DEBRE MARKOS UNIVERSITY INSTITUTE OF TECHNOLOGY

SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING

ELECTRICAL SECOND YEAR STUDENTS

COURSE NAME: SIGNAL AND SYSTEM ANALYSIS

2012 E.C

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CHAPTER ONE

SIGNALS AND SYSTEMS

Introduction

A signal is a function representing a physical quantity or variable, and typically it contains information about the behavior or nature of the phenomenon.

Mathematically, a signal is represented as a function of an independent Variable t. usually t represents time. Thus, a signal is denoted by x (t).

1. **CLASSIFICATION OF SIGNALS**

A. **Continuous-Time and Discrete-Time Signals:**

A signal x(t) is a continuous-time signal if t is a continuous variable. If t is a discrete variable, that is, x(t) is defined at discrete times, then x(t) is a discrete-time signal.
Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as

a sequence of numbers, denoted by {x,) or x[n], where n = integer.

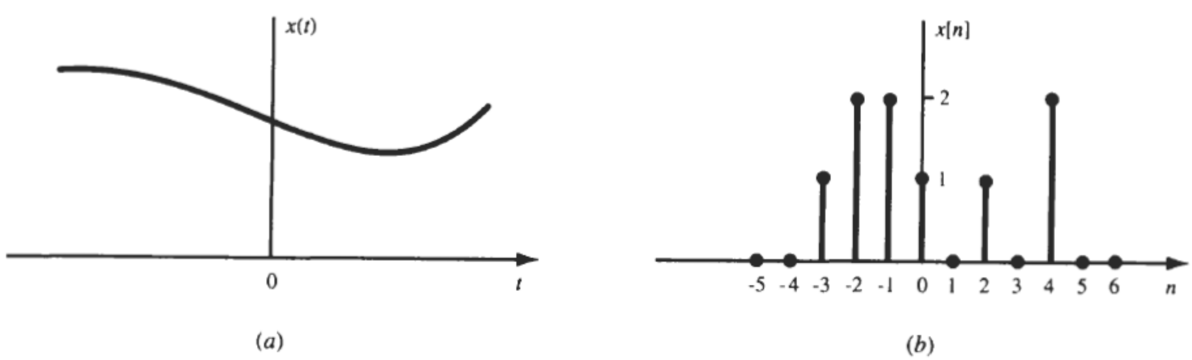


Fig. 1-1 Graphical representation of (a) continuous-time and (b) discrete-time signals.

A discrete-time signal x[n] can be defined in two ways:

1. We can specify a rule for calculating the nth value of the sequence. For example,

$$x[n] = x_n = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\{x_n\} = \left\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, \left(\tfrac{1}{2}\right)^n, \ldots\right\}$$

2. We can also explicitly list the values of the sequence. For example, the sequence Shown in Fig. l-l(b) can be written as

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2, 0, 0, \ldots\}$$

or

$$\{x_n\} = \{1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2\}$$

The sum and product of two sequences are defined as follows:

$$\{c_n\} = \{a_n\} + \{b_n\} \longrightarrow c_n = a_n + b_n$$

$$\{c_n\} = \{a_n\}\{b_n\} \longrightarrow c_n = a_n b_n$$

$$\{c_n\} = \alpha\{a_n\} \longrightarrow c_n = \alpha a_n \qquad \alpha = \text{constant}$$

B. Analog and Digital Signals

If a continuous-time signal $x(t)$ can take on any value in the continuous interval (a, b), where a may be $-\infty$ and b may be $+\infty$, then the continuous-time signal $x(t)$ is called an *analog* signal. If a discrete-time signal $x[n]$ can take on only a finite number of distinct values, then we call this signal a *digital* signal.

C. Real and Complex Signals:

A signal $x(t)$ is a *real* signal if its value is a real number, and a signal $x(t)$ is a *complex* signal if its value is a complex number. A general complex signal $x(t)$ is a function of the form

$$x(t) = x_1(t) + jx_2(t)$$

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$.

D. Deterministic and Random Signals:

Deterministic : signals are those signals whose values are completely specified for any Given time. Thus, a deterministic signal can be modeled by a known function of time ***I.***

Random signals are those signals that take random values at any given time and must be Characterized statistically

D. Even and Odd Signals:

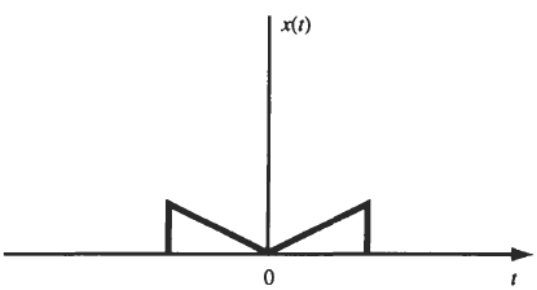
A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

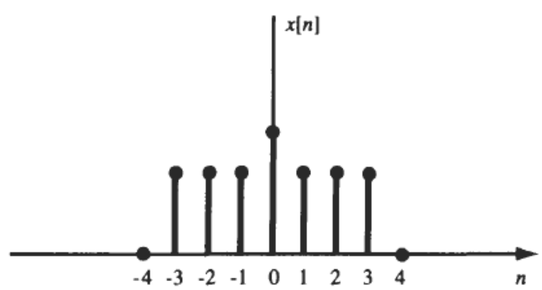
A signal $x(t)$ or $x[n]$ is referred to as an *odd* signal if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Examples of even signals

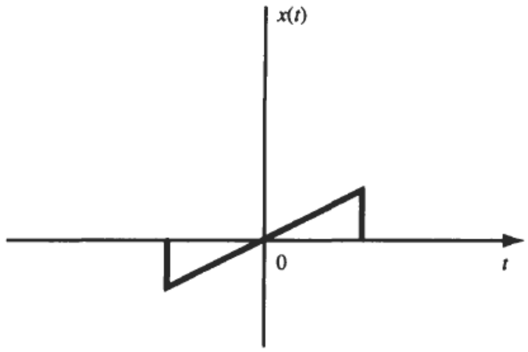


(a)



(b)

Examples of odd signals



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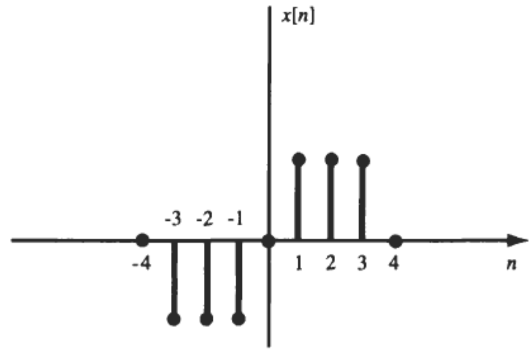


(d)

Any signal ***x(t)*** or ***x[n]*** can be expressed as a sum of two signals, one of which is even
And one of which is odd. That is,

$$x(t) = x_e(t) + x_o(t)$$
$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e(t) = \tfrac{1}{2}\{x(t) + x(-t)\} \qquad \text{even part of } x(t)$$
$$x_e[n] = \tfrac{1}{2}\{x[n] + x[-n]\} \qquad \text{even part of } x[n]$$
$$x_o(t) = \tfrac{1}{2}\{x(t) - x(-t)\} \qquad \text{odd part of } x(t)$$
$$x_o[n] = \tfrac{1}{2}\{x[n] - x[-n]\} \qquad \text{odd part of } x[n]$$

Note that the product of two even signals or of two odd signals is an even signal and
That the product of an even signal and an odd signal is an odd signal.

E. Periodic and Non periodic Signals:

F. A continuous-time signal ***x (t)*** is said to be ***periodic with period T*** if there is a positive Nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t$$

And also

$$x(t+mT) = x(t)$$

for all ***t*** and any integer ***m.*** The ***fundamental period T,*** of ***x (t)*** is the smallest positive value of T. Any continuous-time signal which is not periodic is called a nonperiodic (or aperiodic) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) x[n] is periodic with period N if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n$$

An example of such a sequence is given by:

$$x[n+mN] = x[n]$$

for all n a Any sequence which is not periodic is called a non periodic (or aperiodic sequenced any integer m.

G. Energy and Power Signals:

Consider v(t) to be the voltage across a resistor R producing a current dt). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\,dt \quad \text{joules}$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} i^2(t)\,dt \quad \text{watts}$$

For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

The *normalized average power* P of $x(t)$ is defined as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$$

Similarly, for a discrete-time signal x[n], the normalized energy content ***E*** of x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

The normalized average power P of $x[n]$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

the following classes of signals are defined:

1. $x(t)$ (or $x[n]$) is said to be an *energy* signal (or sequence) if and only if $0 < E < \infty$, and so $P = 0$.
2. $x(t)$ (or $x[n]$) is said to be a *power* signal (or sequence) if and only if $0 < P < \infty$, thus implying that $E = \infty$.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period.

BASIC CONTINUOUS-TIME SIGNALS

A. The Unit Step Function:

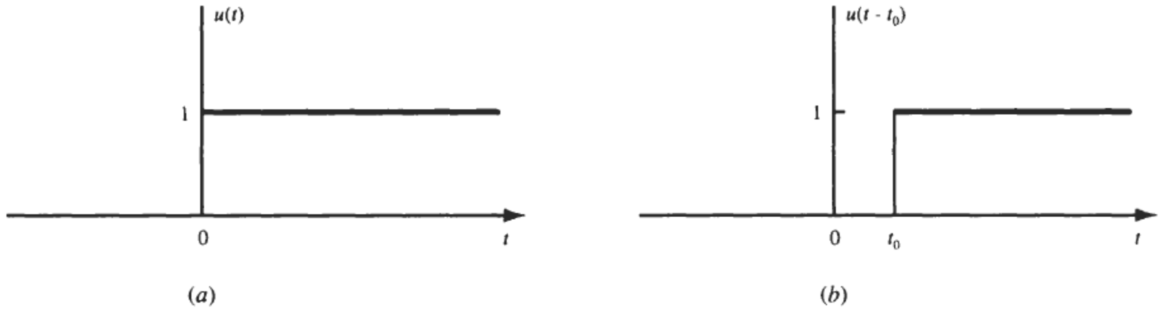
The unit step function u(t), also known as the Heaciside unit function, is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Note that it is discontinuous at t = 0 and that the value at

t = 0 is undefined. Similarly, the shifted unit step function u(t - to) is defined as

$$u(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$$



(a) Unit step function; ***(b)*** shifted unit step function

B. The Unit Impulse Function:

The unit impulse function 6(t), also known as the Dirac delta function, plays a central role in system analysis.

