Chapter Four.

Time domain Response Analysis of Control Systems.

May 3

4.1 Introduction

A manner in which a dynamic system responds to an input, expressed as a function of time, is called time response. The theoretical evaluation of this response is said to be undertaken in the time domain, and is referred to as time domain analysis.

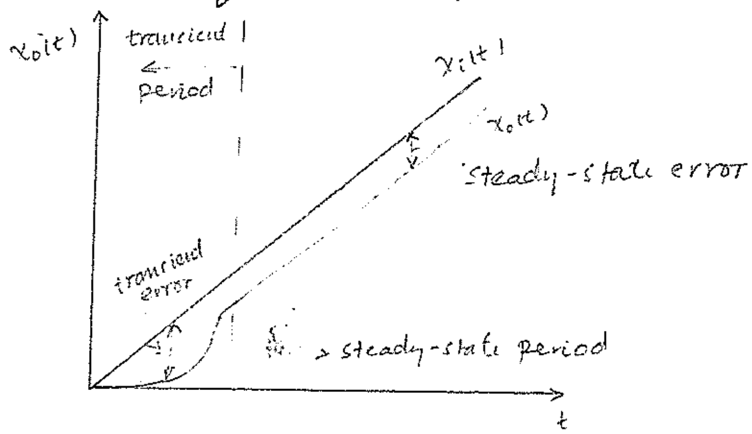
It is possible to compute the time response of a system if the following is known

- The nature of the input(s), expressed as a function of time.
- The mathematical model of the system.

Control systems are inherently dynamic, their performance is usually specified interms of both the transient-response and the steady-state response.

a) Transient response: this response will (for a stable system) decay, usually exponentially, to zero as time increases. It is a function only of the system dynamics, and is independent of the input quantity.

b) Steady-state response: this is the response of the system after the transient component has decayed and is a function of both the system dynamics and the input quantity.



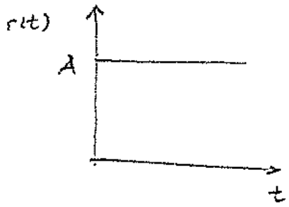
$x_o(t)$ - output
$x_i(t)$ - input / reference

Transient and steady state periods of time.

4.2 Standard Test Signals.

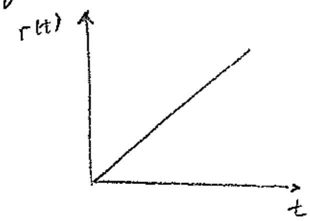
Usually the input signal to control system are not known fully a head of time. Therefore, system dynamics behaviour for analyses and design is judged and compared under application of standard test signals - an impulse, a step, a constant velocity (a ramp input) and a constant acceleration (a parabolic input). Another standard test signal is sinusoidal input.

a) Step signal.



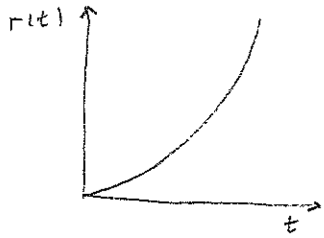
$$r(t) = \begin{cases} A, & t > 0 \\ 0, & t \le 0 \end{cases} \qquad R(s) = \frac{A}{s}$$

b) Ramp signal / velocity



$$r(t) = \begin{cases} At, & t > 0 \\ 0, & t \le 0 \end{cases} \qquad R(s) = \frac{A}{s^2}$$

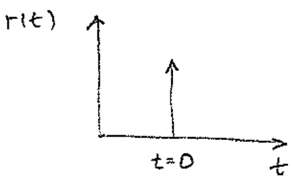
c) Parabolic Signal



$$r(t) = \begin{cases} \frac{At^2}{2}, & t > 0 \\ 0, & t \le 0 \end{cases} \qquad R(s) = \frac{A}{s^3}$$

d) A/unit Impulse signal

(A unit impulse is defined as a signal which has zero value everywhere except at $t=0$, where its magnitude is infinite.



$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\epsilon}^{+\epsilon} \delta(t)\,dt = 1$$

mathematically, an impulse is the derivative of a step function.

$$\mathcal{L}[\delta(t)] = 1 = R(s)$$

The impulse response of a system with transfer function $\frac{Y(s)}{R(s)} = G(s)$ is given by

$$Y(s) = G(s)R(s)$$
$$= G(s)$$

or $\quad y(t) = \mathcal{L}^{-1}[G(s)] = g(t)$

Thus the impulse response of a system, indicated by $g(t)$, is the inverse Laplace transform of its transfer function.

It should be noted that the ramp signal is the integral of of the step input, and the parabolic signal is simply the integral of the ramp input.
Hence one may relate the response to one test signal to another test signal using the general formula form

$$r(t) = t^n$$

$$\mathcal{L}\left[r(t)\right] = R(s) = \frac{n!}{s^{n+1}}$$

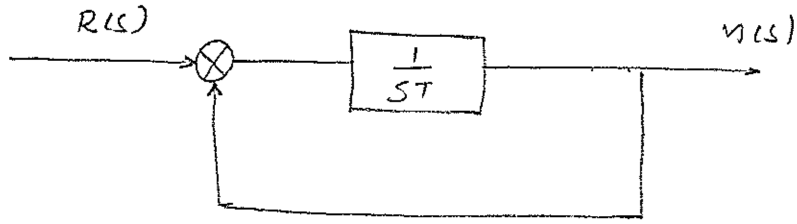
The step input signal is the easiest to generate and evaluate, and usually choosen for performance test.

The nature of the transient response is revealed by any of the test signal mentioned above as this nature is dependent upon system poles only and not on the type of input. Steady-state response is then examined with respect to this particular test signal as well as other test signals.

The time response performance of a control system is measured by computing several time response performance indices as well as steady state accuracy. These indices give quantitative method to compare the performance of alternative system configuration or to adjust the parameter of a given system.

4.3 Time Response of First-Order System.

Consider as a first order system shown in the figure, (which mathematically represents the pneumatic system whose dynamics is described by)



The transfer function is given by

$$T(s) = \frac{y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{1}{ST+1}$$

Analyze the system response for the following inputs

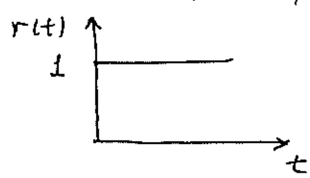
A) unit step input

B) unit-ramp input

C) unit-impulse

The initial conditions are assumed to be zero.

A) Unit-Step Response



$$r(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}$$

$$\mathcal{L} \Downarrow$$

$$R(s) = \frac{1}{S}$$

Transfer function $T(s) = \dfrac{y(s)}{R(s)}$

$$y(s) = R(s)\,T(s)$$

Substituting $R(s)$

$$Y(s) = \frac{1}{s(sT+1)}$$

Expanding $Y(s)$

$$Y(s) = \frac{A}{s} + \frac{B}{sT+1} ,$$

where $A = sY(s)\Big|_{s=0} = \frac{1}{sT+1}\Big|_{s=0}$

$= 1$

$B = (sT+1)\,Y(s)\Big|_{s=-\frac{1}{T}} = \frac{1}{s}\Big|_{s=-\frac{1}{T}}$

$= -T$

* Poles determine the nature of the time response. Poles of the input fn determines the form of the forced response, and poles of the transfer fn determine the form of the natural response

$$Y(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$R - 1$

$E = R - Y$

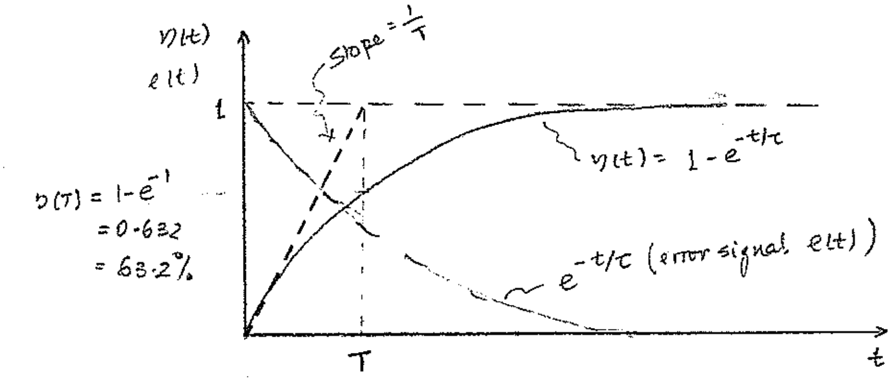
$Y = R - E$

Taking the inverse Laplace transform

$$Y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right]$$

$$= \left[1 - e^{-t/T}\right] u(t) \quad \text{for } t \geq 0$$

- A pole of the input fn generates the form of the forced response
- * A pole of the transfer fn generates the form of the natural response



Unit-step response of first order system.

The output would reach the final value at $t = T$ if it maintain its initial speed of response.

It is seen that the output rises exponentially from zero value to final value of unity.

The initial slope of the curve at $t=0$ is given by

$$\frac{dy}{dt}\Big|_{t=0} = \frac{1}{T} e^{-t/T}\Big|_{t=0} = \frac{1}{T}$$

$\Rightarrow$ The slope of the response curve $y(t)$ decreases monotonically from $\frac{1}{T}$ at $t=0$ to zero at $t=\infty$

where T is known as the time constant of the system.

The time constant is indicative of how fast the system tends to reach the final value. A large time constant corresponds to a sluggish system and a small time constant corresponds to a fast response.

