H(j\omega_1) + 180^\circ$$

$$= -90^\circ - \tan^{-1}\left(\frac{\omega_1}{2\xi\omega_n}\right) + 180^\circ$$

$$= 90^\circ - \tan^{-1}\left[\frac{1}{\sqrt{(4\xi^4+1)^{\frac{1}{2}} - 2\xi^2}}\right]$$

$\therefore$ The phase margin is ONLY a function of ξ.

Frequency Domain Specification

1- Resonant peak M_r – the max. value of M, the magnitude of the closed loop ~~system~~ frequency response.

→ a large resonance peak corresponds to a large overshoot in transient response

2. Resonant frequency, ω_r – is the frequency at which the resonance peak M_r occurs. ~~This is the freq. at which the resonance response~~

The freq. resp

3. Bandwidth - it is the range of frequencies for which the system gain is more than -3dB. Such gain is considered adequate to ensure good transmission of signal. The closed loop ~~system adequate measure~~ system filters out the signal components whose frequencies are greater than the cut-off frequency (freq at of -3dB gain) and transmits those signals components whose frequencies are lower than the cut-off frequency.

The BW information is a good measure the ability of system to reproduce the input signal and also measures its noise rejection characteristics.

4. cut-off rate - it is the slope of the log-mag. curve near the cutt off frequency. The cut-off rate indicates the ability of the system to distinguish the signal from noise

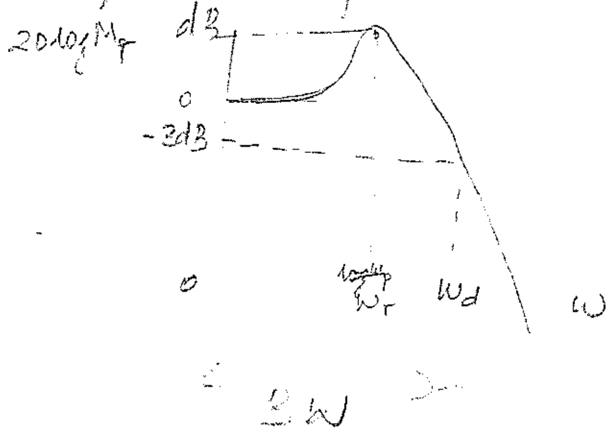
5. GM and PM.

BW - the freq. range $0 \le \omega \le \omega_b$ in which the mag. of the closed loop does not drop -3dB is called BW

specification of BW may be determined by the ff factors

1) The ability to reproduce the input signal. A large bandwidth corresponds to a small rise time, or fast response. BW is proportional to the speed of response

2. The necessary filtering characteristics for high-frequency noise.



Bandwidth - the freq. at which the mag $\left|\frac{T(j\omega)}{\ldots}\right| = \frac{1}{\sqrt{2}}$ or (-3dB)

i.e. $\frac{1}{\sqrt{2}} = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$