Properties:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1$$

Some additional properties of $\delta(t)$ are

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta(-t) = \delta(t)$$

$$x(t)\delta(t) = x(0)\delta(t)$$

if $x(t)$ is continuous at $t = 0$.

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

if $x(t)$ is continuous at $t = t_0$.

any continuous-time signal ***x(t*** can be expressed as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau$$

C. Complex Exponential Signals:

The ***complex exponential*** signal

$$x(t) = e^{j\omega_0 t}$$

Using Euler's formula, this signal can be defined as

$$x(t) = e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

The fundamental period To of x(t) is given by

$$T_0 = \frac{2\pi}{\omega_0}$$

Note that x(t) is periodic for any value of wo.

General Complex Exponential Signals:

Let $s = \sigma + j\omega$ be a complex number. We define $x(t)$ as

$$x(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}(\cos \omega t + j \sin \omega t)$$

D. Sinusoidal Signals:

A continuous-time *sinusoidal* signal can be expressed as

$$x(t) = A \cos(\omega_0 t + \theta)$$

where A is the *amplitude* (real), ω_0 is the *radian frequency* in radians per second, and θ is the *phase angle* in radians. The sinusoidal signal $x(t)$ is shown in Fig. 1-9, and it is periodic with fundamental period

$$T_0 = \frac{2\pi}{\omega_0}$$

The reciprocal of the fundamental period T_0 is called the *fundamental frequency* f_0:

$$f_0 = \frac{1}{T_0} \quad \text{hertz (Hz)} \qquad ($$

we have

$$\omega_0 = 2\pi f_0$$

Which is called the fundamental angular frequency?

Using Euler's formula the sinusoidal

Signal expressed as:

$$A\cos(\omega_0 t + \theta) = A\,\mathrm{Re}\{e^{j(\omega_0 t + \theta)}\}$$

Where "Re" denotes "real part of." We also use the notation "Im" to denote "imaginary Part of." Then

$$A\,\mathrm{Im}\{e^{j(\omega_0 t + \theta)}\} = A\sin(\omega_0 t + \theta)$$

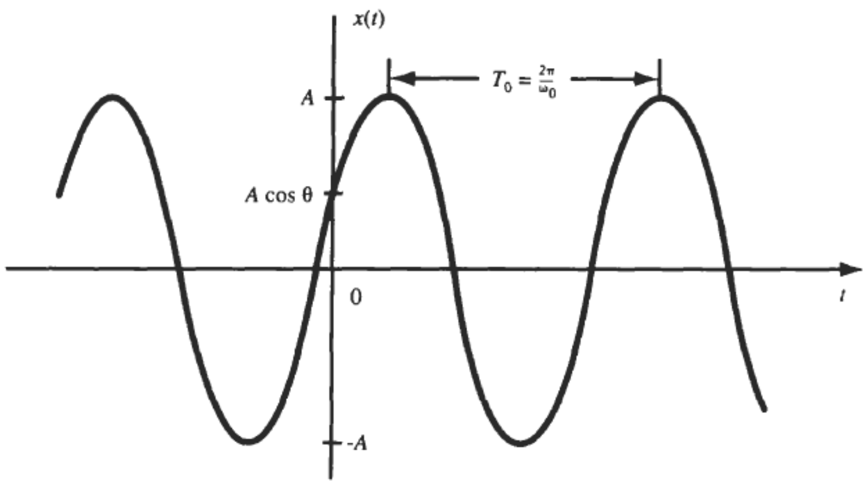


Fig. Continuous-time sinusoidal signal

BASIC DISCRETE-TIME SIGNALS

A. The Unit Step Sequence:

The *unit step* sequence $u[n]$ is defined as

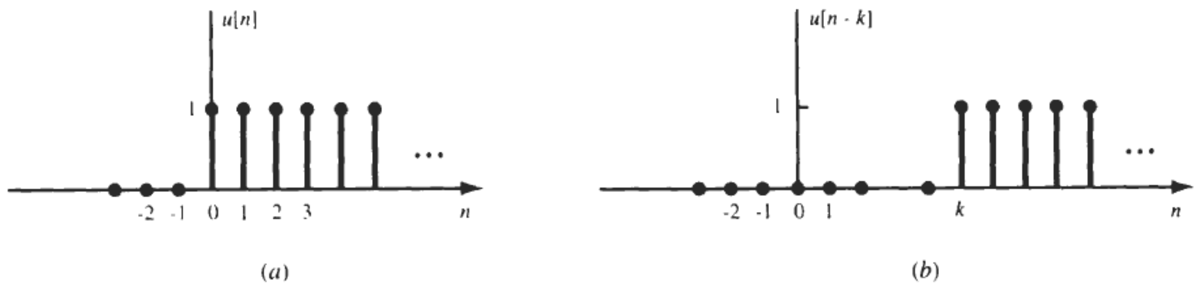
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Similarly, the shifted unit step

Sequence

$u[n-k]$ is defined as

$$u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$



(a) Unit step sequence; (b) shifted unit step sequence

B. The Unit Impulse Sequence:

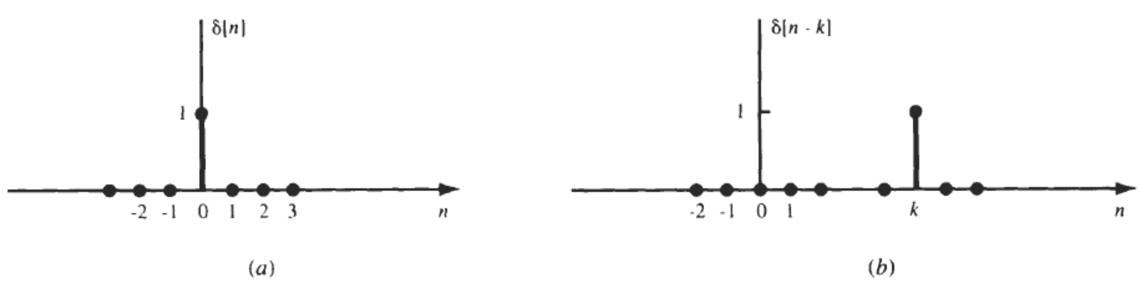
The *unit impulse* (or *unit sample*) sequence $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Similarly, the shifted unit impulse (or sample) sequence

$\delta[n-k]$ is defined as

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$



(a) Unit impulse (sample) sequence; (6) shifted unit impulse sequence.

We have:

$$x[n]\delta[n] = x[0]\delta[n]$$

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

$\delta[n]$ and $u[n]$ are related by

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

any sequence $x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

C. Complex Exponential Sequences:

The *complex exponential* sequence is of the form

$$x[n] = e^{j\Omega_0 n}$$

Again, using Euler's formula, $x[n]$ can be expressed as

$$x[n] = e^{j\Omega_0 n} = \cos \Omega_0 n + j \sin \Omega_0 n$$

Thus $x[n]$ is a complex sequence whose real part is $\cos \Omega_0 n$ and imaginary part is $\sin \Omega_0 n$.

Periodicity of $e^{j\Omega_0 n}$:

In order for $e^{j\Omega_0 n}$ to be periodic with period N (> 0), Ω_0 must satisfy t

The condition:

$$\frac{\Omega_0}{2\pi} = \frac{m}{N} \qquad m = \text{positive integer}$$

Thus the sequence $e^{j\Omega_0 n}$ is not periodic for any value of Ω_0. It is periodic only if $\Omega_0/2\pi$ is a rational number. Note that this property is quite different from the property that the

continuous-time signal $e^{j\omega_0 t}$ is periodic for any value of ω_0.

N_0 given by
$$N_0 = m\left(\frac{2\pi}{\Omega_0}\right)$$

General Complex Exponential Sequences:

The most general complex exponential sequence is often defined as

$$x[n] = C\alpha^n$$

D. Sinusoidal Sequences:

A *sinusoidal* sequence can be expressed as

$$x[n] = A\cos(\Omega_0 n + \theta)$$

If n is dimensionless, then both Ω_0 and θ have units of radians.

SYSTEMS AND CLASSIFICATION OF SYSTEMS

A. System Representation:

A *system* is a mathematical model of a physical process that relates the *input* (or *excitation*) signal to the *output* (or *response*) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a *transformation* (or *mapping*) of x into y. This transformation is represented by the mathematical notation

$$y = \mathbf{T}x$$

Where **T** is the operator representing some well-defined rule by which **x** is transformed into y.

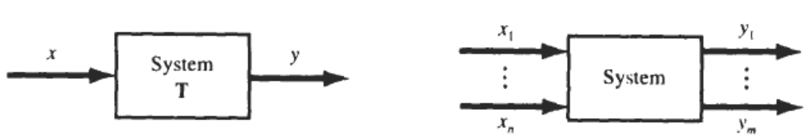


Fig. System with single (a) or multiple input and output signals (b).

B. Continuous; Time and Discrete-Time Systems:

If the input and output signals x and p are continuous-time signals, then the system is called a ***continuous-time system.***

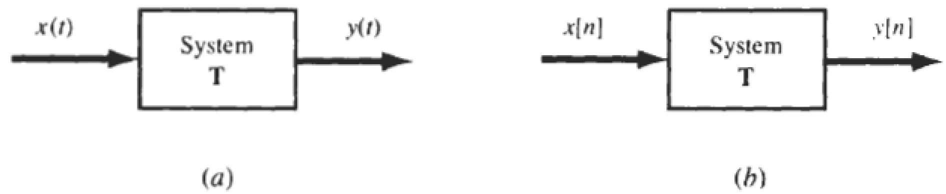


Fig. (a) Continuous-time system; *(b)* discrete-time system.

B. Systems with Memory and without Memory

A system is said to be ***memory less*** if the output at any time depends on only the input at that same time. Otherwise, the system is said to have ***memory.***

A second example of a system with memory is a discrete-time system whose input and output sequences are related by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

D. Causal and Noncausal Systems:

A system is called *causal* if its output $y(t)$ at an arbitrary time $t = t_0$ depends on only the input $x(t)$ for $t \le t_0$. That is, the output of a causal system at the present time depends on only the present and/or past values of the input, not on its future values.