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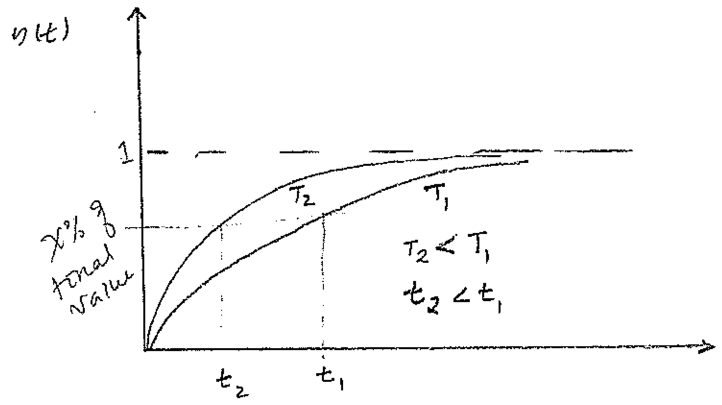


The response of the system reach in
$2T = 86.5\%$
$3T = 95\%$
$4T = 98.2\%$
$5T = 99.3\%$

for $t \geq 4T$, the response remains with in 2% of the final value.

A large time constant corresponds to sluggish system and a small time constant corresponds to a fast response as shown.

Consider, the error respons of the system which is given by

$$e(t) = r(t) - y(t)$$
$$= 1 - (1 - e^{-t/\tau})$$
$$= e^{-t/T}$$

The steady state error is given by

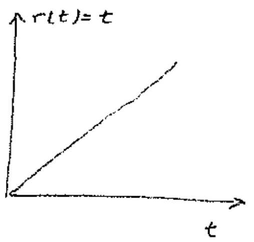
$$e_{ss} = \lim_{t \to \infty} e(t) = 0$$

Thus the system tracks the unit step input with zero steady-state error.

B) Unit - Ramp input Respons.

Transfer function $T(s) = \frac{1}{sT+1}$

Input



$$r(t) = \begin{cases} t, & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$$R(s) = \frac{1}{s^2}$$

$$Y(s) = R(s)\, T(s)$$
$$= \frac{1}{s^2(sT+1)}$$

Expanding $Y(s)$ into partial fractions

$$Y(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$A = s^2 Y(s)\Big|_{s=0} = \frac{1}{sT+1} = F(s)$$
$$= \frac{1}{sT+1}\Big|_{s=0} = 1$$

$$B = \frac{\partial}{\partial s} F(s)\Big|_{s=0} = \frac{-T}{(sT+1)^2}\Big|_{s=0} = -T$$

$$B = \frac{d}{ds}\left[s^2 Y(s)\right]$$
$$\frac{d}{ds}\left[\frac{1}{sT+1}\right]$$

$$c = (sT+1)\,Y(s)\Big|_{s=-\frac{1}{T}} = \frac{1}{s^2}\Big|_{s=-\frac{1}{T}} = T^2$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1}$$

$$= \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Taking the inverse laplace transform

$$y(t) = \left[t - T + Te^{-t/T}\right] \qquad t \geq 0$$

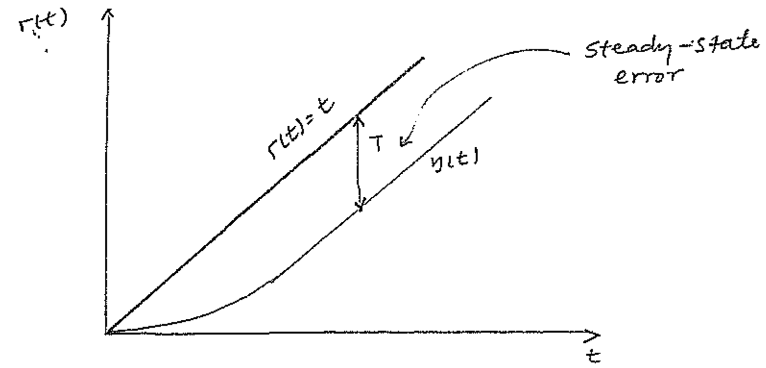
The error signal is

$$e(t) = r(t) - y(t)$$
$$= t - (t - T + Te^{-t/T})$$
$$= T(1 - e^{-t/T})$$

and the steady state error is given by

$$e_{ss} = \lim_{t\to\infty} e(t) = T$$

Thus the first-order system under consideration will track the unit-ramp input with steady-state error T, which is equal to the time constant of the system.



unit-ramp response of a first-order system.

* Reducing the system time constant (T) therefore not only improves its speed of response but also reduces its steady-state error to a ramp input.

If we consider the derivative of $y(t)$

$$\dot{y}(t) = 1 - e^{-t/T}$$

we find that it is identical to the system response to the unit-step input. The transient response to the ramp input signal thus yields no additional information about the speed of response of the system. We therefore need only examine the steady state error to the ramp input which can be obtained directly from the final value theorem

e

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$
$$= \lim_{s\to 0} s\left[R(s) - Y(s)\right]$$
$$= \lim_{s\to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(sT+1)}\right] \quad \text{check}$$
$$= T$$

This avoids the need to obtain the inverse laplace transform.

c) Unit - impulse Response.

$$\delta(t-t_0) \xrightarrow{\mathcal{L}} R(s) = 1$$

The transfer f^n of the system

$$T(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ts+1}$$

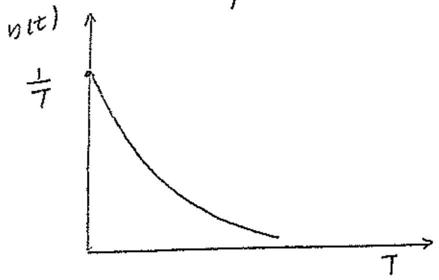
$$Y(s) = R(s)\,T(s) \qquad \text{where } R(s) = 1$$

$$Y(s) = \frac{1}{Ts+1} = \frac{1}{T(s+\frac{1}{T})} = \frac{\frac{1}{T}}{s+\frac{1}{T}}$$

the inverse laplace transform

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \mathcal{L}^{-1}\left[\frac{\frac{1}{T}}{s+\frac{1}{T}}\right]$$
$$= \frac{1}{T} e^{-t/T} \quad \text{for } t \ge 0$$

The response curve



An important Property of linear time - invariant system.

In the analysis above.

1- Unit-ramp input ~~input~~, the output $y_R(t)$ is given by

$$y_R(t) = t - T + Te^{-t/T} \quad \text{for } t \ge 0$$

2- Unit- step input, the output $y_s(t)$ is the derivative the unit ramp input.

$$Y_s(t) = \frac{\partial}{\partial t} Y_R(t) = 1 - e^{-t/T} \quad \text{for } t \ge 0$$

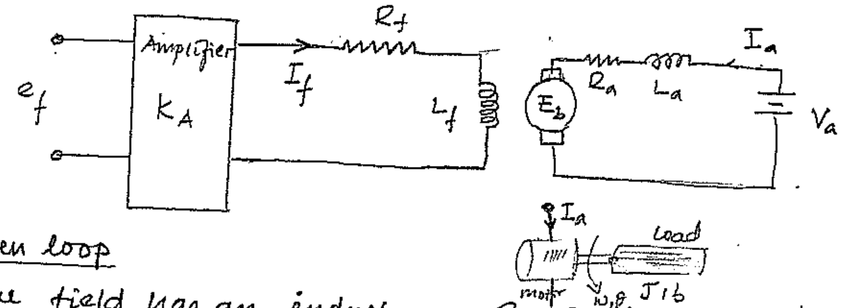
3- Unit - impulse input, the output $Y_I(t)$ is the derivative of the step input

$$Y_I(t) = \frac{\partial}{\partial t} Y_s(t) = \frac{1}{T} e^{-t/T}, \text{ for } t \ge 0$$

comparison of the system respons to these three inputs clearly indicates that the respons to the derivatives of an input signal can be obtained by differentiating the respons of the system to the origional signal. It can also be seen that the response to the integral of the origional signal can be obtained by integrating the response of the system to the origional signal and by determing the integration constand from the zero output intial cond.

Example.

Consider a control system used in steel mills for rolling the steel sheets and moving the steel through the mill using DC servo motors. The Field of a DC servomotor is separately exited by means of a DC amplifier of gain $K_A = 90$, as shown below.



Constant V_a
Variable field current

Open loop

a) The field has an inductance of 2-Henry and a resistance of 50-ohm

i) Determine the field time constant

b) closed loop.

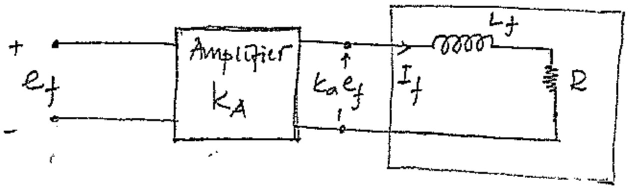
A closed-loop speed control is easily obtained by utilizing a Tachometer to generate a Voltage (e_a) proportional to the speed (the ~~speed~~ field current)

ii) Draw the block diagram ~~diagram~~ the negative-feedback control system.

iii) Determine the value of the feedback ~~control~~ constant K to reduce the field time constant to 4m second.

iii) Draw a practical transistor Amplifier circuit for accomplishing this feedback in low power application

a) Open-loop speed control system.



