Chapter Six

Root Locus Technique.

6.1 Introduction

In the preceding chapter it was discussed how the performance of a feedback system can be described interms of the location of the roots of characteristics eqn in the s-plane. It is known that the ~~the~~ response of a closed loop feedback system can be adjusted to achive the desired performance by judious selection of one or more system parameter.

judicious = wise

It is frequently necessary to adjust one or more system parameter in order to obtain suitable root locations. Therefore it is worthwile to determine how the roots of the characteristics eqn of a given system migrate about the s-plane as the parameter are varied; i.e, it is useful to determine the locus of roots of in the s plane as parameter is varied.

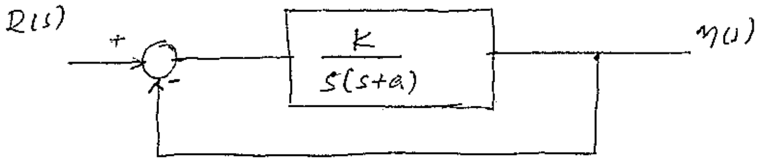
The root locus technique (introduced by Evan in 1948) is a graphical method for sketching the locus of roots in the s-plane as parameter varied.

Root locus also provides a measure of sensetivity of roots to the variation in the parameter being varied / considered.

This technique provides a graphical method of plotting the locus of the roots in the s-plane as given system parameter is varied over the complete range of values (may be zero to infinity). The root corresponding to a particular value of the system parameter can then be located on the locus or the value of the parameter for a desired root location can be determined from the locus.

The Root Locus Concept.

To understand the concepts underlying the root locus technique, consider the simple second-order system below.



1- The closed loop transfer function $T(s) = \frac{Y(s)}{R(s)} = \frac{GH}{1+GH} = \frac{K}{s^2+as+K}$

$= G$

2- The characteristic eqn of the system.

$1 + GH = 0$

$\Rightarrow 1 + G\cancel{H} = 0 \qquad s^2 + as + k = 0$

The second order system under consideration is always stable for +ve values of a and k but its dynamic behaviour is controlled by the roots of the char. eqn

the roots is given by

$$s_1, s_2 = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - k}$$

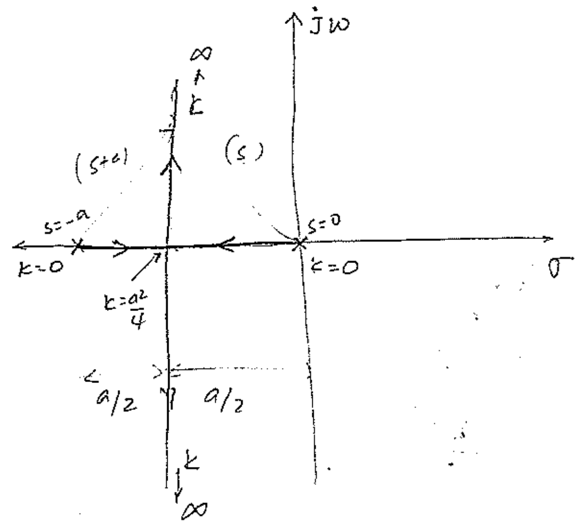
3- Taking $\underline{a}$ fixed, as k is varied from zero to infinity, the two roots (s_1, s_2) describe loci in the s-plane.

Root location for various k

i) $0 \le k < \frac{a^2}{4}$, the roots are are $s_1 = 0$, $s_2 = -a$ i.e they coincide with the open loop poles of the system (i.e for $k=0$)

ii) $k = \frac{a^2}{4}$, the roots are real and equal in value i.e $s_1 = s_2 = -a/2$

iii) $\frac{a^2}{4} < k < \infty$ the roots are complex conjugate with real part $= -\frac{a}{2}$ i.e unvarying real part.

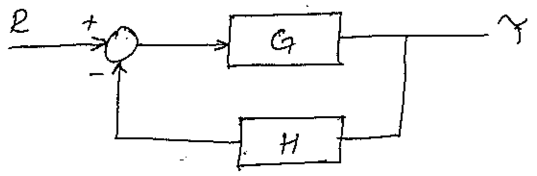


k = 0

k	s_1	s_2
0	0	$-a$
↓		
$\frac{a^2}{4}$	$a/2$	$a/2$
↓	fixed real part $a/2$ but complex conjugate	