A system is called ***noncausal*** if it is not causal. Examples of noncausal system are

$$y(t) = x(t+1)$$

$$y[n] = x[-n]$$

Note that all memoryless systems are causal, but not vice versa.

D. Linear Systems and Nonlinear Systems:

two conditions:

the system represented by a linear operator T is called a linear system:

1. ***Additivity:***

Given that $\mathbf{T}x_1 = y_1$ and $\mathbf{T}x_2 = y_2$, then

$$\mathbf{T}\{x_1 + x_2\} = y_1 + y_2$$

for any signals x_1 and x_2.

2. ***Homogeneity*** (or ***Scaling***)***:***

$$\mathbf{T}\{\alpha x\} = \alpha y$$

for any signals x and any scalar α.

The system combined as;

$$\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2$$

where α_1 and α_2 are arbitrary scalars.

Examples of nonlinear systems are

$$y = x^2$$

$$y = \cos x$$

F. Time-Invariant and Time-Varying Systems:

A system is called rime-inuariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal. Thus, for a continuous-time system, the system is time-invariant if

$$\mathbf{T}\{x(t-\tau)\} = y(t-\tau)$$

for any real value of T. For a discrete-time system, the system is time-invariant (or shift-invariant) if

$$\mathbf{T}\{x[n-k]\} = y[n-k]$$

For any integer ***k.***

G. Linear Time-Invariant Systems

If the system is linear and also time-invariant, then it is called a linear rime-invariant (LTI) system.

H. Stable Systems:

A system is ***bounded-input/bounded-output (BIBO) stable*** if for any bounded input ***x*** defined by

$$|x| \leq k_1$$

the corresponding output y is also bounded defined by

$$|y| \leq k_2$$

where k_1 and k_2 are finite real constants.

H. Feedback Systems:

A special class of systems of great importance consists of systems having ***feedback.*** In a ***feedback system,*** the output signal is fed back and added to the input to the system as shown

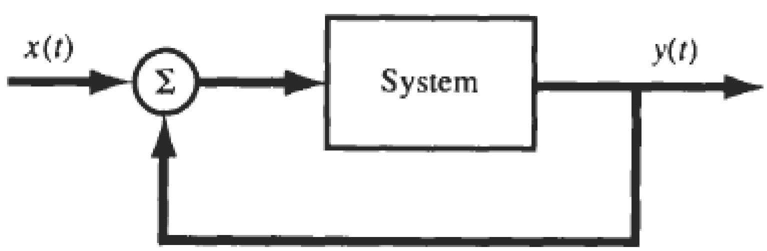


Fig.: Feedback system.

<u>EXERCISES</u>

<u>(1).</u> A continuous-time signal ***x (t)*** is shown in Fig. bellow. Sketch and label each of the following signals.

$(a)\ x(t-2);\ (b)\ x(2t);\ (c)\ x(t/2);\ (d)\ x(-t)$

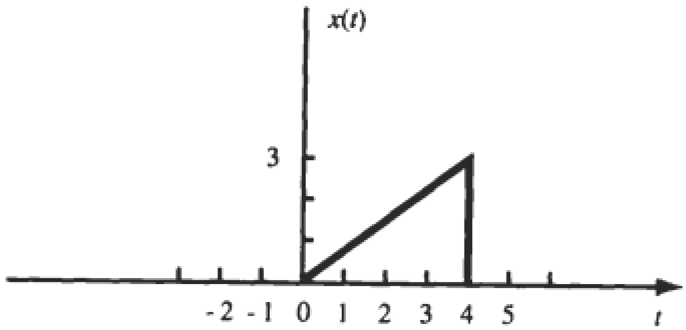


Fig 1. Continuous Time Graph

(2). **A discrete-time signal $x[n]$ is shown in Fig 2. Sketch and label each of the following signals.**

$(a)\ x[n-2];\ (b)\ x[2n];\ (c)\ x[-n];\ (d)\ x[-n+2]$

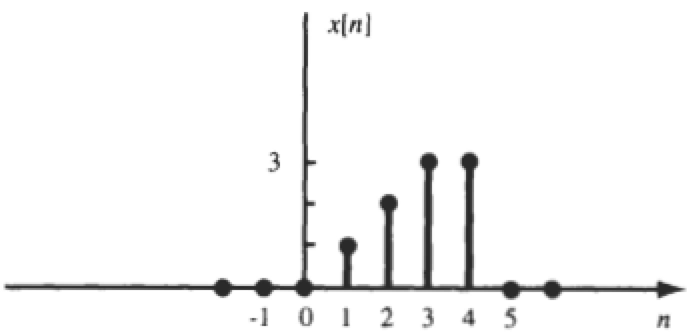


Fig 2. Discrete Time Graph

(3).

Find the even and odd components of $x(t) = e^{jt}$.

(4). Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

A.

$$x(t) = \cos\left(t + \frac{\pi}{4}\right)$$

B.

$$x(t) = \cos t + \sin \sqrt{2}\, t$$

(5). **Determine whether the following signals are energy signals, power signals, or neither**

A.

$$x(t) = e^{-at}u(t), \quad a > 0$$

B,

$$x(t) = A\cos(\omega_0 t + \theta)$$

C.

$$x(t) = tu(t)$$

(6). **A continuous-time signal *A (t)* is shown in Fig. Sketch and label each of the following signals**

$$(a)\ x(t)u(1-t);\ (b)\ x(t)[u(t)-u(t-1)];\ (c)\ x(t)\delta(t-\tfrac{3}{2})$$

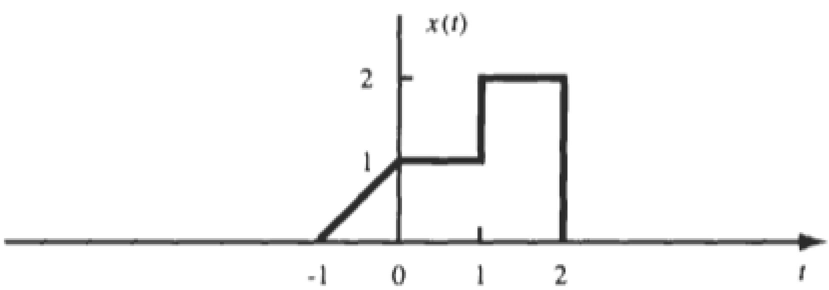


Fig 3. CTSG

(7). **A discrete-time signal *x [n]* is shown in Fig. Sketch and label each of the following signals.**

$$(a)\ x[n]u[1-n];\ (b)\ x[n]\{u[n+2]-u[n]\};\ (c)\ x[n]\delta[n-1]$$

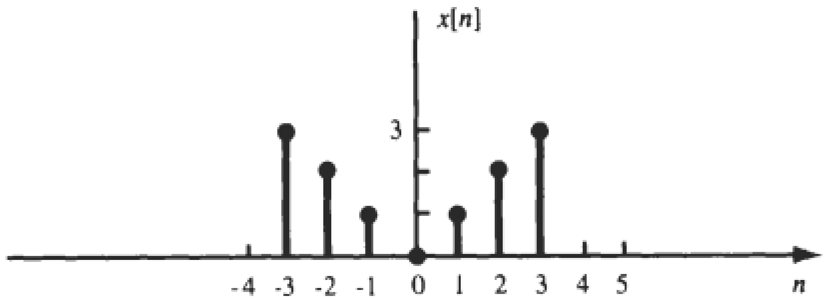


FIG 4. DTSG

(8). Evaluate the following integrals:

A.

$$\int_{-1}^{1} (3t^2 + 1)\delta(t)\, dt$$

B.

$$\int_{-\infty}^{\infty} e^{-t}\delta(2t - 2)\, dt$$

(9). Find and sketch the first derivatives of the given signals:

$$(a) \quad x(t) = u(t) - u(t - a),\ a > 0$$

Chapter two

Convolution

Linear Time-Invariant Systems

Introduction

Two most important attributes of systems are linearity and time-invariance. the input-output relationship for LTI systems is described in terms of a convolution operation.

Response of a continuous-time lti system and

The convolution integral

A. Impulse Response:

The *impulse response* $h(t)$ of a continuous-time LTI system (represented by **T**) is defined to be the response of the system when the input is $\delta(t)$, that is,

$$h(t) = \mathbf{T}\{\delta(t)\}$$

B. Response to an Arbitrary Input:

From Eq. (*1.27*) the input $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Since the system is linear, the response $y(t)$ of the system to an arbitrary input $x(t)$ can be expressed as

$$y(t) = \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\}$$

In general we have:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

D. Convolution Integral:

the *convolution* of two continuous-time signals *x (t)* and *h (t)*
denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

is commonly called the *convolution integral.*
Thus, we have the fundamental
result that *the output of any continuous-time LTI system is the convolution of the input x (t) with the impulse response h (t) of the system.*

Fig .Continuous-time LTl system.

D. Properties of the Convolution Integral:

The convolution integral has the following properties.

1. Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

2. Associative:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

3. Distributive:

$$x(t) * \{h_1(t)\} + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

E. Convolution Integral Operation:

Applying the commutative property

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

we observe that

the convolution integral operation involves the following four steps:

1. The impulse response $h(\tau)$ is time-reversed (that is, reflected about the origin) to obtain $h(-\tau)$ and then shifted by t to form $h(t-\tau) = h[-(\tau - t)]$ which is a function of τ with parameter t.
2. The signal $x(\tau)$ and $h(t-\tau)$ are multiplied together for all values of τ with t fixed at some value.
3. The product $x(\tau)h(t-\tau)$ is integrated over all τ to produce a single output value $y(t)$.
4. Steps 1 to 3 are repeated as t varies over $-\infty$ to ∞ to produce the entire output $y(t)$.

F. Step Response:

The step response s(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is 41); that is,

$$s(t) = \mathbf{T}\{u(t)\}$$

s(t) can be easily determined by

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

And

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus, the impulse response h(t) can be determined by differentiating the step response s(t).

Properties of continuous-time LTI systems

A. Systems with or without Memory

Since the output y(t) of a memoryless system depends on only the present input x(t), then, if the system is also linear and time-invariant, this relationship can only be of the

From

$$y(t) = Kx(t)$$

where K is a (gain) constant. Thus, the corresponding impulse response h(f) is simply

$$h(t) = K\delta(t)$$

Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, the continuous-time LTI system has memory.