The voltage equation governing the field current.

$$K_A e_f = L_f \frac{dI_f}{dt} + R_f I_f$$

Taking the Laplace transform, assuming zero initial conditions, we have

$$K_A E_f(s) = s L_f I_f(s) + R_f I_f(s)$$

$$= [sL_f + R_f]\, I_f(s)$$

$$I_f(s) = \frac{K_A E_f(s)}{sL_f + R_f} = \frac{K_A}{L_f} \cdot \frac{E_f(s)}{\left(s + \frac{R_f}{L_f}\right)}$$

Let $K = \frac{K_A}{L_f}$. $\alpha = \frac{R_f}{L_f}$

$$I_f(s) = \frac{K E_f(s)}{s+\alpha}$$

The open loop transient respons of field current is

$$\mathcal{L}^{-1}[I_f(s)] = \mathcal{L}^{-1}\left[K \frac{E_f(s)}{s+\alpha}\right]$$

$$= K e_f(t) e^{-\alpha t}$$

$$= \frac{K_A}{L_f} e_f(t) e^{-\frac{R_f t}{L_f}}$$

The field-circuit time constant τ is given as

$$\tau = \frac{1}{\alpha} = \frac{L_f}{R_f} = \frac{2}{50}$$

$$\tau = 0.04\,\text{sec}$$

Given $L_f = 2H$
$R_f = 50\Omega$
$K_A = 90$

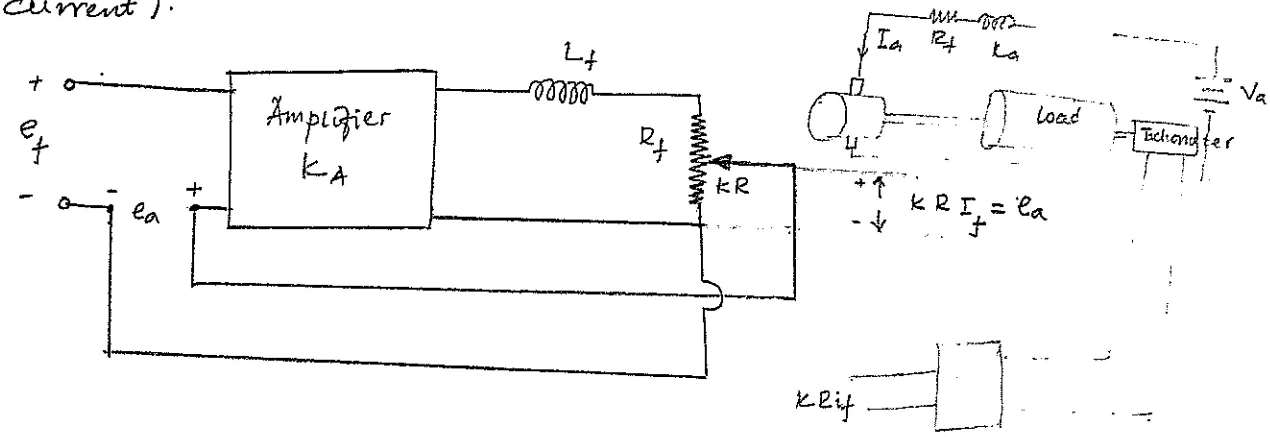
$$i_f(t) = \frac{K_A}{L_f} e_f(t) e^{-R_f/L_f\, t}$$

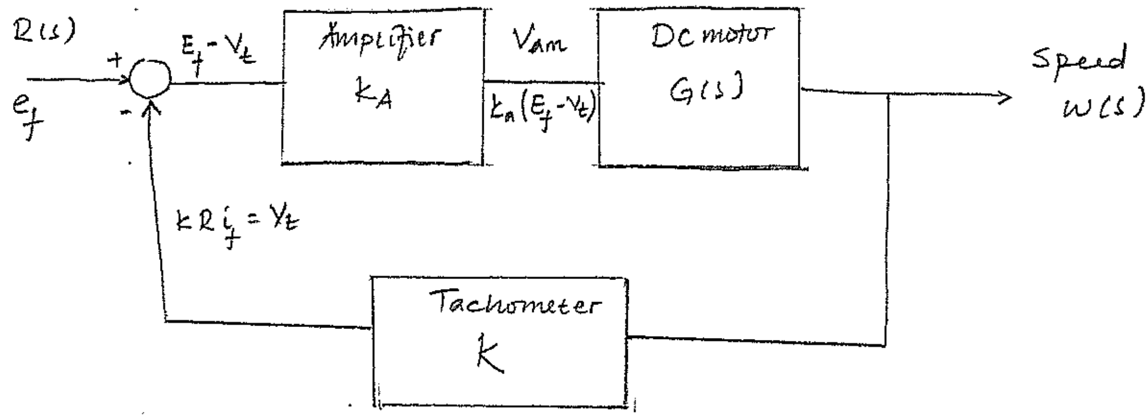
$$= \frac{90}{2} e_f(t) e^{-\frac{50}{2} t}$$

$$i_f(t) = 45\, e_f(t) e^{-25t} \text{ Amper.}$$

b) Closed loop control system.

Tachometer (speed) generates a voltage e_a proportional to the speed (the field current).





Input $R(s) = E_f(s)$

Tachometer Voltage $= V_t(s) = KR_fI_f(s)$

Error-Voltage $= E_a(s) = E_f(s) - V_t(s)$

$$= E_f(s) - KR_fI_f(s)$$

Output of amplifier and input for the servo motor

$$V_a(s) = K_A(E_f(s) - V_t(s))$$

$$= K_A[E_f - KRI_f]$$

The voltage eqn governing DC servo motor

$$V_a(s) = (sL_f + R_f)I_f \qquad V_a(t) = L\frac{di_f}{dt} + Ri_f$$

$$K_AE_f - K_AKR_fI_f = (sL_f + R_f)I_f$$

$$I_f(s) = \frac{K_A}{L}\left[\frac{E_f}{s + \left[\frac{R_f}{L_f} + \frac{K_AKR}{L_f}\right]}\right]$$

The transient respons

$$i_f(t) = \mathcal{L}^{-1}[I_f(s)]$$

$$i_f(t) = \frac{k_A}{L_f} e_f(t)\, e^{-\frac{(R_f + k_A k R_f)}{L_f} t}$$

The closed loop time constant

$$\tau = \frac{L_f}{R_f + k_A k R_f} = \frac{L_f}{R_f}\left(\frac{1}{1+k_A k}\right)$$

substituting $L = 2H$ $R = 50$ $k_A = 90$

$$\tau = \frac{2}{50}\left[\frac{1}{1+90k}\right]$$

To reduce the field time constant, the value of k

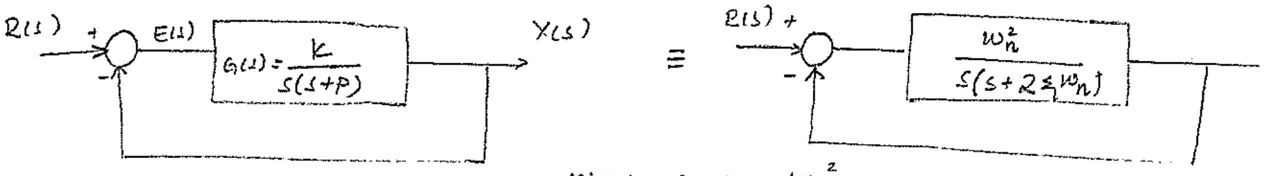
$$\tau = 4\times10^{-3} = \frac{2}{50}\left[\frac{1}{1+90k}\right]$$

$$k = 0.1$$

iii)

4.4 Performance of a Second Order System.

Consider a second order system block diagram



where $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

And ω_n = undamped natural frequency
ξ = damping factor (ratio)

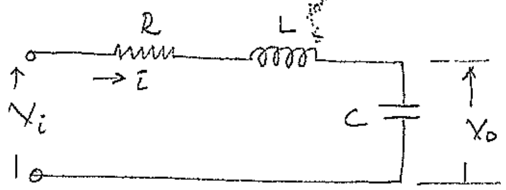
The closed-loop transfer function is given by

$$T(s) = \frac{G}{1+G} = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

It should be noted that the closed-loop transfer function of a second-order system can ~~be~~ always be represented in the general form given above. The dynamic behaviour of the system can be described in terms of the two parameters ω_n and ξ.

Example.

Consider a series RLC circuit shown below



a) Determine the dynamics of the system in OD's

b) Obtain the ~~transfer function~~ $\frac{V_o}{V_i}$ Laplace transform of a)

c) Draw ~~a unity~~ the block diagram of series RLC ckt using unity feedback.

d) Determine the closed loop transfer fn

e) Determine the undamped natural frequency (ω_n) and damping ratio (ξ)

finally f) Determine the unit-step response of a series RLC system.

sol/n.

a) $V_i = Ri + L\frac{di}{dt} + V_o$ ~~V_o~~

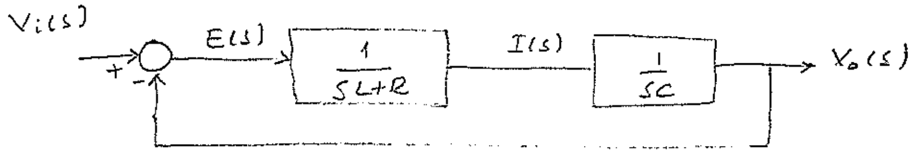
$V_o = \frac{1}{C}\int i\,dt$

b) The Laplace transform

$E(s) = V_i(s) - V_o(s) = (sL + R)\,I(s)$

$V_o(s) = \frac{1}{sC} I(s)$

c) The block diagram representation



d) The ~~Block~~ closed-loop transfer function can be obtained as

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{G}{1+G} = \frac{\frac{1}{sC(sL+R)}}{1+\frac{1}{sC(sL+R)}} = \frac{1}{s^2LC+sRC+1}$$

$$= \frac{1/LC}{s^2+s\left(\frac{R}{L}\right)+\frac{1}{LC}} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

e) By comparing the two transfer function, the natural frequency and damping ratio

$$\omega_n^2 = \frac{1}{LC} \Rightarrow \omega_n = \frac{1}{\sqrt{LC}}$$

and

$$2\zeta\omega_n = \frac{R}{L}$$

$$2\left(\frac{1}{\sqrt{LC}}\right)\zeta = \frac{R}{L}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

f) The unit-step Response of a series RLC

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \quad V_i(s) = \frac{1}{s}$$

$$V_o = V_i(s)\,T(s)$$

$$= \frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$
= called damped natural frequency.