6.2 Basic Properties of the Root-loci

Consider a general closed-loop transfer function



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The Characteristics eqn is given by

$$1 + GH = 0$$
$$\Rightarrow GH = -1$$

It is seen that the roots of the Char. eqn (i.e the closed loop poles of the system) occurs only for those values of 's' where

$$G(s)H(s) = -1$$

since s is a complex variable, the root-loci satisfied the following two Evan conditions

i) Magnitude condition

$$|GH| = 1$$

ii) Angle condition

$$\angle GH = \pm 180 \pm 360k$$
$$= \pm 180(1+2k), \quad k = 0, \pm 1, 2$$

~~Writing GH/ interms of system poles and zeros~~

If GH can be written as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2) + \cdots + (s+z_m)}{(s+P_1)(s+P_2) + \cdots + (s+P_n)} \qquad m < n$$

where the magnitude and angle condn become

iii) $$|GH| = \frac{K|s+z_1||s+z_2| + \cdots |s+z_m|}{|s+P_1||s+P_2| \cdots |s+P_n|} = \frac{K\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}^{n}|s+P_j|} = 1$$

$$|GH| = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+P_i|} = \frac{1}{K}, \quad -\infty < K < \infty$$

iv) The angle of GH

$$\angle GH = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{i=1}^{n}\angle(s+P_i) = \pm 180(1+2k)$$
$$= \pm\pi(1+2k) \qquad k = 0, 1, \ldots$$

After Once the complete root-loci are constructed the value of k along the loci can be determined, for the value of k for a particular root location s_0 can be found from the magnitude criterion

$$K = \frac{\prod_{j=1}^{n} |s_0 + P_j|}{\prod_{i=1}^{m} |s_0 + z_i|}$$

6.3 Construction Rules for Root-locus of a system.

Rule 1: The root-locus starts at the poles of GH, where $k=0$

proof. From the magnitude cond^n

$$|GH| = \frac{\prod_{i=1}^{m} |s + z_i|}{\prod_{j=1}^{n} |s + P_j|} = \frac{1}{|K|}$$

As k approaches, to zero, $|GH|$ approaches to infinity, so s must approche the poles of $GH \Rightarrow s \to -P_j$ (open loop poles)

—/—

The characteristics eq^n can be written as

$$K \prod_{i=1}^{m} |s+z_i| + \prod_{j=1}^{n} |s+P_j| = 0$$

$$1 + GH = 0$$

$$1 + \frac{K \prod (s+z_i)}{\prod (s+P_i)} = 0$$

as $k \to 0$ the eq^n becomes

$\prod_{j=1}^{n} |s+P_j|$, this eq^n has roots at $-P_j$ ($j=1,2,\ldots n$)

which are the open-loop poles. The root locus branches therefore start at the open-loop poles.

—/—

Rule 2: The root-locus ends (complets) at the zeros of GH, where $k = \pm\infty$

proof: consider the magnitude cond^n

$$|GH| = \frac{\prod_{i=1}^{m} |s+z_i|}{\prod_{j=1}^{n} |s+P_j|} = \frac{1}{|k|}$$

As $k \to \pm\infty$ $|GH|$ approaches to zero; this corresponds to s must approache the zeros of $GH \Rightarrow s \to -z_i$

—/—

—1—

* $1+GH=0$ begins at the poles of GH and ends at the zeros of GH as k increases from ~~0~~ to infinity.

The char. eqn can be written as

$$\frac{1}{k}\prod_{j=1}^{n}(s+P_j)+\prod_{i=1}^{m}(s+z_i)=0$$

As k tends to infinity, the first term vanishes and the roots of char. eqn are located at $-z_j$ $(i=1,2,\dots m)$ which are the open loop zeros of the system. Therefore m branches of the root locus terminate on the open-loop zeros.

* in case $m<n$, the GH has $(n-m)$ zeros at infinity, therefore $(n-m)$ branches of the root locus terminates on infinity.

—1—

Rule 3. The number ~~loci~~ separate loci | number of branches is equal to the number poles (since $m \le n$) or the order of the char. eqn poly

~~xxx~~

Rule. 4: The root locus is symmetrical about the real axis (σ- axis)

The roots of the char. eqn are either real or complex roots appear as pair of complex conjugate roots.

Rule 5: The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angle ϕ_A, when the number of finite zeros of GH, m, is less than the number of poles n, by the number $N=n-m$, then N section of loci must end at zeros at infinity.