B. Causality:

a causal system does not respond to an input event until that
event actually occurs. Therefore, for a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0$$

LTI system is expressed as

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\,d\tau$$

any signal x(t) is called causal if

$$x(t) = 0 \qquad t < 0$$

and is called anticausal if

$$x(t) = 0 \qquad t > 0$$

when the input x(t) is causal, the output y(t) of a causal continuous-time LTI system is given by

$$y(t) = \int_0^{t} h(\tau)x(t-\tau)\,d\tau = \int_0^{t} x(\tau)h(t-\tau)\,d\tau$$

C. Stability:

The BIBO **(bounded-input/bounded-output)**stability of an LTI system is readily ascertained from its impulse response.
that a continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$$

Response of a discrete-time lti system and convolution sum

A. Impulse Response:

The impulse response (or unit sample response) ***h [n]*** of a discrete-time LTI system (represented by **T)** is defined to be the response of the system when the input is 6[n], that

is,

$$h[n] = \mathbf{T}\{\delta[n]\}$$

B. Response to an Arbitrary Input:

the input ***x [n]***can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Since the system is linear, the response ***y [n]***of the system to an arbitrary input ***x [n]***can be

expressed as

$$y[n] = \mathbf{T}\{x[n]\} = \mathbf{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\mathbf{T}\{\delta[n-k]\}$$

Since the system is time-invariant, we have

$$h[n-k] = \mathbf{T}\{\delta[n-k]\}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

And

C. Convolution Sum:

the ***convolution*** of two sequences ***x [n]*** and ***h [n]***denoted by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

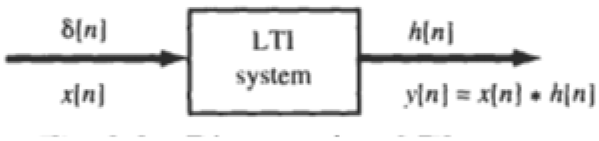


Fig. Discrete-time LTI system.

D. Properties of the Convolution Sum:

The following properties of the convolution sum are analogous to the convolution integral properties are

1. Commutative:

$$x[n] * h[n] = h[n] * x[n]$$

2. Associative:

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

3. Distributive:

$$x[n] * \{h_1[n]\} + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

E. Convolution Sum Operation:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

operation involves the following four steps:

1. The impulse response $h[k]$ is time-reversed (that is, reflected about the origin) to obtain $h[-k]$ and then shifted by n to form $h[n-k] = h[-(k-n)]$ which is a function of k with parameter n.
2. Two sequences $x[k]$ and $h[n-k]$ are multiplied together for all values of k with n fixed at some value.
3. The product $x[k]h[n-k]$ is summed over all k to produce a single output sample $y[n]$.
4. Steps 1 to 3 are repeated as n varies over $-\infty$ to ∞ to produce the entire output $y[n]$.

F. **Step Response:**

The *step response s[n]* of a discrete-time LTI system with the impulse response *h [n]* is

readily obtained from

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

And we have

$$h[n] = s[n] - s[n-1]$$

PROPERTIES OF DISCRETE-TIME LTI SYSTEMS

A. Systems with or without Memory:

Since the output *y[n]* of a memoryless system depends on only the present input *x [n]*.

$$y[n] = Kx[n]$$

where K is a (gain) constant. Thus, the corresponding impulse response is simply

$$h[n] = K\delta[n]$$

Therefore, if $h[n_0] \neq 0$ for $n_0 \neq 0$, the discrete-time LTI system has memory.

B. Causality:

Similar to the continuous-time case, the causality condition for a discrete-time LTI

system is

$$h[n] = 0 \qquad n < 0$$

the output of a causal discrete-time

LTI system is expressed as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

Alternatively,

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

As in the continuous-time case, we say that any sequence ***x[n]***is called ***causal*** if

$$x[n] = 0 \qquad n < 0$$

and is called ***anticausal*** if

$$x[n] = 0 \qquad n \geq 0$$

Then, when the input ***x[n]***is causal, the output ***y[n]***of a causal discrete-time LTI system
is given by

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} x[k]h[n-k]$$

C. Stability:

a discrete-time LTI system is B I B 0 stable if its
impulse response is absolutely summable, that is,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

<u>EXERCISES</u>

(1).

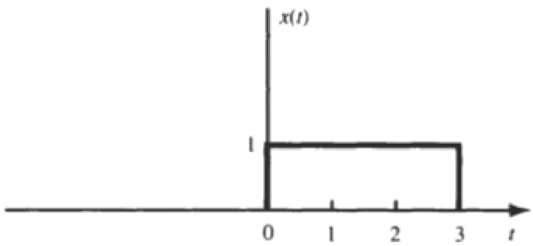
Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t}u(t) \qquad x(t) = e^{\alpha t}u(-t) \qquad \alpha > 0$$

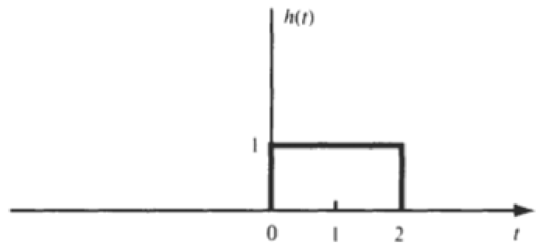
(2). Evaluate $y(t)=x(t)*h(t)$, where $x(t)$ and $h(t)$ are shown in Fig.

A. **analytical technique**

B. by a graphical method.



(a)



(b)

(3). Compute $y[n]=x[n]*h[n]$, where

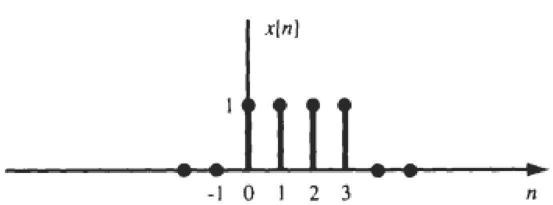
(a)

$$x[n]=\alpha^{n}u[n],\ h[n]=\alpha^{-n}u[-n],\ 0<\alpha<1$$

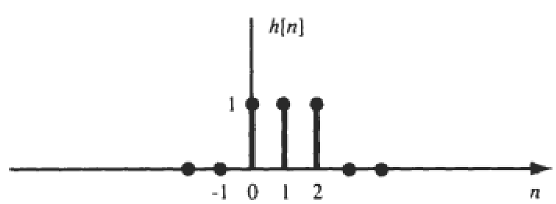
(4). Evaluate $y[n]=x[n]*h[n]$, where $x[n]$ and $h[n]$ are shown in Fig.

(a). by Analythical

(b). by graphical



(a)



(b)

(5).

Consider a discrete-time LTI system with impulse response $h[n]$ given by

$$h[n] = \alpha^n u[n]$$

(a) Is this system causal?

(b) Is this system BIBO stable?

(6).

Find the impulse response $h[n]$ for each of the causal LTI discrete-time systems satisfying the following difference equations and indicate whether each system is a FIR or an IIR system.

(a) $y[n] = x[n] - 2x[n-2] + x[n-3]$

(b) $y[n] + 2y[n-1] = x[n] + x[n-1]$

CHAPTER THREE

FOURIER SERIES AND TRANSFORM ANALYSIS OF SIGNAL AND SYSTEMS

Introduction

In this chapter and the following one, we shall introduce other transformations known as Fourier series and Fourier transform which convert time-domain signals into frequency-domain (or ***spectral)*** representations.

Fourier series representation of periodic signals

a. **Periodic Signals:**

   In Chap. 1 we defined a continuous-time signal ***x (t)*** to be periodic if there is a positive nonzero value of ***T*** for which

$$x(t+T)=x(t) \qquad \text{all } t$$

The fundamental period T_0 of $x(t)$ is the smallest positive value of T is satisfied, and $1/T_0 = f_0$ is referred to as the *fundamental frequency*.

Two basic examples of periodic signals are the real sinusoidal signal

$$x(t) = \cos(\omega_0 t + \phi)$$

and the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the *fundamental angular frequency*.

a. Complex Exponential Fourier Series Representation:

The complex exponential Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

where c_k are known as the *complex Fourier coefficients* and are given by

$$c_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

where $\int_{T_0}$ denotes the integral over any one period and 0 to T_0 or $-T_0/2$ to $T_0/2$ is

Commonly used for the integration. Setting $\boldsymbol{k} = 0$

$$c_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

Which indicates that ***co*** equals the average value of ***x (t)*** over a period.

When ***x(t)*** is real, it follows that

$$c_{-k} = c_k^*$$

D. Trigonometric Fourier Series:

The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

where a_k and b_k are the Fourier coefficients given by

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t \, dt$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t \, dt$$

The coefficients a_k and b_k and the complex Fourier coefficients c_k are related by

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

we obtain

$$c_k = \tfrac{1}{2}(a_k - jb_k) \qquad c_{-k} = \tfrac{1}{2}(a_k + jb_k)$$

When $x(t)$ is real, then a_k and b_k are real and

$$a_k = 2\,\mathrm{Re}[c_k] \qquad b_k = -2\,\mathrm{Im}[c_k]$$

Even and Odd Signals

If a periodic signal $x(t)$ is even, then $b_k = 0$ and its Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

If $x(t)$ is odd, then $a_k = 0$ and its Fourier series contains only sine terms:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

D. Harmonic Form Fourier Series:

Another form of the Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t - \theta_k) \qquad \omega_0 = \frac{2\pi}{T_0}$$

The coefficients C_k and the angles θ_k are called the *harmonic amplitudes* and *phase angles*, respectively, and they are related to the Fourier coefficients a_k and b_k by

$$C_0 = \frac{a_0}{2} \qquad C_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

E. Convergence of Fourier Series:

It is known that a periodic signal $x(t)$ has a Fourier series representation if it satisfies the following Dirichlet conditions:

1. $x(t)$ is absolutely integrable over any period, that is,

$$\int_{T_0} |x(t)|\, dt < \infty$$

2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

E. Amplitude and Phase Spectra of a Periodic Signal:

Let the complex Fourier coefficients c_k can expressed as

$$c_k = |c_k|\, e^{j\phi_k}$$

For a real periodic signal $x(t)$ we have $c_{-k} = c_k^*$. Thus,

$$|c_{-k}| = |c_k| \qquad \phi_{-k} = -\phi_k$$

***C*. Power Content of a Periodic Signal:**

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$$

If ***x (t)*** is represented by the complex exponential Fourier series in

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

The Fourier transform

A. From Fourier Series to Fourier Transform:

Let $x(t)$ be a nonperiodic signal of finite duration, that is,

$$x(t) = 0 \qquad |t| > T_1$$

If we let $T_0 \longrightarrow \infty$, we have

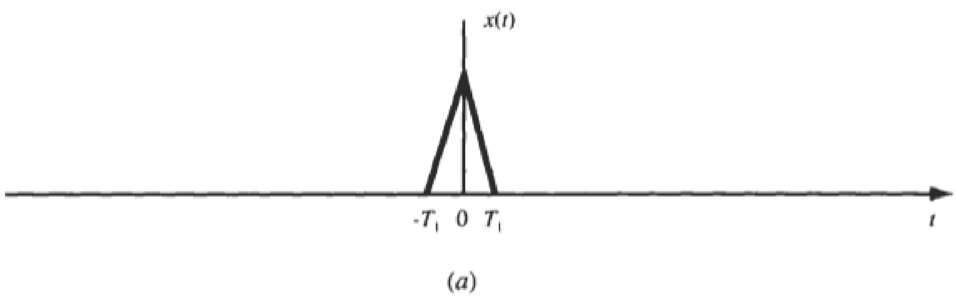
$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

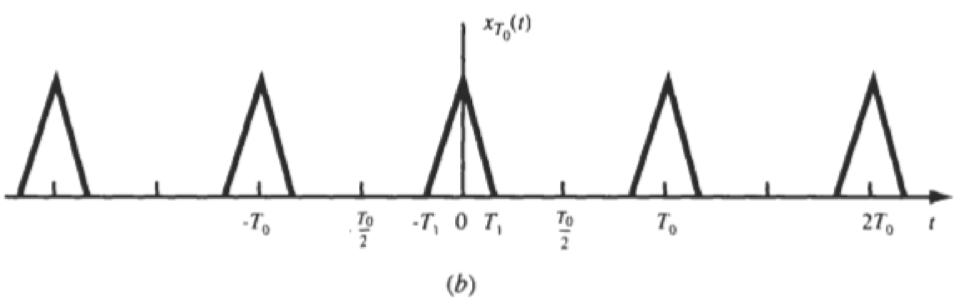
The complex exponential Fourier series of $x_{T_0}(t)$ is given by

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

where $$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

Since $x_{T_0}(t) = x(t)$ for $|t| < T_0/2$ and also since $x(t) = 0$ outside this interval





-1 (a) Nonperiodic signal $x(t)$; (b) periodic signal formed by periodic extension of $x(t)$.

can be rewritten as

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

Let us define $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

the complex Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{T_0} X(k\omega_0)$$

And

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

B. Fourier Transform Pair:

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

and we say that $x(t)$ and $X(\omega)$ form a Fourier transform pair denoted by

$$x(t) \longleftrightarrow X(\omega)$$

D. Fourier Spectra:

The Fourier transform X(w) of x(t) is, in general, complex, and it can be expressed as

$$X(\omega) = |X(\omega)| e^{j\phi(\omega)}$$

The quantity $|X(\omega)|$ is called the *magnitude spectrum* of $x(t)$, and $\phi(\omega)$ is called the *phase spectrum* of $x(t)$.

If $x(t)$ is a real signal, then

$$X(-\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$

Then it follows that

$$X(-\omega) = X^*(\omega)$$

and

$$|X(-\omega)| = |X(\omega)| \qquad \phi(-\omega) = -\phi(\omega)$$

Hence, as in the case of periodic signals, the amplitude spectrum $|X(\omega)|$ is an even function and the phase spectrum $\phi(\omega)$ is an odd function of ω.

E. Convergence of Fourier Transforms:

1. $x(t)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval, and each of these discontinuities is finite.

F. Connection between the Fourier Transform and the Laplace Transform:

the Fourier transform of $x(t)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

The bilateral Laplace transform of x(t), as defined in

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

the Fourier transform is a special case of the Laplace transform in which $s = j\omega$, that is,

$$X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}$$

Setting $s = \sigma + j\omega$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt = \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t}\right] e^{-j\omega t}\, dt$$

or

$$X(\sigma + j\omega) = \mathscr{F}\{x(t)\, e^{-\sigma t}\}$$

Example:

Consider the unit impulse function $\delta(t)$.

Then find its lap laplace and Fourier transform.

Solution.

the Laplace transform of $\delta(t)$ is

$$\mathscr{L}\{\delta(t)\} = 1 \qquad \text{all } s$$

And

the Fourier transform of $\delta(t)$ is

$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$

Thus, the Laplace transform and the Fourier transform of $\delta(t)$ are the same.

Properties of the continuous-time Fourier transform

A. Linearity:

$$a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(\omega) + a_2X_2(\omega)$$

B. Time Shifting:

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(\omega)$$

C. Frequency Shifting:

$$e^{j\omega_0 t}x(t) \leftrightarrow X(\omega - \omega_0)$$

D. Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

E. Time Reversal:

$$x(-t) \leftrightarrow X(-\omega)$$

G. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

F. Duality (or Symmetry):

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

G. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

H. Differentiation in the Frequency Domain:

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega}$$

I. Integration in the Time Domain:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \pi X(0)\, \delta(\omega) + \frac{1}{j\omega} X(\omega)$$

J. Convolution:

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega) X_2(\omega)$$

K. Multiplication:

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

L. Additional Properties:

If $x(t)$ is real, let

$$x(t) = x_e(t) + x_o(t)$$

where $x_e(t)$ and $x_o(t)$ are the even and odd components of $x(t)$, respectively.

$$x(t) \leftrightarrow X(\omega) = A(\omega) + jB(\omega)$$

Then
$$X(-\omega) = X^*(\omega)$$

$$x_e(t) \leftrightarrow \mathrm{Re}\{X(\omega)\} = A(\omega)$$

$$x_o(t) \leftrightarrow j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$$

The frequency response of continuous-time lti systems

A. Frequency Response:

the output ***y (t)*** of a continuous-time LTI system equals the convolution of the input ***x(t)*** with the impulse response ***h(t***1; that is,

$$y(t) = x(t) * h(t)$$

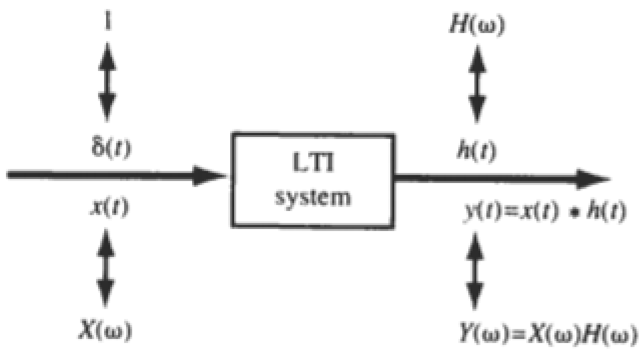
Applying the convolution property

$$Y(\omega) = X(\omega)H(\omega)$$

We have

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

The function H (w) is called the frequency response of the system.



Relationships between inputs and outputs in an LTI system.

Consider the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

With Fourier transform of

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

And we have

$$Y(\omega) = 2\pi H(\omega_0)\,\delta(\omega - \omega_0)$$

Taking the inverse Fourier transform of $Y(\omega)$, we obtain

$$y(t) = H(\omega_0)\, e^{j\omega_0 t}$$

if the input x(t) is periodic with the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

then the corresponding output $y(t)$ is also periodic with the Fourier series

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0)\, e^{jk\omega_0 t}$$

If *x (t)* is not periodic, then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

And the corresponding output *y(t)* can be expressed as

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)\, e^{j\omega t}\, d\omega$$

Thus, the behavior of a continuous-time LTI system in the frequency domain is completely characterized by its frequency response $H(\omega)$. Let

$$X(\omega) = |X(\omega)|e^{j\theta_X(\omega)} \qquad Y(\omega) = |Y(\omega)|e^{j\theta_Y(\omega)}$$

Then we have

$$|Y(\omega)| = |X(\omega)||H(\omega)|$$

$$\theta_Y(\omega) = \theta_X(\omega) + \theta_H(\omega)$$

Fourier analysis of discrete time signal and systems

DISCRETE FOURIER SERIES

A. Periodic Sequences:

In Chap. 1 we defined a discrete-time signal (or sequence) $x[n]$ to be periodic if is a positive integer N for which

$$x[n+N]=x[n] \qquad \text{all } n$$

B. Discrete Fourier Series Representation:

The discrete Fourier series representation of a periodic sequence $x[n]$ with fundamental period N_0 is given by

$$x[n]=\sum_{k=0}^{N_0-1} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0=\frac{2\pi}{N_0}$$

where c_k are the Fourier coefficients and are given by (Pr

$$c_k=\frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$$

C. **Convergence of Discrete Fourier Series:** Since the discrete Fourier series is a finite series, in contrast to the continuous-time case, there are no convergence issues with discrete Fourier series.

D. Properties of Discrete Fourier Series:

I. *Periodicity of Fourier Coefficients:*

$$c_{k+N_0}=c_k$$

2. Duality

$$c[k]=\sum_{n=\langle N_0\rangle}\frac{1}{N_0}x[n]\, e^{-jk\Omega_0 n}$$

3. Other Properties:

Even and Odd Sequences:

When $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$, respectively.

THE FOURIER TRANSFORM

A. From Discrete Fourier Series to Fourier Transform:

Let $x[n]$ be a nonperiodic sequence of finite duration. That is, for some positive integer N_1,

$$x[n] = 0 \qquad |n| > N_1$$

B. Fourier Transform Pair: The function X(R) defined by

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$x[n] = \mathscr{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

and we say that $x[n]$ and $X(\Omega)$ form a Fourier transform pair denoted by

$$x[n] \leftrightarrow X(\Omega)$$

C. Fourier Spectra:

The Fourier transform $X(\Omega)$ of $x[n]$ is, in general, complex and can be expressed as

$$X(\Omega) = |X(\Omega)|e^{j\phi(\Omega)}$$

D. Convergence of $X(\Omega)$:

Just as in the case of continuous time, the sufficient condition for the convergence of $X(\Omega)$ is that $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

F. Connection between the Fourier Transform and the z-Transform:

the Fourier transform of $x[n]$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

The z-transform of x[n], as defined in

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Example1:

Consider the causal exponential sequence

$$x[n] = a^n u[n] \qquad a \text{ real}$$

Find.