$$V_o(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}\left[(s+\zeta\omega_n)^2+\omega_d^2\right]}$$

The inverse Laplace transform

$$v_o(t) = \mathcal{L}^{-1}[V_o(s)]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_d}{\sqrt{1-\xi^2}\,[(s+\xi\omega_n)^2+\omega_d^2]}\right]$$

$\omega_n = (-\xi\omega_n) + j\omega_d$

$|\omega_n|^2 = (\xi\omega_n)^2 + \omega_d^2$

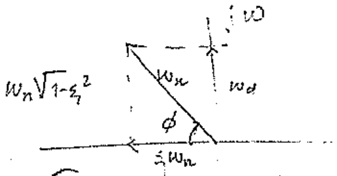
$\omega_d^2 = \omega_n^2 - (\xi\omega_n)^2$

$= \omega_n^2[1-\xi^2]$

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}}\,e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos(\omega_d t) + \xi\sin(\omega_d t)\right]$$



$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\phi\cos(\omega_d t) + \cos\phi\sin(\omega_d t)\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi) \quad \text{for } t \geq 0$$

$\sin\phi = \dfrac{\omega_n\sqrt{1-\xi^2}}{\omega_n}$

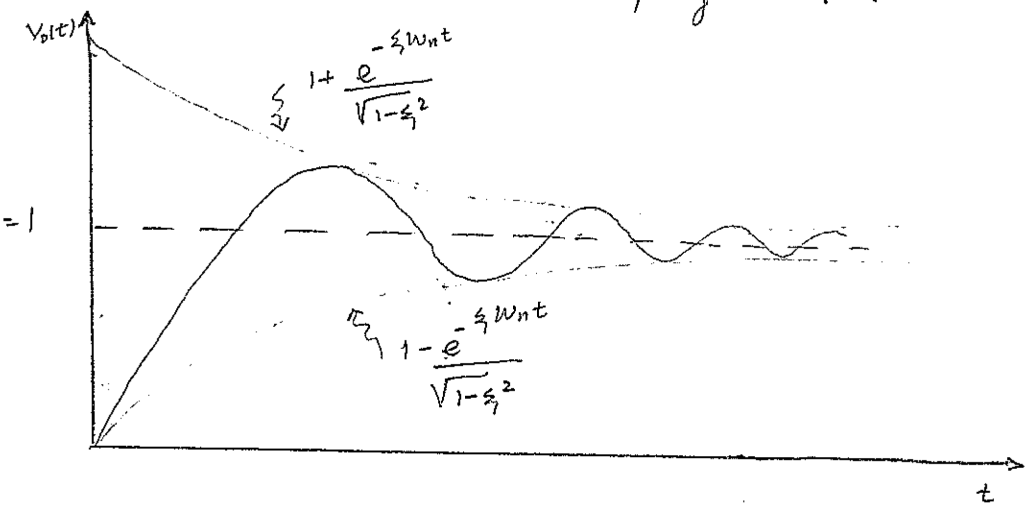
$= \sqrt{1-\xi^2}$

$\cos\phi = \xi\dfrac{\omega_n}{\omega_n} = \xi$

$\phi = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$

From the above eqs, it can be seen that the frequency of transient oscillation is the damped natural frequency (ω_d) and thus varies with the damping ratio ξ ($\omega_d = \omega_n\sqrt{1-\xi^2}$)

Step input $V_o(t) = 1$



The error signal for this system is given by

$e(t) = r(t) - V_o(t)$ where $r(t)$ - unit step input $= 1$

$$= 1 - \left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right]$$

$$= \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi), \quad t \geq 0$$

At steady state $e_{ss} = \lim_{t\to\infty} e(t) = 0$

$V_o = y_{ss} = \lim_{t\to\infty} y(t) = 1$

Roots of Characteristics eqn and their relationship ~~with~~ to damping in second order ~~eqn~~ system.

The time response of any system is characterized by the roots of the denominator polynomial, which in fact are the poles of the transfer function. The denominator polynomial $q(s)$ is therefore called the Characteristic polynomial and

$$\left[\; T(s) = \frac{X(s)}{R(s)} = \frac{p(s)}{q(s)} \; \right]$$

$q(s) = 0$ is called the characteristics eqn.

The Characteristics eqn of the second order system

$$T(s) = \frac{\omega_n^2}{s^2 + 2\omega_n \xi s + \omega_n^2} =$$

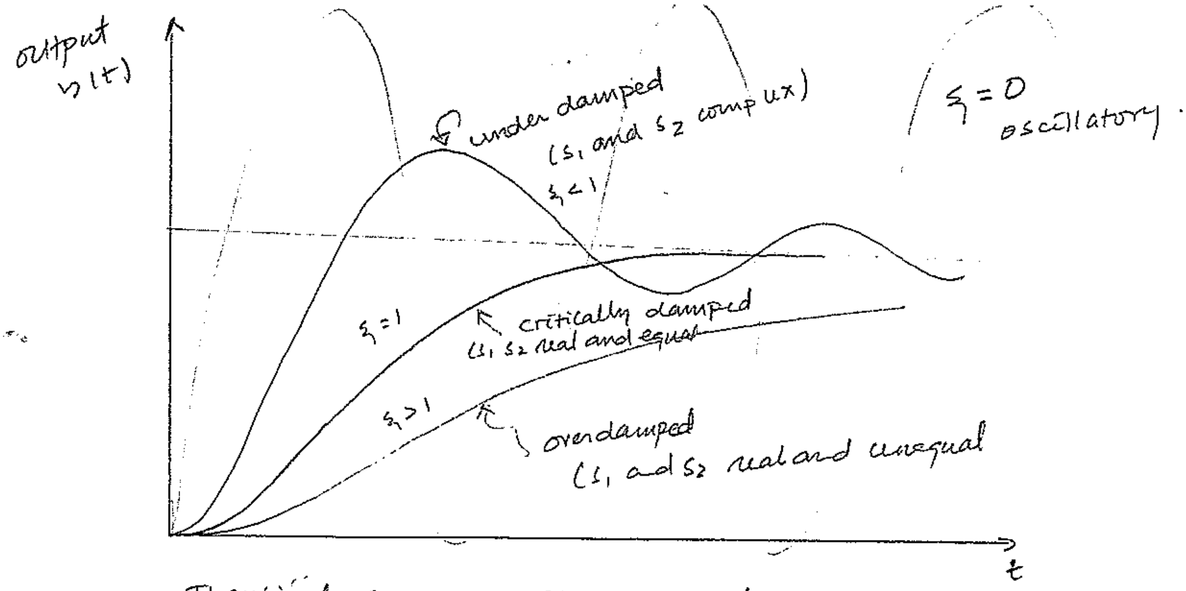
is $\quad s^2 + 2\omega_n \xi s + \omega_n^2 = 0$

The roots of the Characteristic eqn is given by

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s - s_1)(s - s_2)$$

for

Discriminant $b^2 - 4ac$ $= 4\omega_n^2(\xi^2 - 1)$ ($4\xi^2\omega_n^2 - 4(1)\omega_n^2$)	Roots	Transient Response type.
$\xi^2 > 1$	s_1 and s_2 real and unequal ($-ve$)	Overdamped transient response
$\xi^2 = 1$	s_1 and s_2 real and equal ($-ve$)	Critically damped transient Response
$\xi^2 < 1$	s_1 and s_2 complex conjugate of the form $s_1, s_2 = -\sigma \pm j\omega_n$	underdamped transient response.
$\xi = 0$	roots $s_1, s_2 = \pm j\omega_n$ imaginary	undamped. Oscillatory transient response does not die out.
$\xi > 1$	negative damping	

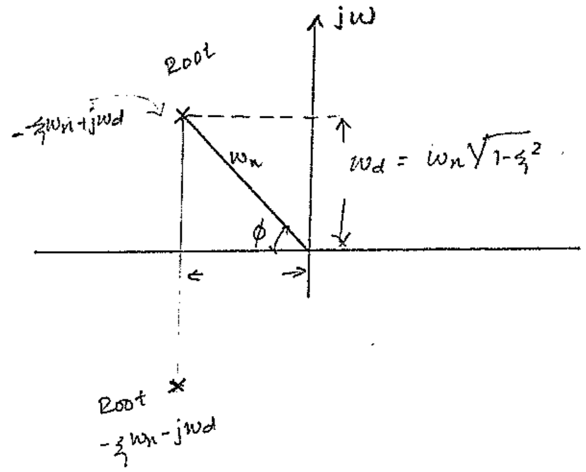


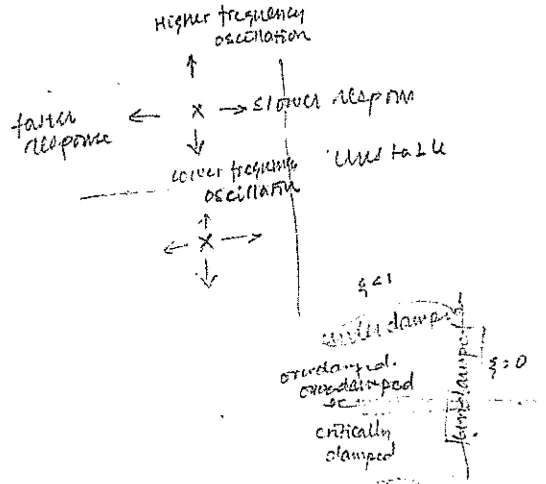
Transient response of a second order system.