This section of loci proceed to the zeros at infinity along asymptotes as k approaches infinity. These linear asymptote are centered at a point on the real axis given by

$$\sigma_A = \frac{\sum \text{poles of } GH - \sum \text{zeros of } GH}{n-m}$$

$$= \frac{\sum_{j=1}^{n}(-P_j) - \sum_{i=1}^{m}(-z_i)}{n-m}$$

*

$$\phi_A = \frac{(2k+1)\,180^0}{n-m}, \quad k=0,1,2,\dots (n-m-1)$$

—1—

Rule 5: The $(n-m)$ branches ~~to~~ of the root locus which tend to infinity, do so along straight line asymptotes whose angles are given by

$$\phi_A = \frac{(2k+1)\,180^0}{n-m} \quad k=0,1,2,\dots (n-m-1)$$

The asymptotes cross the real axis at a point known as centroid, determined by the relationship

$$\sigma_A = \frac{(\text{sum of real parts of poles}) - (\text{sum of real parts of zeros})}{\text{number of poles} - \text{number of zeros}}$$

—1—

Rule 6: A point on the real axis lies on the locus if the number of open-loop (GH) poles ~~and~~ plus zeros on the real axis to the right of this point is odd.

Eg. consider a feedback system with the char. eqn

$$1+\frac{k(s+2)}{s(s+3)(s^2+2s+2)} = 1+GH=0$$

$$=1+\frac{k(s+2)}{s(s+3)[s+(1+j)][s+(1-j)]} \Rightarrow GH = \frac{k(s+2)}{s(s+3)(s+(1+j))(s+(1-j))}$$

Sol^n

1- The root-loci started at the poles of GH, where $k=0$ (Rule 1)

The poles are
$s=0,\ -3,\ -1-j,\ -1+j$

2. The root-loci ends at the zeros of GH, where $k=\pm\infty$

The zeros of GH are

one finite zero at $s=-2$
and ~~two~~ three at infinity

(No poles and zeros are equal if zeros at infinity included)

3- The number of branches of separate loci are equal to the number of poles (or order of the char eqn)

= 4 separate loci

4. The root loci are symmetrical w.r.t the real axis.

5. no poles > no zeros
$n-m$ branches of the root locus tends to infinity (Rule 5)
with asymptote angle

$$\phi_A = \frac{(2k+1)\,180^\circ}{n-m} \qquad k=0,1,2,\ldots(n-m-1)$$

where $n-m-1 = 4-1-1 = 2$

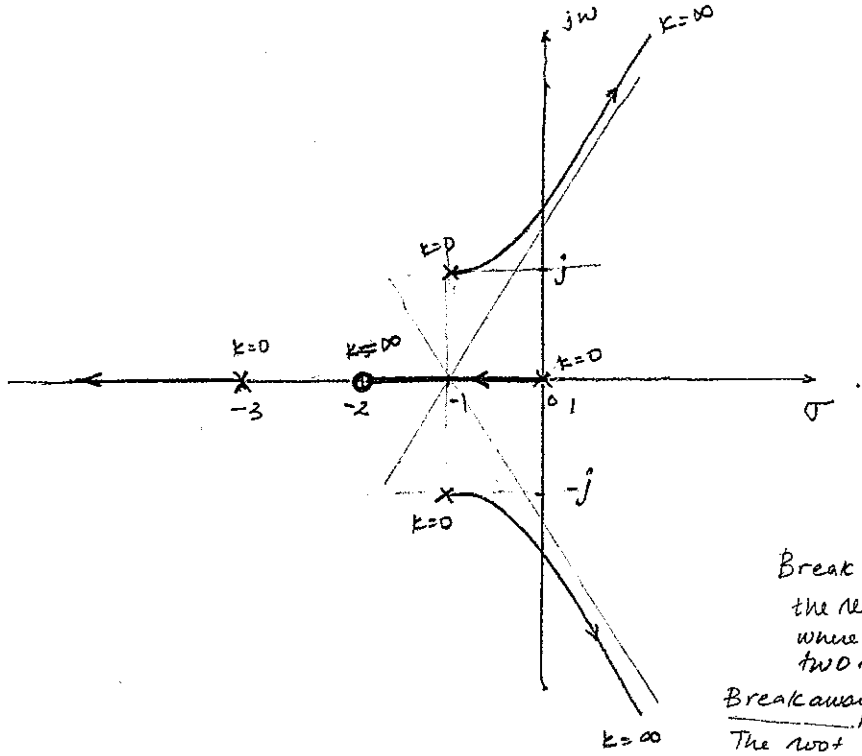
$k=0,1,2 \Rightarrow$ three angles

$$\phi_A = \frac{(0+1)\,180}{4-1} = 60^\circ, \quad \phi_A = \frac{(2+1)\,180^\circ}{3} = 180^\circ, \quad \phi_A = \frac{(2\times2+1)\,180}{3} = 300^\circ$$

And the centroid

$$\sigma_A = \frac{\sum(\text{poles of } GH) - \sum(\text{zeros of } GH)}{n-m}$$

$$= \frac{(0-3-1-j-1+j)-(-2)}{4-1} = \frac{-5+2}{3} = -1$$



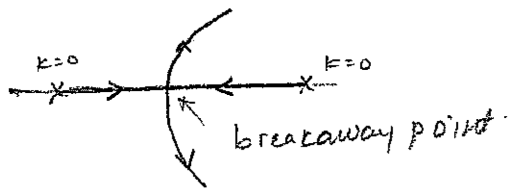
Break away pt - the pt where the root left the real axis. The locus leaves the real axis where there are a multiplicity of roots, typically two.