$X(\Omega)$

Solution:

the z-transform of $x[n]$ is given by

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

First find

Thus, $X(e^{j\Omega})$ exists for $|a| < 1$ because the ROC of $X(z)$ then contains the unit circle. That is,

$$X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

the Fourier transform of ***x [n]*** is

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n$$

$$= \frac{1}{1 - ae^{-j\Omega}} \qquad |ae^{-j\Omega}| = |a| < 1$$

we have

$$X(\Omega) = X(z)|_{z=e^{j\Omega}}$$

Note that ***x [n]*** is absolutely summable.

PROPERTIES OF THE FOURIER TRANSFORM

A. Periodicity:

$$X(\Omega + 2\pi) = X(\Omega)$$

B. Linearity:

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$$

C. Time Shifting:

$$x[n - n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$$

D. Frequency Shifting:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(\Omega - \Omega_0)$$

E. Conjugation:

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

where $*$ denotes the complex conjugate.

F. Time Reversal:

$$x[-n] \leftrightarrow X(-\Omega)$$

G. Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

G. Duality:

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

H

H

I. Differentiation in Frequency:

$$nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

J. Differencing:

$$x[n] - x[n-1] \leftrightarrow (1 - e^{-j\Omega})X(\Omega)$$

Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)X_2(\Omega)$$

Multiplication:

$$x_1[n]x_2[n] \leftrightarrow \frac{1}{2\pi}X_1(\Omega) \otimes X_2(\Omega)$$

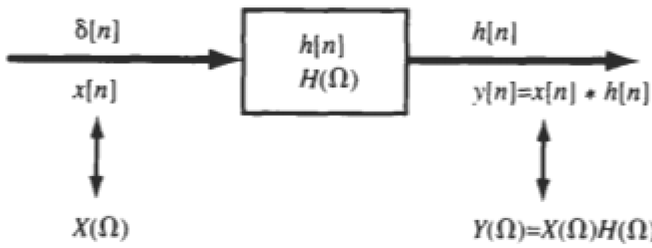
THE FREQUENCY RESPONSE OF DISCRETE-TIME LTI SYSTEMS

A. Frequency Response:

$$y[n] = x[n] * h[n]$$

Applying the convolution property

$$Y(\Omega) = X(\Omega)H(\Omega)$$



Relationships between inputs and outputs in an LTI discrete-time system.

B. LTI Systems Characterized by Difference Equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

C. Periodic Nature of the Frequency Response:

we have

$$H(\Omega) = H(\Omega + 2\pi)$$

EXERCISES

(1). Determine the complex exponential Fourier series representation for each of the following signals:

$(a)\quad x(t) = \cos \omega_0 t$

(b)

$$x(t) = \cos 4t + \sin 6t$$

(2). Consider the periodic square wave ***x (t)*** shown in Fig bellow.

(a) Determine the complex exponential Fourier series of ***x (t)***.

(b) Determine the trigonometric Fourier series of ***x (t)***.

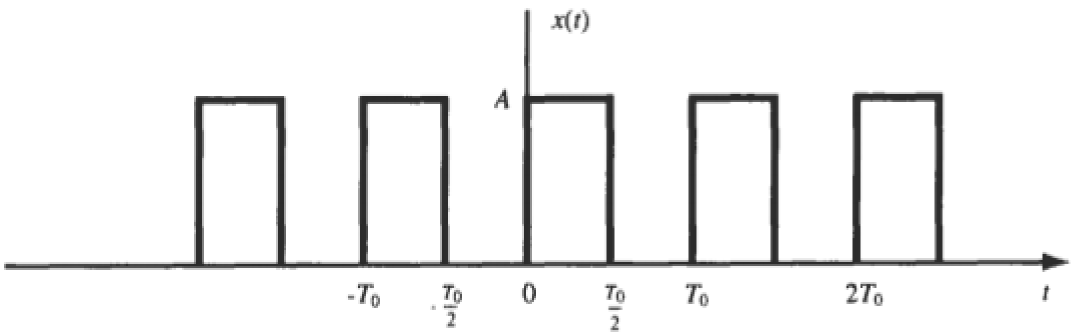


Fig 1.

(3). Find the Fourier transform of the signal

a).

$$x(t) = e^{-a|t|} \qquad a > 0$$

b).

$$x(t) = \frac{1}{a^2 + t^2}$$

c). $x(t) = u(-t)$

(4). **Using the time convolution theorem find the inverse Fourier transform of**

$X(\omega) = 1/(a + j\omega)^2$.

5).

Consider a continuous-time LTI system described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Using the Fourier signals: transform, **find** the output y(t) to each of the following input

(a) $x(t) = e^{-t}u(t)$

(6).

Consider a rectified sine wave signal $x(t)$ defined by

$$x(t) = |A \sin \pi t|$$

(a) Sketch $x(t)$ and find its fundamental period.
(b) Find the complex exponential Fourier series of $x(t)$.
(c) Find the trigonometric Fourier series of $x(t)$.

7).

Find the inverse Fourier transform of

$$X(\omega) = \frac{1}{2 - \omega^2 + j3\omega}$$

.

(8). **Determine the discrete Fourier series representation for each of the given sequences:**

$$x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$$

CHAPTER FOUR

LAPLACE TRANSFORM AND CONTINUOUS-TIME LTI SYSTEMS

Introduction

the Laplace transform is introduced to represent continuous-time signals in the s-domain (s is a complex variable)

THE LAPLACE TRANSFORM

for a continuous-time LTI system with impulse response h(t),

the output $y(t)$ of the system to the complex exponential input of the form e^{st} is

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\, dt$$

A. **Definition:**

For a

general continuous-time signal x(t), the Laplace transform X(s) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt$$

the unilateral (or one-sided) Laplace transform, which is

defined as

$$X_I(s) = \int_{0^-}^{\infty} x(t)e^{-st}\, dt$$

and the signal x(t) and its Laplace transform X(s) are said to form a Laplace transform pair denoted as

$$x(t) \leftrightarrow X(s)$$

B. The Region of Convergence:

The range of values of the complex variables s for which the Laplace transform converges is called the *region of convergence* (ROC). To illustrate the Laplace transform and the associated ROC let us consider some examples.

Example 1:

Consider the signal

$$x(t) = e^{-at}u(t) \qquad a \text{ real}$$

Solution:

the Laplace transform of x(t) is

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(s+a)t}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_{0^+}^{\infty} = \frac{1}{s+a} \qquad \operatorname{Re}(s) > -a$$

because $\lim_{t\to\infty} e^{-(s+a)t} = 0$ only if $\operatorname{Re}(s+a) > 0$ or $\operatorname{Re}(s) > -a$.

C. Poles and Zeros of $X(s)$:

Usually, $X(s)$ will be a rational function in s, that is,

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0}\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

C. Properties of the ROC:

Property 1: The ROC does not contain any poles.

Property 2: If $x(t)$ is a *finite-duration* signal, that is, $x(t) = 0$ except in a finite interval $t_1 \le t \le t_2$ $(-\infty < t_1$ and $t_2 < \infty)$, then the ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.

Property 3: If $x(t)$ is a *right-sided* signal, that is, $x(t) = 0$ for $t < t_1 < \infty$, then the ROC is of the form

$$\operatorname{Re}(s) > \sigma_{\max}$$

where $\sigma_{\max}$ equals the maximum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the right of the vertical line $\text{Re}(s) = \sigma_{\max}$ in the s-plane and thus to the right of all of the poles of $X(s)$.

Property 4: If $x(t)$ is a *left-sided* signal, that is, $x(t) = 0$ for $t > t_2 > -\infty$, then the ROC is of the form

$$\text{Re}(s) < \sigma_{\min}$$

where $\sigma_{\min}$ equals the minimum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the left of the vertical line $\text{Re}(s) = \sigma_{\min}$ in the s-plane and thus to the left of all of the poles of $X(s)$.

Property 5: If $x(t)$ is a *two-sided* signal, that is, $x(t)$ is an infinite-duration signal that is neither right-sided nor left-sided, then the ROC is of the form

$$\sigma_1 < \text{Re}(s) < \sigma_2$$

where σ_1 and σ_2 are the real parts of the two poles of $X(s)$. Thus, the ROC is a vertical strip in the s-plane between the vertical lines $\text{Re}(s) = \sigma_1$ and $\text{Re}(s) = \sigma_2$.

PROPERTIES OF THE LAPLACE TRANSFORM

A. Linearity:

If

$$x_1(t) \longleftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

Then $$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(s) + a_2 X_2(s) \qquad R' \supset R_1 \cap R_2$$

B. Time Shifting:

If

$$x(t) \longleftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(t - t_0) \longleftrightarrow e^{-st_0} X(s) \qquad R' = R$$

C. Shifting in the s-Domain:

If

$$x(t) \longleftrightarrow X(s) \qquad \text{ROC} = R$$

then $$e^{s_0 t} x(t) \longleftrightarrow X(s - s_0) \qquad R' = R + \text{Re}(s_0)$$

D. Time Scaling:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR$$

E. Time Reversal:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

And

$$x(-t) \leftrightarrow X(-s) \qquad R' = -R$$

F. Differentiation in the Time Domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$\frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$$

G. Differentiation in the *s*-Domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

H. Integration in the Time Domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X(s) \qquad R' = R \cap \{\text{Re}(s) > 0\}$$

I. Convolution:

If

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

THE INVERSE LAPLACE TRANSFORM

Inversion of the Laplace transform to find the signal ***x (t)*** from its Laplace transform X(s) is called the inverse Laplace transform, symbolically denoted as

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$

A. Inversion Formula:

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds$$

B. **Use of Tables of Laplace Transform Pairs:** In the second method for the inversion of X(s), we attempt to express X(s) as a sum

$$X(s) = X_1(s) + \cdots + X_n(s)$$

C. **Partial-Fraction Expansion:** If X(s) is a rational function, that is, of the form

$$X(s) = \frac{N(s)}{D(s)} = k \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

Two pole cases(for proper rational function)