(unit-step response of a second order system for various ξ values)

Relationship between Second-order system roots. ξ, ω_n and ω_d.

$\omega_n = -\xi\omega_n \pm j\omega_d$



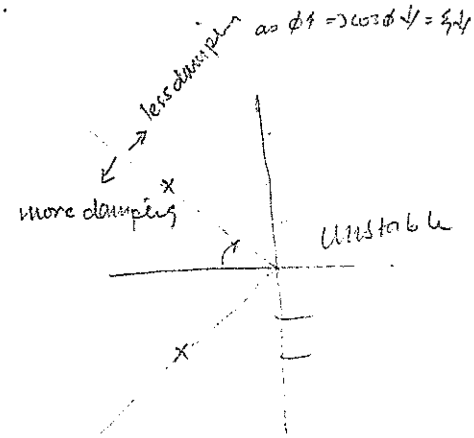


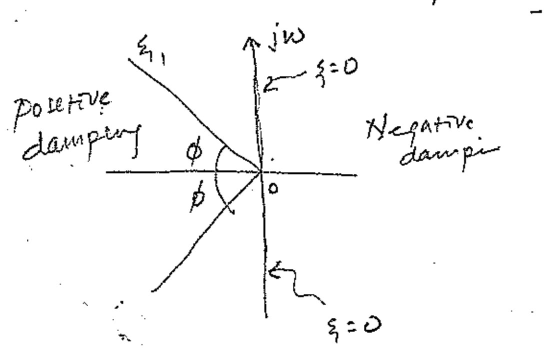
1- ω_n - radial distance from the roots to the origin of the s-plane
- called Natural undamped frequency.

2. α the real parts of the root
$= -\xi\omega_n$

3 - ω_d - the imaginary part of the roots
- called damped frequency
$= \omega_n\sqrt{1-\xi^2}$

4 - ξ - damping ratio
$= \cos\phi = \frac{\xi\omega_n}{\omega_n} = \xi$





4.1. The left-half s-plane corresponds to positive damping (i.e. the damping ratio is +ve).

$\Rightarrow$ positive damping causes the unit-step response to settle to a constant final value in the steady state due to the negative component exponent ($e^{-\xi\omega_n t}$)

$\hookrightarrow$ The system is stable.

4.2 The right-half s-plane corresponds to negative damping
Negative damping gives a response that grows in magnitude without bound with time
↳ The system is unstable.

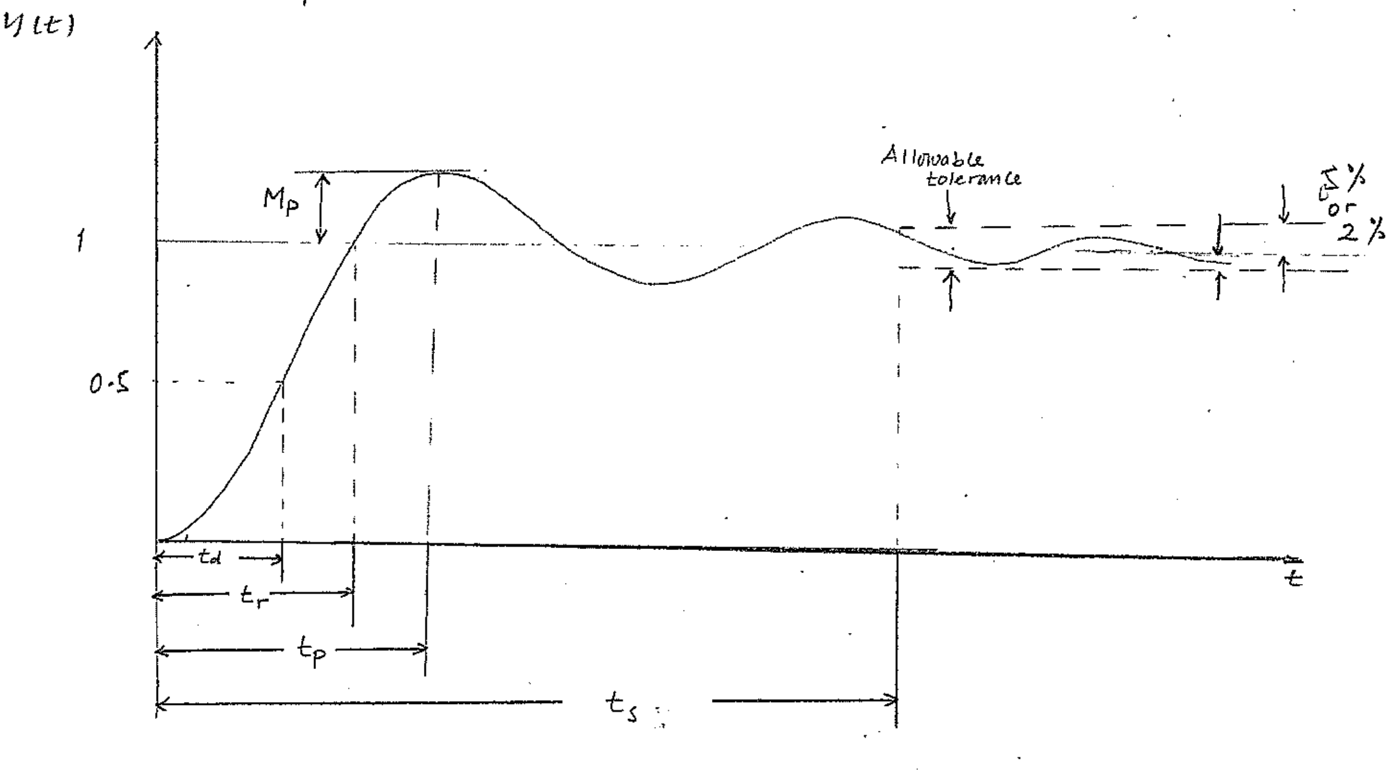
4.3 The imaginary axis corresponds to zero damping ($\xi = 0$)
Zero damping results in a sustained oscillation response, and the system is marginally stable or marginally unstable.

4.5 Transient-Response Specification

The desired performance characteristics of control systems are usually defined in terms of the unit-step response of the system.
The transient response of a system often exhibits damped oscillations before reaching steady state.

In specifying the transient-response characteristics of a control system to a unit step input, it is common to specify the following:

1- Delay time, t_d
2- Rise time, t_r
3- Peak time, t_p
4- Maximum overshoot, M_p
5- settling time, t_s



Unit-step response

1. Delay time (t_d) - it is the time required for the response to reach half of the final value for the very first time.

2. Rise time (t_r) :- it is the time required for the response to rise from 0% to 100% its final value. — For under damped
[10% to 90% of its final value — For overdamped syste

we obtain t_r by setting $V(t_r) = 1$

$$V(t_r) = 1 - \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \phi) = 1$$

$$\Rightarrow \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \phi) = 0$$

$$b/s \quad \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \neq 0$$

we get $\sin[\omega_d t_r + \phi] = 0$ since $\sin \pi = 0$

$$\sin(\omega_d t_r + \phi) = \sin(\pi)$$

$$\omega_d t_r + \phi = \pi$$

$$t_r = \frac{\pi - \phi}{\omega_d} \quad \text{where } \omega_d = \omega_n \sqrt{1-\xi^2}$$

The rise time is given by

$$t_r = \frac{\pi - \phi}{\omega_n \sqrt{1-\xi^2}} \quad \text{and } \cos\phi = \xi$$

3. Peak time (t_p) - it is the time required for the response to reach the first peak of the overshoot.

we may obtain the peak time by differentiating $V_o(t)$ w.r.t time and letting this derivative equal to zero.

$$V_o(t) = 1 - \frac{e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \phi)$$

$$\frac{dV_o(t)}{dt} = \frac{\xi \omega_n}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t_p} \sin(\omega_d t_p + \phi) - \frac{\omega_d}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t_p} \cos(\omega_d t + \phi)$$

* (substituting $\omega_d = \omega_n \sqrt{1-\xi^2}$)

$$= \frac{\xi w_n}{\sqrt{1-\xi^2}} e^{-\xi w_n t_p} \sin(w_d t_p + \phi) - \frac{w_n\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} e^{-\xi w_n t_p} \cos(w_d t_p + \phi) = 0$$

$$\Rightarrow \frac{w_n}{\sqrt{1-\xi^2}} e^{-\xi w_n t_p} \left[\xi \sin(w_d t_p + \phi) - \sqrt{1-\xi^2} \cos(w_d t_p + \phi) \right] = 0$$

$$\Rightarrow \xi \sin(w_d t + \phi) - \sqrt{1-\xi^2} \cos(w_d t_p + \phi) = 0$$

substituting $\xi = \cos\phi$ and $\sqrt{1-\xi^2} = \sin\phi$ we obtain

$$\cos\underset{\beta}{\phi} \sin(\underbrace{w_d t_p + \phi}_{\alpha}) - \sin\phi \cos(w_d t_p + \phi) = 0$$

From trignometry

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

$$\cos\phi \sin(w_d t_p + \phi) - \sin\phi \cos(w_d t_p + \phi) = \sin(w_d t_p + \phi - \phi) = 0$$

$$\Rightarrow \sin(w_d t_p) = 0 = \sin(n\pi) \quad \text{where } n = 0, 1, 2, \ldots$$

$$w_d t_p = n\pi$$

since the peak-time t_p corresponds to the first overshoot $n=1$

$$w_d t_p = \pi$$

$$t_p = \frac{\pi}{w_d}; \qquad w_d = w_n\sqrt{1-\xi^2}$$

$$t_p = \frac{\pi}{w_n\sqrt{1-\xi^2}}$$

4. Maximum Overshoot (M_p) : It is the maximum peak value of the response curve measured from unity.

it is defined as:

$$\text{Maximum Percent overshoot} = \frac{V_o(t) - V_o(\infty)}{V_o(\infty)} \times 100$$

The maximum overshoot directly indicates the relative stability of the system.