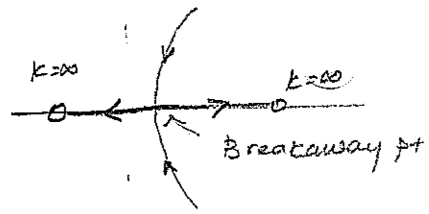
Breakaway Directions of root locus Branches
The root locus branches must approach or leave the breakaway on the real axis at an angle of $\pm 180/r$, where r is the no of root locus branches approaching or leaving the point.

Rule 7: Break away point / saddle point.

Breakaway point (points at which multiple roots of the characteristics eqn occur) of the root locus are the solution of $\frac{dk}{ds} \neq 0$ $\quad \frac{d}{ds}\left[G(s)H(s)\right] = 0$

→ the point where two branches of the root loci meet at the breakaway point on the real axis and then depart from the axis in the opposite direction





The breakaway point on the complete root loci of $1+GH=0$ must satisfy $\frac{d}{ds}\left[G(s)H(s)\right] = 0$

* The actual breakaway points are those roots of the eqn at which the root locus angle criterion is met

$$1+G(s)H(s) = 0 = 1+K\frac{B(s)}{A(s)} = 0$$

$$\frac{d}{ds}\left[G(s)H(s)\right] = k\,\frac{A(s)B'(s) - A'(s)B(s)}{[A(s)]^2} = 0$$

The break away point given by roots of

$$A(s)B'(s) - A'(s)B(s) = 0$$

We can write $1+K\frac{A(s)}{B(s)} = 0 \implies K = \frac{-A(s)}{B(s)}$

differentiating K w.r.t s we get

$$\frac{dk}{ds} = \frac{A(s)B'(s) - A'(s)B(s)}{[B(s)]^2}$$

∴ The break away point of the original eqn are determined by

$$\frac{dk}{ds} = 0$$

eg. the Char. eq of the feedback loop system is given

$$1 + \frac{K}{s(s+1)(s+2)} = 0 \qquad 1+GH=0$$

Then Find the break away pt.

There is a break away point on the real axis b/n 0 and -1 as the two real-root branches are oppositely directed on this segment.

$$\frac{K}{s(s+1)(s+2)} = -1$$

$$GH = \frac{K}{s(s+1)(s+2)}$$

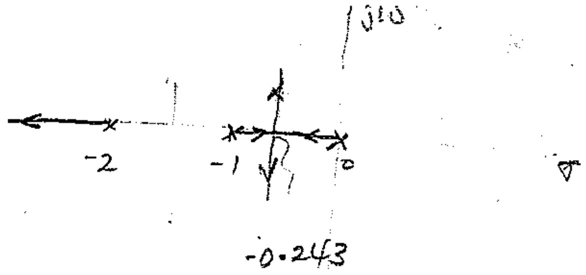
poles at $s = 0, -1, -2$

$$K = -s(s+1)(s+2)$$

$$\frac{dK}{ds} = \frac{d}{ds}\left[-(s^3+3s^2+2s)\right] = 0$$

$$= -(3s^2+6s+2) = 0$$

$$s_{1,2} = -0.423, -1.577$$



since the break away point must lie b/n 0 and -1, it is clear that $s = -0.423$ corresponds to the actual breakaway pt.

Eg. The open loop transfer function of a feedback system is

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

1- The poles of GH $s = 0, -4, -2+j4, -2-j4$

2- 4 poles $\Rightarrow$ 4 branches originate at the poles of GH, $s = 0, -4, -2 \pm j4$

3- since there are no open-loop (GH) zeros in the finite region, all the four ($n-m$, i.e $4-0$) branches terminate on infinity along whose angles with the real axis are

$$\phi_A = \frac{(2K+1)180}{n-m=4} = \frac{1}{4} * 180^\circ = 45^\circ \qquad \begin{matrix} n-m-1 \\ 4-0-1 = 3 \end{matrix}$$

$$K = 0, 1, 2, 3$$

$$= 45^\circ, 135^\circ, 225, 315$$

4- The centroid

$$\sigma_A = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{n-m} = \frac{(0-4-2-2)-(0)}{4}$$