1. **Simple Pole Case:** If all poles of X(s), that is, all zeros of D(s), are simple (or distinct), then X(s) can be written as

$$X(s) = \frac{c_1}{s-p_1} + \cdots + \frac{c_n}{s-p_n}$$

where coefficients c_k are given by

$$c_k = (s-p_k)X(s)\big|_{s=p_k}$$

2. Multiple pole cases

$$\frac{\lambda_1}{s-p_i} + \frac{\lambda_2}{(s-p_i)^2} + \cdots + \frac{\lambda_r}{(s-p_i)^r}$$

where $$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left[(s-p_i)^r X(s)\right]\Big|_{s=p_i}$$

When $X(s)$ is an improper rational function, that is, when $m \geq n$:
If $m \geq n$, by long division we can write $X(s)$ in the form

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

THE SYSTEM FUNCTION

A. The System Function:

the output ***y (t)*** of a continuous-time LTI system equals the convolution of the input ***x (t)***with the impulse response ***h(t);***that is,

$$y(t) = x(t) * h(t)$$

Applying the convolution property

$$Y(s) = X(s)H(s)$$

B. Characterization of LTI Systems:

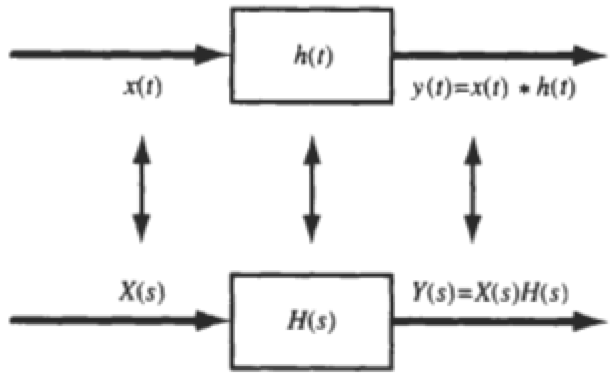


Fig. Impulse response and system function.

1. Causality:

For a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0$$

Since $h(t)$ is a right-sided signal, the corresponding requirement on $H(s)$ is that the ROC of $H(s)$ must be of the form

$$\mathrm{Re}(s) > \sigma_{\max}$$

That is, the ROC is the region in the s-plane to the right of all of the system poles. Similarly, if the system is anticausal, then

$$h(t) = 0 \qquad t > 0$$

and $h(t)$ is left-sided. Thus, the ROC of $H(s)$ must be of the form

$$\mathrm{Re}(s) < \sigma_{\min}$$

That is, the ROC is the region in the s-plane to the left of all of the system poles.

3. Stability:

a continuous-time LTI system is B I B 0 stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Systems Interconnection:

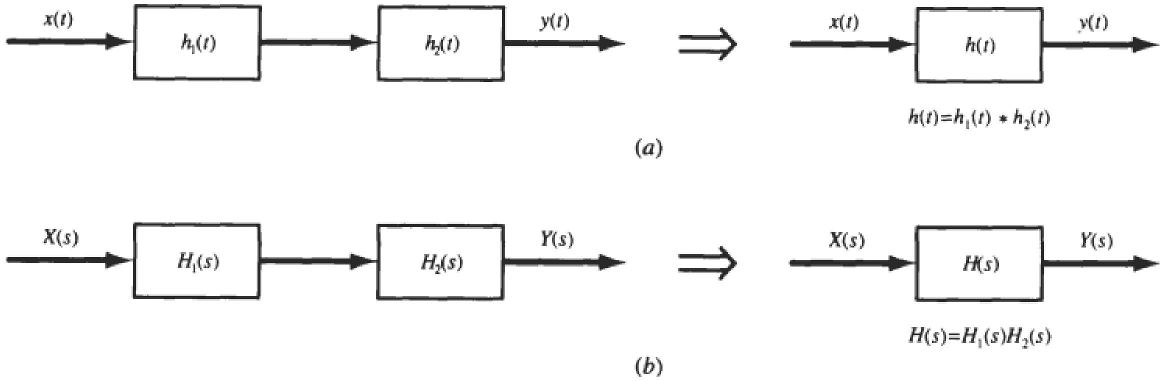


Fig. Two systems in cascade. ***(a)***Time-domain representation; ***(b)***s-domain representation

A. Definitions:unilaTERAL LAPLACE TRANSFORM

The unilateral (or one-sided) Laplace transform X,(s) of a signal x(t) is defined as

$$X_I(s) = \int_{0^-}^{\infty} x(t) e^{-st}\, dt$$

B. Basic Properties:

1. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow sX_I(s) - x(0^-)$$

2. Integration in the Time Domain:

$$\int_{0^-}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X_I(s)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X_I(s) + \frac{1}{s} \int_{-\infty}^{0^-} x(\tau)\, d\tau$$

Transform Circuits:

Signal Sources:

$$v(t) \leftrightarrow V(s) \qquad i(t) \leftrightarrow I(s)$$

where $v(t)$ and $i(t)$ are the voltage and current source signals, respectively.

2. *Resistance R:*

$$v(t) = Ri(t) \leftrightarrow V(s) = RI(s)$$

3. *Inductance L:*

$$v(t) = L\frac{di(t)}{dt} \leftrightarrow V(s) = sLI(s) - Li(0^-)$$

4. *Capacitance C:*

$$i(t) = C\frac{dv(t)}{dt} \leftrightarrow I(s) = sCV(s) - Cv(0^-)$$

EXERCISES

(1). Find the Laplace transform of the following ***x(t***).

(a) $x(t) = \sin \omega_0 t u(t)$

b). $x(t) = e^{-at}u(t) - e^{at}u(-t)$

2).

Find the inverse Laplace transform of the following $X(s)$;

(a) $X(s) = \dfrac{1}{s(s+1)^2}$, $\operatorname{Re}(s) > -1$

b).

$X(s) = \dfrac{s+1}{s^2+4s+13}$, $\operatorname{Re}(s) > -2$

3).

Find the output $y(t)$ of the continuous-time LTI system with

$$h(t) = e^{-2t}u(t)$$

for the each of the following inputs:

(a) $x(t) = e^{-t}u(t)$

CHAPTER FIVE

THE Z-TRANSFORM AND DISCRETE-TIME LTI SYSTEMS

Introduction

The z-transform is introduced to represent discrete-time signals (or sequences) in the z-domain (
z is a
complex variable). And The properties of the z-transform closely parallel those of the Laplace transform.

THE Z-TRANSFORM for a discrete-time LTI system with impulse response h[n], the output y[n] of the system to the complex exponential input of the form

z^n is

$$y[n] = \mathbf{T}\{z^n\} = H(z)z^n$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

A. Definition:

the z-transform of h[n]. For a general
discrete-time signal x[n], the z-transform X (z) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The variable z is generally complex-valued and is expressed in polar form as

$$z = re^{j\Omega}$$

the unilateral (or one-sided) z-transform, which is defined as

$$X_I(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

The $x[n]$ and $X(z)$ are said to form a z-transform pair denoted as

$$x[n] \leftrightarrow X(z)$$

B. The Region of Convergence:

the range of values of the complex variable z

for which the z-transform converges is called the region of convergence (ROC).

Example 1. Consider the sequence

$$x[n] = a^n u[n] \qquad a \text{ real}$$

Solution

the z-transform of ***x [n]***is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

For the convergence of $X(z)$ we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Then

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

Alternatively, by multiplying the numerator and denominator

$$X(z) = \frac{z}{z - a} \qquad |z| > |a|$$

C. Properties of the ROC:

the ROC of ***X (z)*** depends on the nature of ***x [n] .*** The properties of the ROC are summarized below. We assume that ***X (Z)*** is a rational function of z.

Property 1: The ROC does not contain any poles.

Property 2: If $x[n]$ is a finite sequence (that is, $x[n] = 0$ except in a finite interval $N_1 \le n \le N_2$, where N_1 and N_2 are finite) and $X(z)$ converges for some value of z, then the ROC is the entire z-plane except possibly $z = 0$ or $z = \infty$.

Property 3: If $x[n]$ is a right-sided sequence (that is, $x[n] = 0$ for $n < N_1 < \infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| > r_{\max} \qquad \text{or} \qquad \infty > |z| > r_{\max}$$

where $r_{\max}$ equals the largest magnitude of any of the poles of $X(z)$. Thus, the ROC is the exterior of the circle $|z| = r_{\max}$ in the z-plane with the possible exception of $z = \infty$.

Property 4: If $x[n]$ is a left-sided sequence (that is, $x[n] = 0$ for $n > N_2 > -\infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| < r_{\min} \qquad \text{or} \qquad 0 < |z| < r_{\min}$$

where $r_{\min}$ is the smallest magnitude of any of the poles of $X(z)$. Thus, the ROC is the interior of the circle $|z| = r_{\min}$ in the z-plane with the possible exception of $z = 0$.

Property 5: If $x[n]$ is a two-sided sequence (that is, $x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of z, then the ROC is of the form

$$r_1 < |z| < r_2$$

where r_1 and r_2 are the magnitudes of the two poles of $X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z| = r_1$ and $|z| = r_2$ not containing any poles.

z-TRANSFORMS OF SOME COMMON SEQUENCES

A. Unit Impulse Sequence $\delta[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^{-0} = 1 \qquad \text{all } z$$

Thus,

$$\delta[n] \leftrightarrow 1 \qquad \text{all } z$$

B. Unit Step Sequence $u[n]$:

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1$$

PROPERTIES OF THE 2-TRANSFORM

A. Linearity:

If

$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

then

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(z) + a_2 X_2(z) \qquad R' \supset R_1 \cap R_2$$

where a_1 and a_2 are arbitrary constants.

B. Time Shifting:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$x[n-n_0] \leftrightarrow z^{-n_0} X(z) \qquad R' = R \cap \{0 < |z| < \infty\}$$

Special Cases:

$$x[n-1] \leftrightarrow z^{-1} X(z) \qquad R' = R \cap \{0 < |z|\}$$
$$x[n+1] \leftrightarrow z X(z) \qquad R' = R \cap \{|z| < \infty\}$$

C. Multiplication by z_0^n:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0|R$$

In particular, a pole (or zero) at $z = z_k$ in $X(z)$ moves to $z = z_0 z_k$ after multiplication by z_0^n and the ROC expands or contracts by the factor $|z_0|$.

Special Case:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0} z) \qquad R' = R$$

In this special case, all poles and zeros are simply rotated by the angle Ω_0 and the ROC is unchanged.