The maximum overshoot occur at peak time t_p

$$t = t_p = \frac{\pi}{w_n\sqrt{1-\xi^2}}$$

Maximum overshoot

$$M_p = V_o(t_p) - 1$$

$$= 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \phi) - 1$$

$$= -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin\left[\omega_d t_p + \phi\right]$$

substituting $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$M_p = -e^{-\xi\omega_n\pi/\omega_n\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\cdot\frac{\pi}{\omega_n\sqrt{1-\xi^2}} + \phi\right]$$

$$= \frac{-e^{-\xi\pi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \sin(\pi+\phi)$$

$$= \frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \sin\phi \qquad \text{where } \sin\phi = \sqrt{1-\xi^2}$$

$$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

The max. percent overshoot given by

$$\% M_p = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

check

$$\xi = \frac{-\ln\left(\frac{M_p\%}{100}\right)}{\sqrt{\pi^2 + \left[-\ln\left(\frac{M_p\%}{100}\right)\right]^2}}$$

5. Settling-time (t_s) - it is the time required for the respons curve to reach and stay within a certain percentage of the final value (usually 2% or 5%).

The settling time is related to the largest time constant of the control system.

Considering only the exponentially decaying envelop of the response of the underdamped system, for 2% tolerance

$$e^{-\xi\omega_n t_s} = 0.02$$

$$\ln(e^{-\xi\omega_n t_s}) = \ln(0.02)$$

$$-\xi\omega_n t_s = -4$$

$$t_s = \frac{4}{\xi\omega_n} = 4\tau$$

Similarly for the 5% tolerance band

$$e^{-\xi \omega_n t_s} = 0.05$$

$$t_s = \frac{3}{\xi \omega_n} = 3\tau$$

we define the time constant $\tau = \frac{1}{\xi \omega_n}$ of the dominant roots of the characteristics equation.

Generally the transient response of the system may be described interms of two factors

transient 1 - The swiftness of respons, as represented by the rise-time and the peak time.

steady state 2. The ~~less~~ closness of the response to the desired response as represented by the overshoot and settling time.

Note: For a desired transient response of a second order system, the damping ratio (ξ) must be b/n 0.4 and 0.8

- small values of ξ ($\xi < 0.4$) yield exessive overshoot in transient response
- large value of ξ ($\xi > 0.8$), the system becomes sluggish.

Settling time primarily determined by the undamped natural frequency ω_n.

Example.

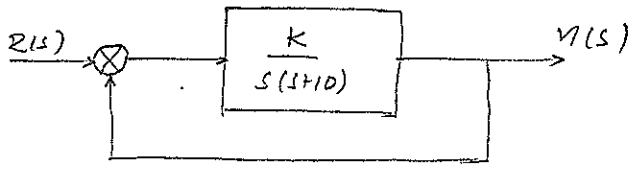
A unity feedback control system is characterized by an open-loop transfer function

$$G(s) = \frac{k}{s(s+10)}.$$

a) Determine the gain k so that the system will have a damping ratio of 0.5

b) Determine the settling time, peak overshoot and time to peak overshoot for a unit step input.

Soln



The closed loop transfer function of the system is given by

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad \text{where } G(s) = \frac{k}{s(s+10)} \qquad H(s)=1$$

$$T(s) = \frac{\frac{k}{s(s+10)}}{1+\frac{k}{s(s+10)}} = \frac{k}{s^2+10s+k}$$

The closed-loop transfer function of a 2nd order closed loop transfer function is given by

$$T(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Comparing

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{k}{s^2+10s+k}$$

1) $k = \omega_n^2 \Rightarrow \omega_n = \sqrt{k}$

2) $10 = 2\xi\omega_n$

Given $\xi = 0.5$

$$\omega_n = \frac{10}{2\xi} = \frac{10}{1} = 10$$

a) $\omega_n^2 = k$

$k = 10^2 = 100$

b) The settling time is given by

i) $t_s = \dfrac{4}{\xi\omega_n}$ for 2% tolerance.

$$t_s = \frac{4}{0.5*10} = 0.8\text{ sec.}$$

ii) The peak-overshoot (M_p) is given by

$$M_p = 100\, e^{-\pi \xi / \sqrt{1-\xi^2}}$$

$$= 100\, e^{\frac{-\pi * 0.5}{\sqrt{1-0.5^2}}}$$

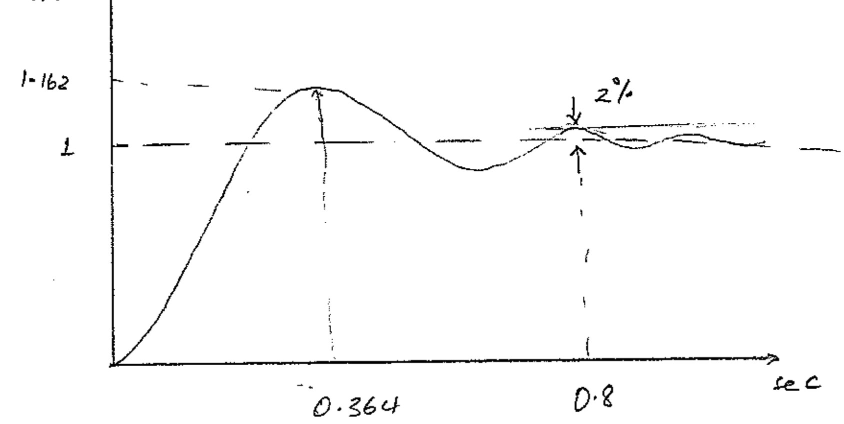
$$= 16.2\%$$

iii) Time to peak-overshoot (peak time) t_p

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$= \frac{\pi}{10\sqrt{1-0.5^2}} = \frac{\pi}{10\sqrt{0.75}}$$

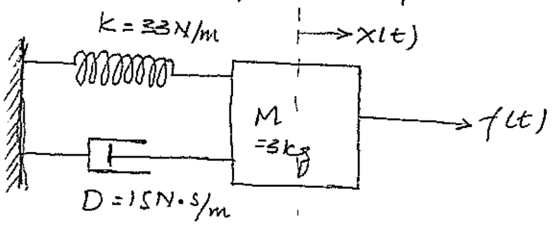
$$= 0.364 \text{ sec}$$



Example. For the system shown in the figure

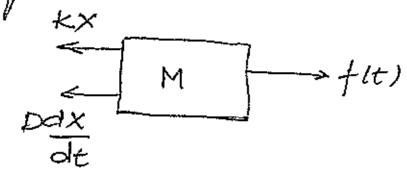
a) find the transfer function $G(x) = \frac{X(s)}{F(s)}$

b) find ξ, ω_n, $\%M_p$, t_s, t_p, and t_r



sol:

Free body Diagram



$$f(t) - Kx - D\frac{dx}{dt} = ma$$

$$f(t) = m\frac{d^2x}{dt^2} + Kx + D\frac{dx}{dt}$$

$$F(s) = s^2 m X(s) + sD X(s) + K X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 m + sD + K}$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 m + sD + K}$$

$$= \frac{\frac{1}{m}}{s^2 + s\frac{D}{m} + \frac{K}{m}}$$

$$= \frac{\frac{1}{3}}{s^2 + 5s + 11} = \frac{\omega_n^2}{s^2 + 2\omega_n \xi s + \omega_n^2}$$

b) $\omega_n^2 = 11 \Rightarrow \omega_n = 3.32$

$2\xi\omega_n = 5 \Rightarrow \xi = \frac{5}{2\omega_n} = 0.754$

$t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.754 * 3.32} = 1.6\ \text{sec}$

C $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1.44\text{sec}$

$\% M_p = 100\ e^{-\xi\pi/\sqrt{1-\xi^2}} = 2.7\%$

$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\xi^2}}$ where $\cos\phi = \xi$

$\phi = \cos^{-1}(\xi)$ in rad $= 0.716$

$t_r = \frac{\pi - 0.716}{3.32\sqrt{1-(0.754)^2}}$

? $= 1.105$ sec

C

Design

Ex. Given the translational mechanical system of figure the above example where $K = 1$ and $f(t)$ is a unit step, find the values of M and D to yield a response with 30% overshoot and settling time of 10 sec

sol/n: $T(s) = \frac{X(s)}{F(s)} = \frac{1/M}{s^2 + \frac{D}{M}s + \frac{1}{M}}$

since $t_s = 10 = \frac{4}{\xi\omega_n}$, $\xi\omega_n = 0.4$

But $\frac{D}{M} = 2\xi\omega_n = 2(0.4) = 0.8$

Also

$\% M_p = 100\ e^{-\xi\pi/\sqrt{1-\xi^2}} \Rightarrow \xi = \frac{-\ln\frac{M_p\%}{100}}{\sqrt{\pi^2 + (-\ln\frac{M_p\%}{100})^2}} = 0.358$