Symmetry

$$= -2$$

5. The point b/n 0 and -4 on the real axis lie on the root locus.

6- From the char. eqn $\quad K = -s(s+4)(s^2+4s+20) \qquad 1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$

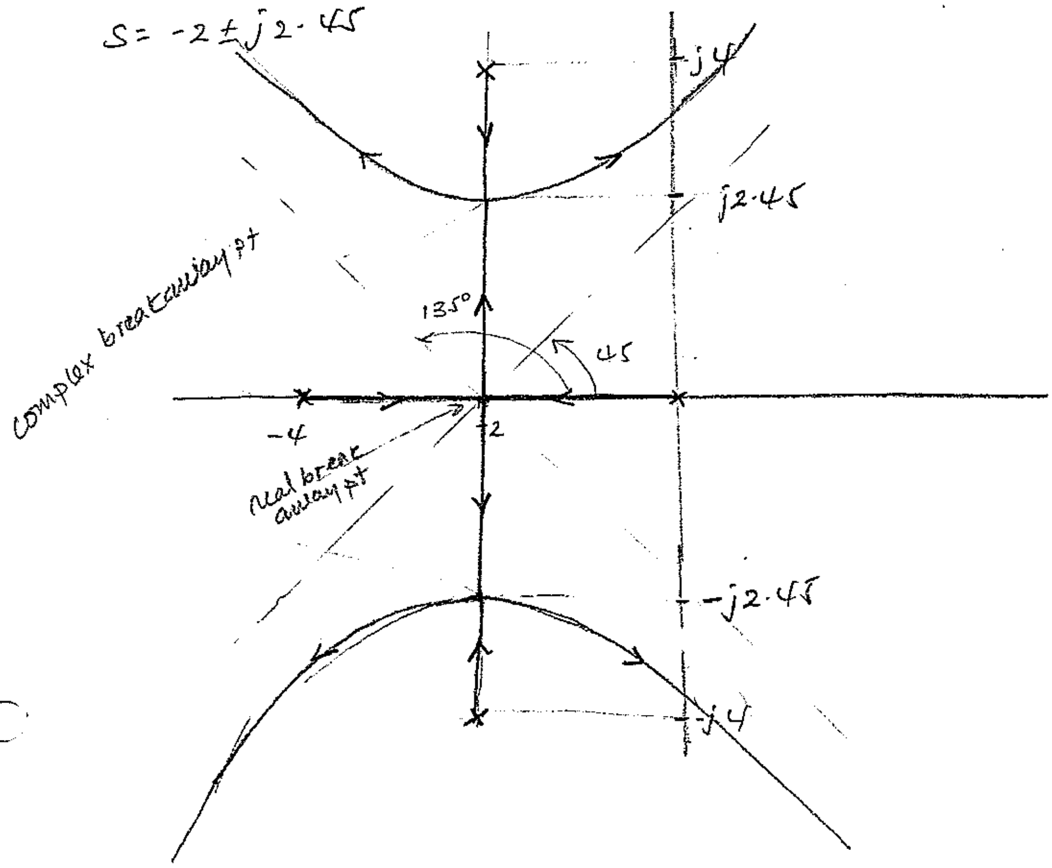
$$\frac{dK}{ds} = -(4s^3+24s^2+72s+80) = 0 \qquad K = -1(s(s+4)(s^2+4s$$

$$s = -2 \text{ and } s = -2 \pm j2.45$$

Therefore, there is one breakaway point on the real axis at $s = -2$ and two complex conjugate breakaway pt on the real axis at $s = -2 \cancel{\pm j2}$

c)

$s = -2 \pm j2.45$



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<u>Note</u>

A breakaway on the <u>real</u> axis may occur in two ways.

First, a breakaway point may result from two real-root branches moving towards each other as K increased. After a breakaway pt these branches become complex-root branch.

Secondly, a real axis breakaway pt may occur with complex-root branches moving towards the real axis and meeting at the breakaway point. These branches then become real-root branches and moves in the opposite direction.

Breakaway Direction of Root locus Branches

The root locus branches must approach or leave the breakaway point on the real axis at an angle of $\pm \frac{180}{r}$, where r is the number of root locus branches approaching or leaving the pt.