D. Time Reversal:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \qquad R' = \frac{1}{R}$$

E. Multiplication by n (or Differentiation in z):

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad R' = R$$

F. Accumulation:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{1}{1-z^{-1}}X(z) = \frac{z}{z-1}X(z) \qquad R' \supset R \cap \{|z| > 1\}$$

G. Convolution:

If

$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

then

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad R' \supset R_1 \cap R_2$$

THE INVERSE Z-TRANSFORM

Inversion of the z-transform to find the sequence $x[n]$ from its z-transform $X(z)$ is called the inverse z-transform, symbolically denoted as

$$x[n] = \mathfrak{Z}^{-1}\{X(z)\}$$

A. Inversion Formula: As in the case of the Laplace transform, there is a formal expression for the inverse z-transform in terms of an integration in the z-plane; that is,

$$x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1}\, dz$$

B. Use of Tables of z-Transform Pairs: In the second method for the inversion of X(z), we attempt to express X(z) as a sum

$$X(z) = X_1(z) + \cdots + X_n(z)$$

C. Power Series Expansion:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2] z^2 + x[-1] z + x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

D. Partial-Fraction Expansion:

$$X(z) = \frac{N(z)}{D(z)} = k \frac{(z - z_1) \cdots (z - z_m)}{(z - p_1) \cdots (z - p_n)}$$

Assuming $n \geq m$ and all poles p_k are simple, then

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \cdots + \frac{c_n}{z - p_n} = \frac{c_0}{z} + \sum_{k=1}^{n} \frac{c_k}{z - p_k}$$

where

$$c_0 = X(z)|_{z=0} \qquad c_k = (z - p_k)\frac{X(z)}{z}\bigg|_{z=p_k}$$

Hence, we obtain

$$X(z) = c_0 + c_1 \frac{z}{z - p_1} + \cdots + c_n \frac{z}{z - p_n} = c_0 + \sum_{k=1}^{n} c_k \frac{z}{z - p_k}$$

Thus for *rn* > n, the complete partial-fraction

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} c_k \frac{z}{z - p_k}$$

If $X(z)$ has multiple-order poles, say p_i is the multiple pole with multiplicity r, then the expansion of $X(z)/z$ will consist of terms of the form

$$\frac{\lambda_1}{z - p_i} + \frac{\lambda_2}{(z - p_i)^2} + \cdots + \frac{\lambda_r}{(z - p_i)^r}$$

where

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{dz^k} \left[(z - p_i)^r \frac{X(z)}{z} \right] \Bigg|_{z = p_i}$$

The system function of discrete-time LTI systems

A. The System Function:

$$y[n] = x[n] * h[n]$$

Applying the convolution property

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

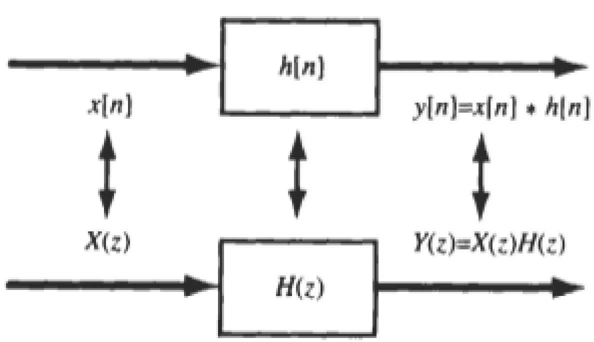


Fig. Impulse response and system function.

B. Characterization of Discrete-Time LTI Systems:

Many properties of discrete-time LTI systems can be closely associated with the characteristics of H(z) in the z-plane and in particular with the pole locations and the ROC.

1. Causality: For a causal discrete-time LTI system, we have

$$h[n] = 0 \qquad n < 0$$

since $h[n]$ is a right-sided signal, the corresponding requirement on $H(z)$ is that the ROC of $H(z)$ must be of the form

$$|z| > r_{\max}$$

That is, the ROC is the exterior of a circle containing all of the poles of $H(z)$ in the z-plane. Similarly, if the system is anticausal, that is,

$$h[n] = 0 \qquad n \geq 0$$

then $h[n]$ is left-sided and the ROC of $H(z)$ must be of the form

$$|z| < r_{\min}$$

That is, the ROC is the interior of a circle containing no poles of $H(z)$ in the z-plane.

2. Stability:

a discrete-time LTI system is BIB0 stable if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

3. Causal and Stable Systems:

If the system is both causal and stable, then all of the poles of $H(z)$ must lie inside the unit circle of the z-plane because the ROC is of the form $|z| > r_{\max}$, and since the unit circle is included in the ROC, we must have $r_{\max} < 1$.

EXERCISES

(1).

Find the z-transform of

$(a)\quad x[n] = -a^n u[-n-1]$

(2).

A finite sequence $x[n]$ is defined as

$$x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$$

Find $X(z)$ and its ROC.

.

(3). Find the z-transform ***X(z)*** and sketch the pole-zero plot with the ROC for each of the given sequences:

$(a)\quad x[n] = (\tfrac{1}{2})^n u[n] + (\tfrac{1}{3})^n u[n]$

4).

Let

$$x[n] = a^{|n|} \qquad a > 0 \qquad (\cdot$$

(a) Sketch $x[n]$ for $a < 1$ and $a > 1$.

(b) Find $X(z)$ and sketch the zero-pole plot and the ROC for $a < 1$ and $a > 1$

5).

Find the inverse z-transform of

$$X(z) = z^2(1 - \tfrac{1}{2}z^{-1})(1 - z^{-1})(1 + 2z^{-1}) \qquad 0 < |z| < \infty$$

6). Using the power series expansion technique and partial-fraction expansion, find the inverse z-transform of the expression X(z):

$$(a) \quad X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < \frac{1}{2}$$

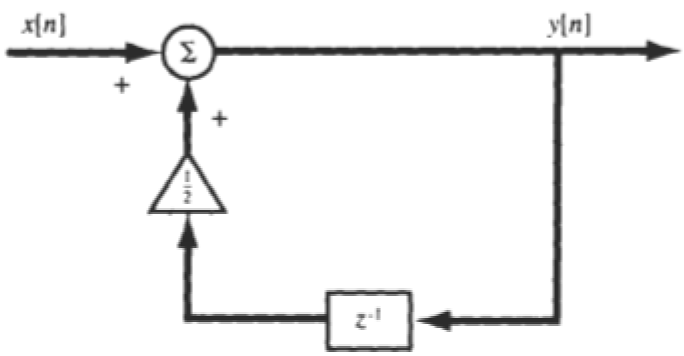
z

7). Using the z-transform find the system function where ***x [n]*** and ***h [n]***are given by

a.

$$x[n] = u[n] \qquad h[n] = \alpha^n u[n] \qquad 0 < \alpha < 1$$

b. Find the system function *H (z)* and its impulse response h(n) shown in fig bellow.



8).

A causal discrete-time LTI system is described by

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = x[n]$$

where $x[n]$ and $y[n]$ are the input and output of the system, respectively.

(a) Determine the system function $H(z)$.

(b) Find the impulse response $h[n]$ of the system.

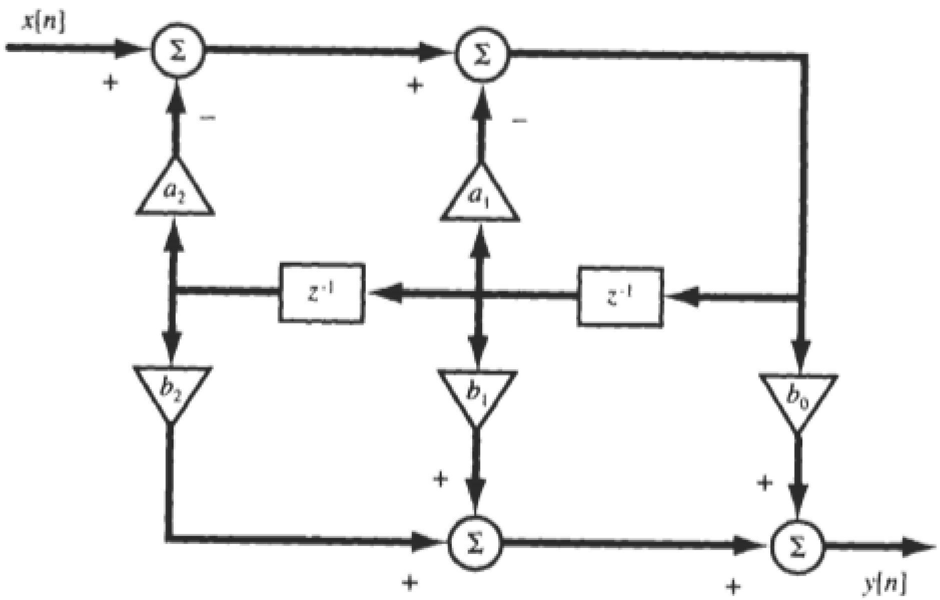
(c) Find the step response $s[n]$ of the system.

Assignments (25%)

(1). **Consider the system shown in Fig bellow.**

(a)* Find the system function *H (z)* *.(2.s pts)

(b)* Find the difference equation relating the output *y [n]* and input *x [n]* *.(2.s pts)



(2). 2.5 pts

Using the relation

$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \qquad |z| > |a|$$

find the z-transform of the following $x[n]$:

$x[n] = n(n-1)a^{n-2}u[n]$

(3).

Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t}u(t) \qquad x(t) = e^{\alpha t}u(-t) \qquad \alpha > 0$$

a). by Analytical method(3 pt)

b). by graphical method (3 pt)

(4). Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.(6 pts)

a).

$$x(t) = (\cos 2\pi t)u(t)$$

b).

$$x[n] = \cos\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{2}\right)$$

(5). Find the inverse Laplace transform of the expression X(s);(2.s pts)

$$X(s) = \frac{s}{s^3 + 2s^2 + 9s + 18}, \ \mathrm{Re}(s) > -2$$

(6).

Consider a rectified sine wave signal $x(t)$ defined by

$$x(t) = |A \sin \pi t|$$

(a) Sketch $x(t)$ and find its fundamental period.

(b) Find the complex exponential Fourier series of $x(t)$.

(c) Find the trigonometric Fourier series of $x(t)$. (3 pts).