Hence $\omega_n = 1.117$

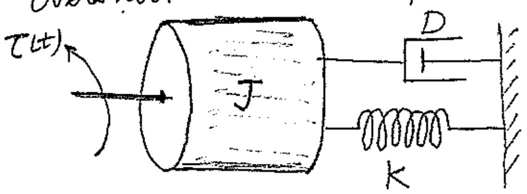
now $\frac{1}{M} = \omega_n^2 = 1.248$

$M = 0.801$

and since $\frac{D}{M} = 2\zeta\omega_n = 0.8$

$D = 0.641$

Ex. Find J and k in Rotational system shown below, to yield a 30% overshoot and settling time of 4 sec for step input in torque.



sol^n $\quad T(s) = (Js^2 + s + k)\,\theta(s)$

$$\frac{\theta(s)}{T(s)} = \frac{1/J}{s^2 + \frac{1}{J}s + \frac{k}{J}}$$

$$\zeta = \frac{-\ln\left(\frac{Mp\%}{100}\right)}{\sqrt{\pi^2 + \left(-\ln\frac{Mp\%}{100}\right)^2}} = 0.358$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\frac{1}{2J}} = 8J = 4$$

Therefore $J = \frac{1}{2}$ Also $t_s = 4 = \frac{4}{\zeta\omega_n}$

$$= \frac{4}{(0.358)(\omega_n)}$$

Hence $\omega_n = 2.793$

Now $\frac{k}{J} = \omega_n^2 = 7.803$

Finally $k = 3.901$

4.6 Steady State error.

One of the fundamental reason for using feedback, despite its cost and increased complexity, is the attendant improvement in the reduction of the steady-state error of the system.

i.e the system output response follows the reference input signal accurately in steady state.

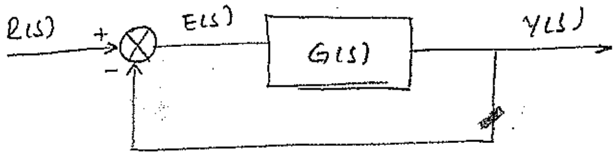
The steady state error is a measure of system accuracy.

The difference b/n the output and the reference input in the steady state defined as steady-state-Error e_{ss}

Steady state error in controll system are almost unavoidable. In a design one of the objectives is to keep e_{ss} to a minimum or below a certain tolerable value.

Steady-state error of unity feedback system.

Lets us consider the unity feedback system shown below.



1. The error of the system is given by

$$E(s) = R(s) - Y(s)$$

2. The output of the system is given by

$$Y(s) = \frac{G}{1+G} R$$

3. Substitute the output

$$E(s) = R(s) - \frac{G}{1+G} R(s)$$

$$= R(s)\left[1 - \frac{G}{1+G}\right]$$

$$E(s) = \frac{1}{1+G} R(s)$$

4. Steady-State Error is given by

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$= \lim_{s \to 0} sE(s) \qquad \text{where } e(t) = r - y$$

$$= \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

It is useful to determine the steady state error of the system for the three standard test inputs signals

a) Step displacement input
b) Step Velocity input
c) Step acceleration input

a) Step Displacement input $\quad r(t) = K u(t)$

$$R(s) = \frac{K}{s}$$

The steady state error becomes

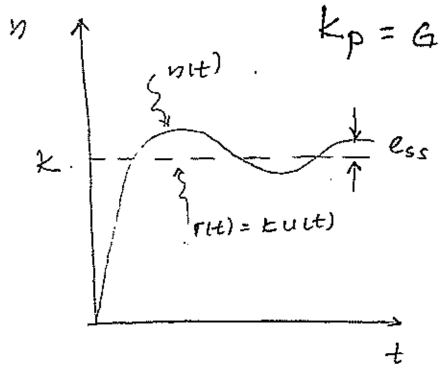
$$e_{ss} = \lim_{s \to 0} \frac{s R(s)}{1 + G(s)}$$

$$= \lim_{s \to 0} \frac{K}{1 + G(s)} = \frac{K}{1 + \lim_{s \to 0} G(s)}$$

$$= \frac{K}{1 + K_p}$$

where $K_p = \lim_{s \to 0} G(s)$ = positional Error constant.

$K_p = G(0)$ is defined as the position error constant.



It is seen that the error decreases as with increasing K_p.

The positional error-constant can be predetermined for a given step-input and a fixed allowable steady-state error

$$K_p = \frac{K - e_{ss}}{e_{ss}}$$

b) Step Velocity input $\quad r(t) = K t$

$$R(s) = \frac{K}{s^2}$$

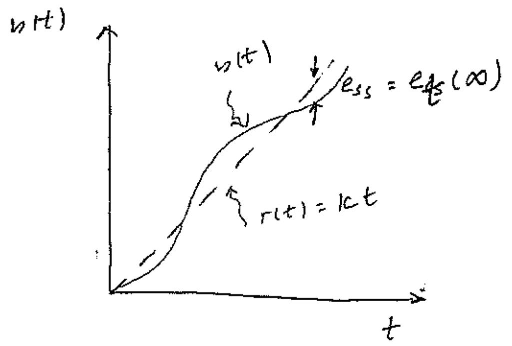
The steady State error becomes

$$e_{ss} = \lim_{s\to 0} \frac{s R(s)}{1+G(s)} = \lim_{s\to 0} \frac{s * \frac{k}{s^2}}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{k}{s(1+G(s))} \simeq \lim_{s\to 0} \frac{k}{s G(s)}$$

$$= \frac{k}{\lim_{s\to 0} s G(s)} = \frac{k}{k_v}$$

where $k_v = \lim_{s\to 0} s G(s)$: Velocity Error Constant.



k_v

c) step acceleration input $r(t) = \frac{1}{2} k t^2$

$$R(s) = \frac{k}{s^3}$$

The steady state error becomes

$$e_{ss} = \lim_{s\to 0} \frac{s R(s)}{1+G(s)} = \lim_{s\to 0} \frac{s * \frac{k}{s^3}}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{k}{s^2(1+G(s))} \simeq \lim_{s\to 0} \frac{k}{s^2 G(s)}$$

$$= \frac{k}{\lim_{s\to 0} s^2 G(s)} = \frac{k}{k_a}$$

where $k_a = \lim_{s\to 0} s^2 G(s)$: Acceleration error constant.

Types of feedback Control system.

Both dynamic and steady state errors depend on the form of $G(s)$

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$$G(s) = \frac{K(s+z_1)\cdots(s+z_m)}{s^n(s+P_{n+1})+\cdots(s+P_n)} \qquad n \geq m$$

$$= \frac{K\prod_{i=1}^{m}(s+z_i)}{s^n \prod_{j=1}^{Q}(s+P_j)}$$

Π - the product of factors.

The number of poles at the origin (i.e n) determines the type of system. The terms s^n in the denominator which corresponds to number of integration in the system. As s tends to zero, this term dominates in determining the steady state error. Control system are therefore classified in accordance with the no of integration in the open loop transfer function $G(s)$)

Thus $n = 0 \rightarrow$ represents a Type 0 system

$n = 1 \rightarrow$ represents a Type 1 system.

Since each pole at the origin signifies pure integration, it is easy to derive the results shown below, for a <u>step input</u>.

1. Type - 0 system :- A system for which $G(s)$ has no-poles at the origin of the s-plane

$$G(s) = \frac{K\Pi(s+z_i)}{\Pi(s+P_i)}$$

and $$e_{ss} = \lim_{s\to 0}\left[\frac{1}{1+G(s)}\right] = \frac{1}{1+G(0)} = \frac{1}{1+K_p}$$

where $$K_p = G(0) = \frac{K z_1 z_2 \cdots z_m}{P_1 P_2 \cdots P_n}$$ = positional error constant.

2. Type 1 system: A system is said to be type 1, if it has one pole at the origin of the s-plane (i.e $n=1$)

$$G(s) = \frac{K\Pi(s+z_i)}{s\,\Pi(s+P_i)}$$

$$e_{ss} = \lim_{s\to 0}\left[\frac{1}{1+G(s)}\right] = \lim_{s\to 0}\left[\frac{1}{1+\frac{K\Pi(s+z_i)}{s\,\Pi(s+P_i)}}\right]$$

$$= \frac{1}{1+\infty} = 0$$

3 - Type-2 system. A system is said to be type-2, if it has two poles at the origin of the s-plane (i.e $n=2$)

$$G(s) = \frac{K \prod (s - z_i)}{s^2 \prod (s + P_i)}$$

$$e_{ss} = \lim_{s \to 0} \left[\frac{1}{1+G(s)} \right] = \lim_{s \to 0} \left[\frac{1}{1 + \frac{K\prod(s-z_i)}{s^2\prod(s-P_i}} \right]$$

$$= \frac{1}{1+\infty} = 0$$

Relation ship between system type and error constant shown in the following table.