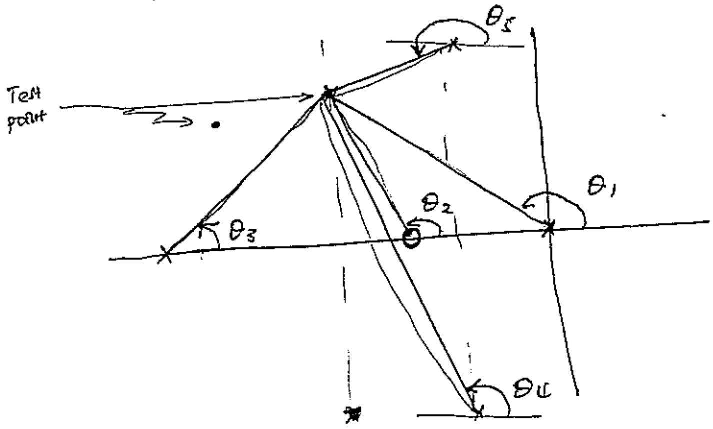
eg/ on the above

Rule 8: Angle of Departure and Angle of Arrival of the root loci.

The angle of Departure or arrival of a root locus at a pole or zero of GH denotes the angle of the tangent to the locus near the point.

- consider the given pole-zero configuration

The locus leaving a pole or entering a zero will have to pass through a test point placed close to the pole or zero, at which the total angle contributions of the poles and zeros, of GH will have to satisfy the angle criterion.



o - zeroes
x - poles.

Determining the angle of Departure θ_5 using angle criterion

$$\sum \text{angle of zeros} - \sum \text{angle of poles} = \pm (2k+1)\pi$$

$$\theta_2 - (\theta_1 + \theta_3 + \theta_4 + \theta_5) = (2k+1)\pi$$

Angle of departure becomes - (departure angle from the pole)

$$\theta_5 = \theta_2 - (\theta_1 + \theta_3 + \theta_4) \pm (2k+1)\pi$$

In general

$$\begin{bmatrix} \text{Angle of} \\ \text{Departure} \end{bmatrix} = \sum \theta_{zeros} - \sum \theta_{poles} \pm (2k+1) * 180^\circ$$

$k = 0, \pm 1, \pm 2 \ldots$

Example. find the angle of Departure at the complex poles.

Example.

Draw the root locus for

$$GH = \frac{k}{s(s+1)(s+2)}$$

s^2+3s+2

s^3+2s^2+2s+0

1- The poles of GH, $s=0$, $s=-1$ and $s=-2$

2- The function GH has three zeros at ∞

3. There are 3-poles indicating, that there are three separate loci.

4. The pole-zero configuration of GH is symmetrical w.r.t the real axis.

$$\sigma_A = \frac{\sum \text{poles of } GH - \sum \text{zeros of } GH}{\text{no poles} - \text{no zeros}}$$

$$= \frac{[(0-1-2)-0]}{3-0} = \frac{-3}{3} = -1$$

$$\phi_A = \frac{(2k+1)180}{n-m} =$$

$k = n-m-1$

$3-0-1=2$

$0,1,2$

$$= \frac{(2k+1)180}{3}, \Rightarrow 60$$

$\Rightarrow 180$

$\Rightarrow 300$

5. The breakaway point on the complete root loci

$$\frac{dk}{ds} = -\frac{d(GH)}{ds} = -\frac{d}{ds}(s^3+3s^2+2s) = 0$$

$$\Rightarrow 3s^2+6s+2=0$$

The breakaway point ~~on the~~

$$S_{b1,2} = \frac{-6 \pm \sqrt{6^2-4\times3\times2}}{6}$$

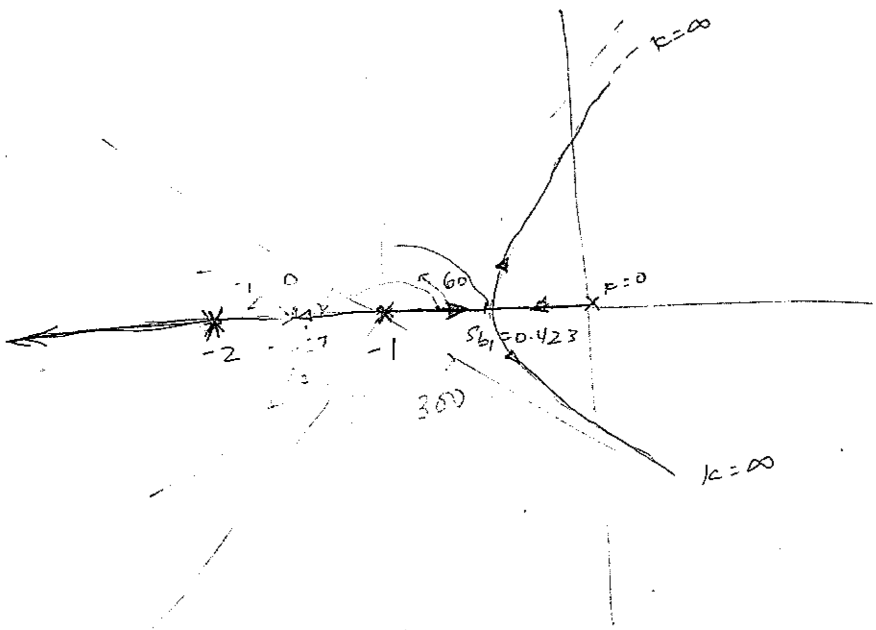
$$= -0.463 \text{ or } -1.57$$

$\pm\frac{180}{n} = \pm\frac{180}{2} = \pm 90$

$S_{b1} = -0.423$

$S_{b2} = -1.57$

6- The root loci



Rule 9: The intersection of root locus branches with imaginary axis can be determined by use of Routh Criterion.

Eg. A feedback control system has an open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+3)(s^2+2s+2)}$$

Find the root locus as K varies from 0 to ∞

1- open loop poles at
$s = 0, -3, (-1+j)$ and $(-1-j1)$

2- There is no finite open loop zeros

3- have four branches, orginates at $K=0$ (and ends at the zeros at ∞, as $K \to \infty$

4- the root locus exists on the real ~~root~~ axis for $-3 \le s \le 0$

5- the four branches tends to infinity along asymptotes whose angles with the real axis are

$$\phi_A = \frac{(2K+1)180}{n-m} \qquad q = 0,1,2,3 \qquad \begin{matrix} n-m-1 \\ 4-0-1 = 3 \end{matrix}$$

$$= \frac{(2K+1)180}{4-0}$$

$$= \underline{45^\circ, 135^\circ, 225^\circ, 315^\circ}$$

The centroid is at

$$\sigma_A = \frac{\sum \text{poles} - \sum \text{zeros}}{\text{poles} - \text{zeros}}$$

$$= \frac{(-3-1-1)-(0)}{4-0} = -1.25$$

6 The break away point.

$$K = -s(s+3)(s^2+2s+2) = -(s^4+5s^3+8s^2+6s)$$

$$\frac{dK}{ds} = -4(s^3+3.75s^2+4s+1.5) = 0$$

Use bisection method b/w $s = 0$ and -3

The possible breakaway point

$$s = -2.3, \ -0.725 \pm j0.365$$

The break away point must occur at $s = -2.3$ as this part of the real axis is on the root locus and the two root locus branchs starting from $s=0$ and $s=-3$ are approaching each other it can be checked that $s = -0.725 \pm j\,0.365$ are not the breakaway points as the angle of criterion is not met at this point.

7 The ~~two~~ two break branches break at an angle of $\pm 90^\circ$

$$\text{i.e. } \pm \frac{180}{r} = \pm \frac{180}{2}$$

r no branches approaching each other.

8. The value of k at the breakaway point is evaluated at

$$k = \frac{\prod_{j=1}^{n} |s_0 + P_j|}{\cancel{\prod_{i=1}^{m} |s_0 + z_i|}} = \Big\{ |s|\,|s+3|\,|s+1+j1|\,|s+1-j| \Big\}_{s=-2.3}$$

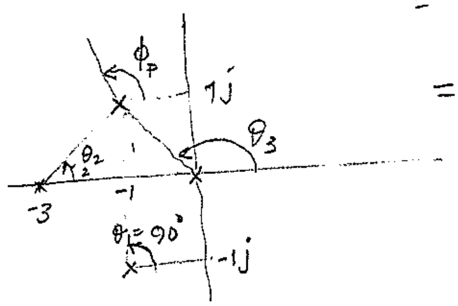
$$= 2.3 \times 0.7 \times \cancel{1.68}\,1.64 \times \cancel{1.68}\,1.64$$

$$= \cancel{4.54}\ 4.33$$

9. The root locus branches leaves the pole at $s = (-1+j1)$ at an angle ϕ_p given by

$$\phi_p = \cancel{180(2k+1)} -$$

$$= \sum \theta_{zeros} - \sum \theta_{poles} - (2k+1) * 180^\circ$$



$$= 0 - (26 + 135 + 90) - (2k+1) * 180^\circ$$

$$= \cancel{-251.6+180}\ -251.6 + 180$$

$$= -71.6^\circ$$

$\theta_1 = 90$
$\theta_3 = 135$
$\theta_2 = \tan^{-1}\left(\frac{1}{2}\right) = 26.5$