System type	Step input $R(s) = \frac{1}{s}$	Ramp input $R(s) = \frac{1}{s^2}$; $e_{ss} = \frac{K}{\lim_{s\to 0} sG(s)}$	Parabolic input $R(s) = \frac{1}{s^3}$; $e_{ss} = \frac{K}{\lim_{s\to 0} s^2G(s)}$
Type-0	$e_{ss} = \frac{1}{1+K_p}$	$e_{ss} = \infty = \frac{k}{\lim_{s\to 0} sG(s)}$	$e_{ss} = \infty$; $\frac{k}{\lim_{s\to 0} s^2G(s)}$
Type-1	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_v}$	$e_{ss} = \infty$
Type-2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_a}$

Note:

$e_{ss} = \infty \Rightarrow$ system can not follow the input

$e_{ss} = 0 \Rightarrow$ system can follow the input without error

and

$$K_p = \lim_{s \to 0} G(s) \qquad K_v = \lim_{s \to 0} sG(s) \qquad K_a = \lim_{s \to 0} s^2 G(s)$$

Example

A unity feedback system characterized by the open-loop transfer function

$$G(s) = \frac{1}{s(0.5s+1)(0.2s+1)}$$

Determine the steady-state-Error for

a) unit-step-input

b) unit-Ramp-Input and

c) unit-Acceleration input

soln

a) a unit-step-input

The position-Error constant K_p is given by

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{1}{s(0.5s+1)(0.2s+1)} = \frac{1}{0} = \infty$$

The steady state error to a unit step input

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

b) The velocity-error constant is given by

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\left[\frac{1}{s(0.5s+1)(0.2s+1)}\right]$$

$$= \lim_{s\to 0}\left[\frac{1}{(0.5s+1)(0.2s+1)}\right] = 1$$

The Steady state error to a unit Ramp-input is given by

$$e_{ss} = \frac{1}{K_v} = \frac{1}{1} = 1$$

c) The acceleration-Error-Constant is given by

$$K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} s^2\left[\frac{1}{s(0.5s+1)(0.2s+1)}\right]$$

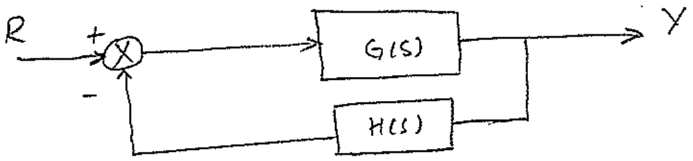
$$= \lim_{s\to 0}\left[\frac{s}{(0.5s+1)(0.2s+1)}\right] = 0$$

The steady state error to a unit acceleration input is given by

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

0.7 Determination of Error Coefficients

Consider the closed-loop feedback control system.



We may obtain

1. System transfer function $= T_R^Y(s) = \dfrac{Y}{R} = \dfrac{G}{1+GH}$

2. Error transfer function $T_R^E(s) = \dfrac{E}{R} = \dfrac{1}{1+GH}$.

in the case of unity feedback control $H(s)=1$

the above transfer function reduces to

3 - $T_R^Y = \dfrac{Y}{R} = \dfrac{G}{1+G}$

4 - $T_R^E = \dfrac{E}{R} = \dfrac{1}{1+G}$

The Error function of a unity Feedback control becomes

5 - $E = \dfrac{1}{1+G} R$

~~If we expand Error of a unity feedback~~ control becomes

If we expand error function in power series about $s=0$, we get the series

6. $E(s) = [C_0 + C_1 s + C_2 s^2 + C_3 s^3 + \cdots] R(s)$

$$= \sum_{i=0}^{n} [C_i s^i] R(s)$$

$$= C_0 R(s) + s C_1 R(s) + s^2 C_2 R(s) + \cdots$$

If we take the inverse Laplace transform of the above error function, we get the error as a function of time

7 - $e(t) = C_0 r(t) + C_1 \dfrac{dr(t)}{dt} + C_2 \dfrac{d^2 r(t)}{dt^2} + \cdots$

Where $C_0, C_1, C_2 \cdots$ are called dynamic error constants.

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power series.

$$C_0 + c_1(x-a)^1 + C_2(x-a)^2 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Comparing Eqn (5) and (6) the dynamic error coefficents evaluated as 34

8. $C_0 + C_1 s + C_2 s^2 + C_3 s^3 + \cdots = \frac{1}{1+G(s)}$

We see that

9. $C_0 = \left. \frac{1}{1+G(s)} \right|_{s=0}$ = positional Error coefficents

10. $C_1 = \frac{d}{ds}\left[\frac{1}{1+G(s)}\right]_{s=0}$ = Velocity Error coefficents

11. $C_2 = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{1+G(s)}\right]_{s=0}$ = Acceleration Error coefficents.

$\vdots$

12. $C_n = \frac{1}{n!}\frac{d^n}{ds^n}\left[\frac{1}{1+G(s)}\right]_{s=0}$

Example

Determine the dynamic Error coefficents for the unity feedback system with

$$G(s) = \frac{10}{s(s+1)}$$

so/n

1- Transfer function $\frac{1}{1+G} = \frac{s(s+1)}{s^2+s+10}$

2- positional Error cofficents

$$C_0 = \left. \frac{1}{1+G(s)} \right|_{s=0} = \left. \frac{s^2+s}{s^2+s+10} \right|_{s=0} = 0$$

3. Velocity error coefficents

$$C_1 = \left. \frac{d}{ds}\left[\frac{1}{1+G(s)}\right] \right|_{s=0} = \left. \frac{d}{ds}\left[\frac{s^2+s}{s^2+s+10}\right] \right|_{s=0}$$

$$= \left. \frac{(2s+1)(s^2+s+10) - (s^2+s)(2s+1)}{(s^2+s+10)^2} \right|_{s=0}$$

$$= \frac{1(10) - 0}{(10)^2} = \frac{10}{100} = 0.1$$

4. Acceleration Error - coeff. (C_2)

$$C_2 = \left. \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{1+G(s)}\right] \right|_{s=0} = 0.09$$

so that the error fn in time domain

$$e(t) = 0.1 \frac{dr}{dt} + 0.09 \frac{d^2 r}{dt^2} + \cdots$$

4.8. Error - Minimizing Performance Indices.

A number of performance measures (measurable quantities) have been introduced in respect of dynamic response to step input (ξ, M_p, t_r, t_p, t_s etc.) and the steady state error e_{ss} to both step and higher order inputs. These measures have to be satisfied simultaneously in design and hence the design necessarily becomes a trial and error procedure. If, however a single performance index could be established on the basis of which one may describe the goodness of the system respons, then the design procedure will become logical and straight forward.

A performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specification.

* A system is considered as an Optimal control system when the system parameters are adjusted so that the index reaches an extremum value, commonly a minimum value. A performance index, to be useful must be a number that is always positive or zero. Then the best system is defined as the system that minimizes this index.

The most popular performance indeces are

1- The integral of the square-error Criterion (ISE)
   i.e performance index (PI)

$$ISE = J_1 = \int_0^\infty e^2(t)\, dt \quad --$$

where $e(t) = r(t) - y(t)$

It is obvious that the ISE converges to a limit as $t \to \infty$

Minimizing of ISE is a good compromise b/n the reduction of rise time to limit the effect of larger initial error, reduction of peak overshoot and reduction of setteling time to limit the effect of small error lasting for a long time.

2 - Integral of Absolute error (IAE)

$$IAE = J_2 = \int_0^T |e(t)|\,dt$$

This index is useful computer simulation method.

3 - Integral of time-multiplied square error criterion (ITSE)

$$ITSE = J_3 = \int_0^T t\,e^2(t)\,dt$$

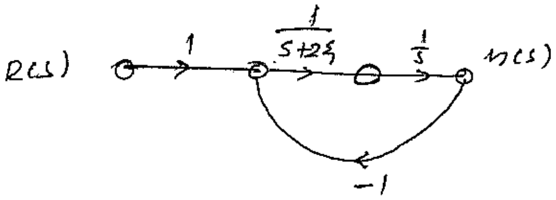
— The advantage of ITSE is providing a higher order weighting to errors occuring later in the reaponse than during the initial part.

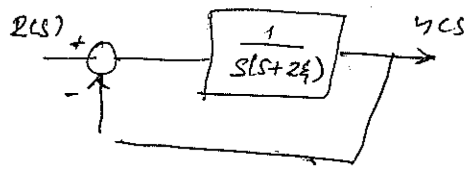
4 - Integral of time-multiplied absolute error criterion (ITAE)

$$ITAE = J_4 = \int_0^T t\,|e(t)|\,dt$$

→ provides the best selectivity of the performance indices.

Eg. A single loop feedback control system is shown below

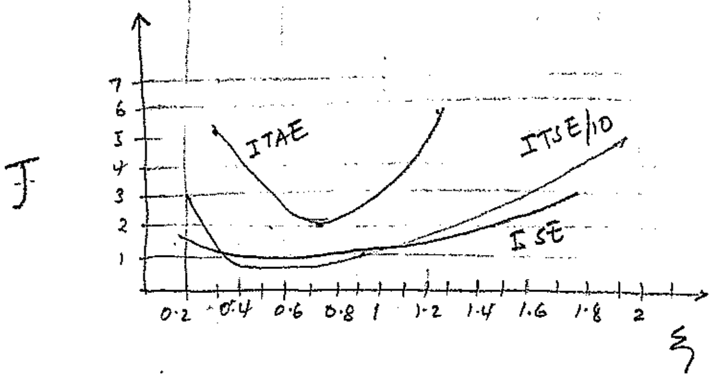




The closed loop transfer function is then

$$T(s) = \frac{1}{s^2 + 2\xi s + 1}$$

Three performance indices ISE, ITSE and ITAE calculated for various values of the damping ratio ξ and for a step input as shown below



The curve shows the selectivity of the ITAE index in comparison with ISE index. The value of damping ratio ξ selected on the basis of ITAE is 0.7, which for a second order system, results in a swift response to a step with a 4.6% overshoot

$$M_p = 100\,e^{-\pi\xi/\sqrt{1-\xi^2}}$$