10. The intersection of the root locus with the ~~real~~ imagi axis. from the Routh criterion.

The system Charecteristics

$1 + GH = 0$

$$1 + \frac{k}{s(s+3)(s^2+2s+2)} = 0$$

$$s(s+3)(s^2+2s+2) + k = 0$$

$$s^4 + 5s^3 + 8s^2 + 6s + k = 0$$

Routh array

s^4	1	8	k
s^3	5	6	
s^2	$34/5$	k	
s^1	$\frac{\frac{204 - 5k}{5}}{34/5}$	0	
s^0	k	0	

$$\frac{34}{5}s^2 + k = 0$$

Examination of the first element in the first column of the Routh array reveals that the above mentioned root locus branches will intersect the imaginary axis at a value of K given by

$$\frac{204}{5} - 5K \leq 0 \qquad \text{marginally stable.}$$

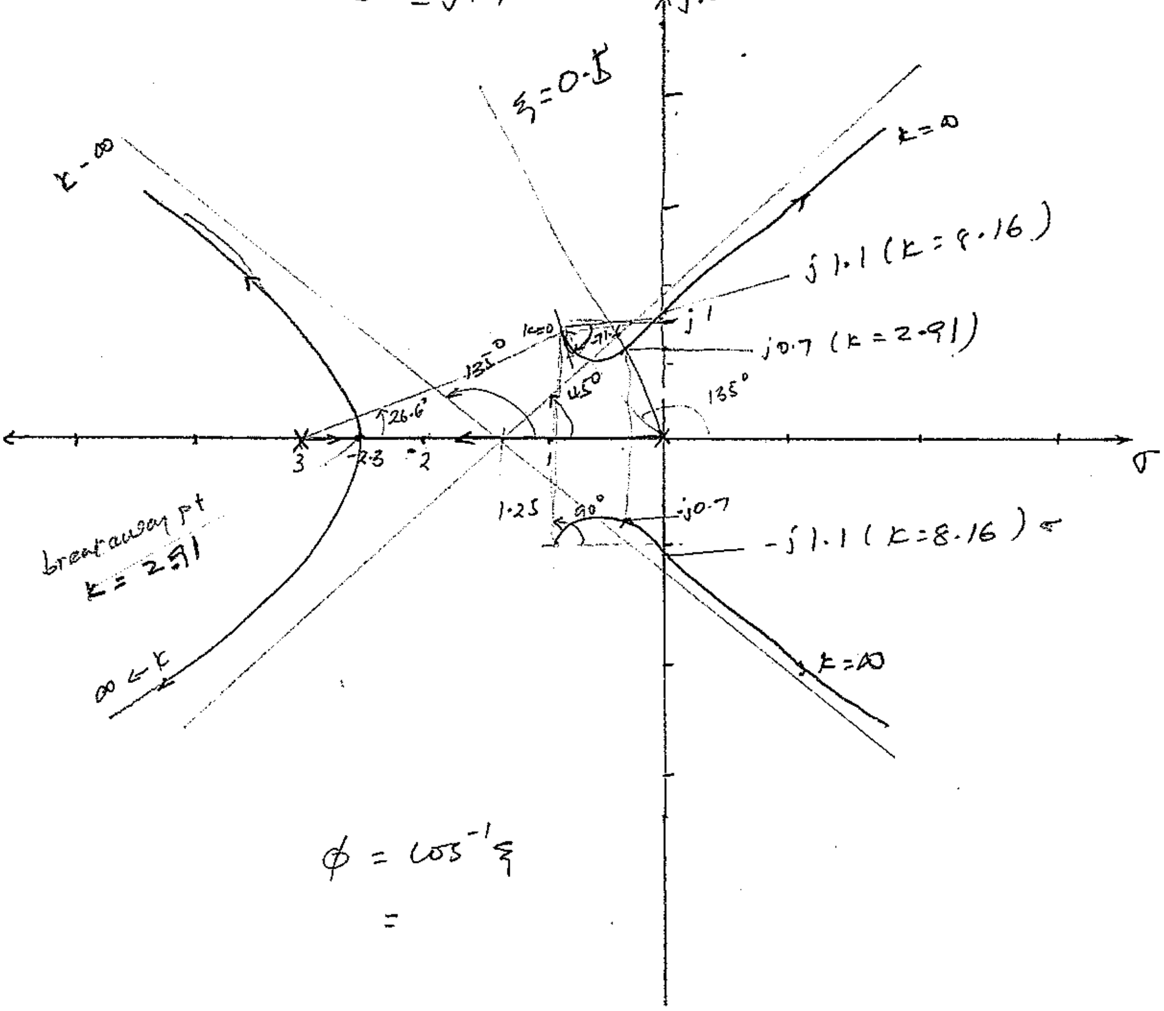
$$K \leq 8.16$$

The auxiliary equation

$$\frac{34}{5} s^2 + 8.16 = 0$$

$$s = \pm j1.1$$

~~The~~ Therefore the purely imaginary closed-loop poles of the system are located at $s = \pm j1.1$



$$\phi = \cos^{-1}\xi$$

$$=$$